Quantization for probability measures

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Abstract

We introduce the notions of quantization number and essential covering rate. Using these notions, we treat the quantization for product measures and give effective upper bounds for the quantization dimension of measures - especially for those with unbounded support. We also introduce the concepts of complete moment condition and limit quantization dimension and study the interesting cases where the r-moment condition holds for all $r \geq 1$.

We then introduce the notions of stability and stabilization for dimensions of measures. Under this framework, we study the stability of the upper and lower quantization dimension. Several examples are given. We prove that the stabilized upper quantization dimension coincides with the packing dimension, and for measures with compact support, it also coincides with the stabilized upper box counting-dimension. The quantization for homogeneous Cantor measures are particularly studied in detail to construct examples showing that the lower quantization dimension is not finitely stable.

We finally introduce the concept of the upper and lower vanishing rates. Using this concept and the upper and lower box-counting dimension, we study the relationship between the quantization and absolute continuity of measures. We give several sufficient conditions to ensure a definite inequality between the quantization dimension of two measures one of which is absolutely continuous with respect to the other. Measures which are absolutely continuous with respect to self-similar measures are particularly studied. A stronger result regarding the quantization coefficient is obtained as an analogue to the case of finite-dimensional Lebesgue measures.

Introduction

Quantization problems originate in information theory and engineering technology such as image compression and data processing. The study of this field goes back to the 1940's. Mathematically, the aim of quantization is to approximate a given probability measure by discrete probability measures with finite supports. Let $r \geq 1$ and μ_1, μ_2 be Borel probability measures on $\mathbb{R}^d$ with $\int \|x\|^r \, d\mu_i(x) < \infty, i = 1, 2$. We denote by $\mathcal{M}(\mu_1, \mu_2)$ the set of all Borel probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ with the i-th marginal $\mu_i,\ i = 1, 2$. Define

$$\rho_r(\mu_1, \mu_2) := \inf \left\{ \int \|x - y\|^r \, d\nu(x, y) : \ \nu \in \mathcal{M}(\mu_1, \mu_2) \right\}^{1/r}.$$

$\rho_r(\cdot, \cdot)$ defines a metric on the set of Borel probability measures on $\mathbb{R}^d$ with the r-moment condition $\int \|x\|^r \, d\mu(x) < \infty$, which is called L_r-minimal metric. Now let μ be a Borel probability measure on $\mathbb{R}^d$ and let $\mathcal{P}_n$ denote the set of all discrete probability measures Q on $\mathbb{R}^d$ with $\operatorname{card}(\operatorname{supp}(Q)) \leq n$. Define the n-th *quantization error* of μ of order r by $e_{n,r}(\mu) := V_{n,r}^{1/r}(\mu)$, where

$$V_{n,r}(\mu) := \inf \left\{ \rho_r^r(\mu, Q) : \ Q \in \mathcal{P}_n \right\}.$$

If the above infimum is attained at some $Q \in \mathcal{P}_n$, we then call Q an n-optimal probability measure since this measure best approximates μ with respect to the L_r-minimal metric. By [**9**, Lemma 6.1], if μ fulfills the r-*moment condition* $\int \|x\|^r \, d\mu(x) < \infty$ then the quantization error $e_{n,r}(\mu)$ tends to zero as n tends to infinity. The rate at which $e_{n,r}(\mu)$ tends to zero gives good information about how well the measure μ can be approximated by discrete measures with finite support. A good way to characterize this convergence rate is to consider the *upper* and *lower quantization dimension* of μ. Define

(0.0.1) $$\overline{D}_r(\mu) := \limsup_{n \to \infty} \frac{\log n}{-\log e_{n,r}(\mu)}, \ \underline{D}_r(\mu) := \liminf_{n \to \infty} \frac{\log n}{-\log e_{n,r}(\mu)}.$$

We call $\overline{D}_r(\mu), \underline{D}_r(\mu)$ the upper and lower quantization dimension of μ of order r. If $\overline{D}_r(\mu)$ coincides with $\underline{D}_r(\mu)$, then we call the common value the quantization dimension of μ of order r and denote it by $D_r(\mu)$.

The upper and lower quantization dimension of μ of *order infinity* $\overline{D}_\infty(\mu), \underline{D}_\infty(\mu)$ are defined similarly by replacing $e_{n,r}(\mu)$ with $e_{n,\infty}(\mu)$ given by

$$e_{n,\infty}(\mu) := \inf\left\{\sup_{x\in\text{supp}(\mu)} \min_{a\in\alpha} \|x-a\| : \ \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n\right\}. \tag{0.0.2}$$

The quantity $e_{n,\infty}(\mu)$ is called the n-th *covering radius* of $\text{supp}(\mu)$. Whenever we consider $\overline{D}_\infty(\mu), \underline{D}_\infty(\mu)$, we assume that $\text{supp}(\mu)$ is compact. Let $\overline{\dim}_B, \underline{\dim}_B$ respectively denote the upper and lower box-counting dimension of sets. Among other significant properties, it is shown in [**9**] that

$$\overline{D}_\infty(\mu) = \overline{\dim}_B(\text{supp}(\mu)), \quad \underline{D}_\infty(\mu) = \underline{\dim}_B(\text{supp}(\mu)). \tag{0.0.3}$$

The notion of quantization dimension was first introduced by ZADOR (cf. [**26**]). In this thesis, we will further study some basic properties of the quantization dimension, including the range and bounds of the quantization dimension and the quantization dimension for product measures. For this purpose we will introduce in chapter 1 the notion of quantization numbers and that of essential covering rate. These notions will also be used to prove an inequality between the upper quantization dimension and the upper rate distortion dimension.

The upper and lower s-dimensional *quantization coefficient* of order r are respectively defined by

$$\overline{Q}_r^s(\mu) := \limsup_{n\to\infty} n^{r/s} V_{n,r}(\mu), \quad \underline{Q}_r^s(\mu) := \liminf_{n\to\infty} n^{r/s} V_{n,r}(\mu).$$

When they coincide, we call the common value the s-dimensional quantization coefficient of μ of order r and denote it by $Q_r^s(\mu)$.

In accordance with the main aim of quantization, people are concerned with the following two objectives. The first objective is to seek for each $n \in \mathbb{N}$ the n-optimal probability measure, which best approximates μ; the second is to study the asymptotic property of the quantization error, including the calculation of the quantization dimension and the estimate of the quantization coefficient. We shall study in detail the quantization for the uniform probability measures on homogeneous Cantor sets and achieve these goals under some suitable conditions (cf. chapter 2). We remark that there are several equivalent definitions of the quantization error (cf. [**9**]). One may go back and forth among these equivalent definitions and choose for each context the most suitable one. We state here one of the equivalent definitions which we will use frequently in this thesis.

$$V_{n,r}(\mu) = \inf\left\{\int \min_{a\in\alpha} \|x-a\|^r \, d\mu(x) : \ \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n\right\}. \tag{0.0.4}$$

If the infimum in (0.0.4) is attained for some $\alpha \subset \mathbb{R}^d$ with $\mathrm{card}\,(\alpha) \leq n$, we then call α an n-optimal set of μ of order r. The collection of all n-optimal sets of μ of order r is denoted by $C_{n,r}\,(\mu)$.

In the following, we recall some recent progress in quantization theory which is closely connected to this work.

ZADOR (cf. [**27**]), BUCKLEW and WISE ([**2**]) proved that if a probability measure μ on $\mathbb{R}^d$ has compact support and its absolutely continuous part does not vanish, then

$$Q_r^d\,(\mu) = C\,(r,d) \left\| \frac{d\mu}{d\lambda^d} \right\|_{\frac{d}{d+r}},$$

where $C\,(r,d)$ is a positive constant depending only on r and d. This result is then proved valid for probability measures μ not necessarily with compact support but fulfilling the $(r+\delta)$-moment condition (cf. [**9**, Theorem 6.2]). So all absolutely continuous measures fulfilling the $(r+\delta)$-moment condition behave nicely with respect to their quantization properties. The only one drawback is that the constant $C\,(r,d)$ is only known for very few special cases and quite difficult to calculate. We study in Chapter 3 the relationship between quantization and absolute continuity of measures. We especially study those measures which are singular with respect to Lebesgue measure. Several counterexamples are constructed and various conditions will be given there to ensure a definite inequality between the quantization dimensions of two measures one of which is absolutely continuous with respect to the other.

GRAF and LUSCHGY have studied in detail the quantization properties of self-similar measures under the open set condition (cf. [**10, 7, 8**]). These results were extended by LINDSAY and MAULDIN to self-conformal measures (cf. [**17**]). In there, the authors point out the relationship between the quantization dimension for self-conformal measures and its multifractal spectrum. This shows that quantization theory should in some nice way be connected with fractal geometry. Indeed, according to [**9**], the upper (lower) quantization dimension of order infinity coincides with the upper (lower) box-counting dimension, which are two of the most popular fractal quantities. GRAF and LUSCHGY even conjecture that the covering problem is just the limit of the quantization problem, while we know that fractal objects, like Hausdorff, packing dimensions and measures, the upper and lower box-counting dimensions are all defined in terms of coverings, packings, or partitions. Let $\dim_H^*\mu, \dim_P^*\mu, \overline{\dim}_B^*\mu, \underline{\dim}_B^*\mu$ respectively denote the Hausdorff, packing, upper and lower box-counting dimension of μ. Then GRAF and LUSCHGY show in [**9**] that

$$\dim_H^*\mu \leq \underline{D}_r\,(\mu) \leq \overline{D}_r\,(\mu) \leq \overline{D}_\infty\,(\mu)\,.$$

Independently, PÖTZELBERGER proves for measures with compact support that

$$\dim_H^* \mu \leq \underline{D}_2(\mu) \leq \underline{\dim}_B^* \mu,\ \dim_p^* \mu \leq \underline{D}_2(\mu) \leq \overline{\dim}_B^* \mu.$$

Analogous to TRICOT's work for sets (cf. [**24**]), we shall make use of these results to effect a stabilization formalism for the upper quantization dimension and prove that the stabilized upper quantization dimension for a measure coincides with the packing dimension (cf. Theorem 2.4.9).

In this thesis, we further studies some basic properties of quantization for general Borel probability measures. We also investigate some particular measures in detail, either because they are interesting themselves or for the purpose of constructing meaningful examples. The thesis is organized as follows.

In Chapter 1, we introduce the notions of quantization number and essential covering rate. By means of the first notion, we first settle a question on the relationship between the upper rate distortion dimension and the upper quantization dimension, then prove an inequality between the upper quantization dimension of a product measure and that of its marginals. After that, we use the second notion to give an effective upper bound for the upper and lower quantization dimension. Finally, we will study the measures satisfying $\int \|x\|^r d\mu(x) < \infty$ for all $r \geq 1$.

In Chapter 2, we introduce the notion of stability and stabilization for dimensions of measures. We prove that the upper quantization dimension is finitely stable but not countably stable, while the lower quantization dimension is even not finitely stable. We also show that the stabilized upper quantization dimension coincides with the packing dimension. In particular, we study in detail the quantization for the uniform probability measures on homogeneous Cantor sets. We give explicit formulae for the upper and lower quantization dimension for these measures. Moreover, we construct examples to show that there exists a probability measure with its upper quantization dimension strictly greater than the lower one and that the lower quantization dimension is not finitely stable.

In Chapter 3, we study the relationship between quantization and absolute continuity of measures. We show that in general the absolute continuity between measures doesnot imply a definite inequality between their quantization dimensions. Then we give sufficient conditions under which a definite inequality is guaranteed. These conditions will be given in terms of the upper and lower box-counting dimension and the so-called upper and lower vanishing rates defined there. In particular, we will consider the quantization for measures which are absolutely continuous with respect to s-regular measures and self-similar measures. We shall prove an analogue of [**9**, Theorem 6.2] for restricted Hausdorff measure on self-similar sets under the strong separation condition.

CHAPTER 1

Quantization numbers and limit quantization dimensions

In this chapter, we give an interpretation of the quantization dimension of finite order in terms of quantization numbers defined in (1.1.1) (cf. [**15**]). As applications, we first solve a question on the upper rate distortion dimension which is left open in [**9**] and then prove an inequality for the upper quantization dimension of product measures. Moreover, we provide an upper bound for the quantization dimension by means of the essential covering rate defined in (1.4.5). This upper bound can be used for measures with unbounded support where known results are not applicable. In the last section, we study the class of measures μ fulfilling the complete moment conditions, i.e., $\int \|x\|^r \, d\mu(x) < \infty$ for all $r \in [1, \infty)$.

1.1. Quantization numbers

In this section, we establish an equivalent definition for quantization dimension in terms of quantization number. This result will be applied to treat the relationship between the upper rate distortion dimension and quantization dimension and the quantization for product measures. First we introduce the quantization number.

For $r \in [1, \infty]$, we call

$$n_{r,\epsilon}(\mu) := \inf\{n \geq 1 : \ e_{n,r}(\mu) \leq \epsilon\} \tag{1.1.1}$$

the *ϵ-quantization number of μ of order r.* GRAF and LUSCHGY showed in [**9**] that

$$\overline{D}_\infty(\mu) = \limsup_{\epsilon \to 0} \frac{\log n_{\infty,\epsilon}(\mu)}{-\log \epsilon}, \ \ \underline{D}_\infty(\mu) = \liminf_{\epsilon \to 0} \frac{\log n_{\infty,\epsilon}(\mu)}{-\log \epsilon}.$$

It is natural to ask whether the quantization dimensions of finite order share a similar property. Theorem 1.1.2 will show that this is the case. For the proof of this theorem, we need the following elementary lemma.

LEMMA 1.1.1. *Let $(\beta_n)_{n \geq 1}$be a non-increasing sequence of real numbers tending to zero. We define*

$$B(\epsilon) := \inf\{n \in \mathbb{N} : \ \beta_n \leq \epsilon\}.$$

Suppose either of the following two conditions holds:

(1) *There exists $N \geq 1$ such that $\beta_n = 0$ for all $n \geq N$.*

(2) *The sequence* $(\beta_n)_{n\geq 1}$ *is strictly decreasing.*

Then we have

$$\limsup_{n\to\infty} \frac{\log n}{-\log \beta_n} = \limsup_{\epsilon\to 0} \frac{\log B(\epsilon)}{-\log \epsilon}, \ \liminf_{n\to\infty} \frac{\log n}{-\log \beta_n} = \liminf_{\epsilon\to 0} \frac{\log B(\epsilon)}{-\log \epsilon}.$$

PROOF. We first assume that condition (1) holds. Without loss of generality, we assume that N is the smallest integer fulfilling the condition. If $N = 1$ then the lemma trivially holds. Otherwise we have $\beta_{N-1} > 0$. Since for any $0 < \epsilon < \beta_{N-1}$, we have $B(\epsilon) = N$ and $\beta_N = 0$, the equalities in the lemma follows.

Now we assume that (β_n) is strictly decreasing. Then for all $n \in \mathbb{N}$ we have $\beta_n > 0$. By the definition of $B(\epsilon)$,

$$\beta_{B(\epsilon)} \leq \epsilon, \ \beta_{B(\epsilon)-1} > \epsilon, \ \text{and}\, B(\beta_n) = n.$$

It follows that

$$\begin{aligned}\limsup_{n\to\infty} \frac{\log n}{-\log \beta_n} &= \limsup_{\epsilon\to 0} \frac{\log(B(\epsilon))}{-\log \beta_{B(\epsilon)}} \leq \limsup_{\epsilon\to 0} \frac{\log B(\epsilon)}{-\log \epsilon} \\ &\leq \limsup_{\epsilon\to 0} \frac{\log B(\epsilon)}{-\log \beta_{B(\epsilon)-1}} = \limsup_{\epsilon\to 0} \frac{\log(B(\epsilon)-1)}{-\log \beta_{B(\epsilon)-1}} \\ &= \limsup_{n\to\infty} \frac{\log n}{-\log \beta_n}.\end{aligned}$$

This proves the first equality in the lemma. For the second one, we need only to replace $\limsup$ with $\liminf$ and use a similar argument. □

THEOREM 1.1.2. *Let* μ *be a Borel probability measure on* $\mathbb{R}^d$ *with* $\int ||x||^r d\mu(x) < \infty$. *Let* $n_{r,\epsilon}(\mu)$ *be defined as above. Then for* $1 \leq r < \infty$ *we have*

$$\overline{D}_r(\mu) = \limsup_{\epsilon\to 0} \frac{\log n_{r,\epsilon}(\mu)}{-\log \epsilon}, \ \underline{D}_r(\mu) = \liminf_{\epsilon\to 0} \frac{\log n_{r,\epsilon}(\mu)}{-\log \epsilon}.$$

PROOF. If $\mathrm{card}(\mathrm{supp}(\mu)) < \infty$, then clearly condition (1) of Lemma 1.1.1 holds. We now assume that $\mathrm{card}(\mathrm{supp}(\mu)) = \infty$. By [**9**, Theorem 4.1, 4.12], we have $e_{n,r}(\mu) < e_{n-1,r}(\mu)$ for all $n \in \mathbb{N}$. Since $\lim_{n\to\infty} e_{n,r}(\mu) = 0$, the condition (2) of Lemma 1.1.1 holds. Combining the two observations, the Theorem follows.

REMARK 1.1.3. For the above theorem, we make the following two remarks.

(1) The property $e_{n,r}(\mu) < e_{n-1,r}(\mu)$ is not shared by the n-th covering radius $e_{n,\infty}(\mu)$. This can be seen from a simple counter example , say, the classical Cantor measure μ, where $e_{n,\infty}(\mu) = e_{n-1,\infty}(\mu)$ for infinitely many n.
(2) In fact, for any finite integer $k \geq 0$, we may choose (ϵ_j) for each sequence (n_j) such that $n_{r,\epsilon_j}(\mu) - k = n_j$, where we also could take $\epsilon_j = e_{n_j+k,r}(\mu)$.

Hence
$$\overline{D}_r(\mu) = \limsup_{\epsilon \to 0} \frac{\log(n_{r,\epsilon}(\mu) - k)}{-\log e_{(n_{r,\epsilon}(\mu))-k,r}(\mu)}.$$
The same equality holds for the lower quantization dimension of order r by replacing $\limsup$ with $\liminf$.

□

1.2. Rate distortion dimension

In this section, we solve a question on the upper rate distortion dimension. Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int ||x||^r d\mu(x) < \infty$ and Q a Borel probability on $\mathbb{R}^d \times \mathbb{R}^d$. Denote by Q_1, Q_2 the first and second marginal of Q respectively. If $Q_1 = \mu$, then the *average mutual information* $I(\mu, Q)$ of Q is given by
$$I(\mu, Q) := \int h(x,y) \log h(x,y) d(\mu \otimes Q_2)(x,y),$$
if Q is absolutely continuous with respect to $\mu \otimes Q_2$ and $h(x,y)$ is the corresponding Radon-Nikodym derivative, otherwise $I(\mu, Q) = \infty$. Now the *upper* and *lower rate distortion dimension* of order r of μ are defined to be
$$\overline{\dim}_R(\mu) = \limsup_{\epsilon \to 0} \frac{R_{\mu,r}(\epsilon^r)}{-\log \epsilon}, \quad \underline{\dim}_R(\mu) = \liminf_{\epsilon \to 0} \frac{R_{\mu,r}(\epsilon^r)}{-\log \epsilon},$$
where $R_{\mu,r}(t)$ is the *rate distortion function* of order r defined by
$$R_{\mu,r}(t) = \inf \left\{ I(\mu, Q) : \ Q_1 = \mu, \int \|x - y\|^r \, dQ(x,y) \leq t \right\}.$$
KAWABATA and DEMBO proved in [**12**] that the upper and lower rate distortion dimension do not depend on r and are equal to the upper and lower Renyi information dimension respectively. It is proved in [**9**, Theorem 11.10] that
$$\underline{\dim}_R(\mu) \leq \underline{D}_1(\mu) \leq \underline{D}_r(\mu) \leq \underline{D}_\infty(\mu) = \underline{\dim}_B(\mathrm{supp}(\mu)).$$
In there the authors leave it open whether corresponding inequalities hold for the upper rate distortion dimension. The following theorem gives a positive answer to this question as a straightforward consequence of Theorem 1.1.2.

THEOREM 1.2.1. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int ||x||^r d\mu(x) < \infty$. Then we have*
$$\overline{\dim}_R(\mu) \leq \overline{D}_1(\mu) \leq \overline{D}_r(\mu).$$

PROOF. It is proved in [**9**] that $e_{n,r}(\mu) \leq \epsilon$ implies $R_{\mu,r}(\epsilon^r) \leq \log n$. Recall that
$$n_{r,\epsilon}(\mu) = \inf\{n \geq 1 : \ e_{n,r}(\mu) \leq \epsilon\}.$$

Then we have $R_{\mu,r}(\epsilon^r) \leq \log n_{r,\epsilon}(\mu)$. It follows that

$$\overline{\dim}_R(\mu) \leq \limsup_{\epsilon \to 0} \frac{\log n_{r,\epsilon}(\mu)}{-\log \epsilon}.$$

So the first inequality follows from Theorem 1.1.2. □

1.3. Quantization dimension for product measures

In this section, we study the quantization dimension for product measures. Let $\mu_i, i = 1, 2$ be Borel probability measures on $(\mathbb{R}^{d_i}, \mathcal{B}_i)$. Recall that the product measure $\mu_1 \otimes \mu_2$ on $(\mathbb{R}^{d_1+d_2}, \mathcal{B}_1 \times \mathcal{B}_2)$ satisfies

$$(\mu_1 \otimes \mu_2)(A \times B) = \mu_1(A)\mu_2(B), \ \ f\text{or all } A \in \mathcal{B}_1, B \in \mathcal{B}_2.$$

Next we will show that the upper quantization dimension of the product measure is not greater than the sum of that of its marginals.

Let $\|\cdot\|_1, \|\cdot\|_2$ be arbitrary norms respectively on $\mathbb{R}^{d_1}$, $\mathbb{R}^{d_2}$. For any $w = (x, y) \in \mathbb{R}^{d_1+d_2}$, define

$$\|w\| = \|x\|_1 + \|y\|_2 \,. \tag{1.3.1}$$

Then $\|\cdot\|$ is a norm on $\mathbb{R}^{d_1+d_2}$.

THEOREM 1.3.1. *Let μ_i be Borel probability measures on $(\mathbb{R}^{d_i}, \mathcal{B}_i)$, $i = 1, 2$. Then*

$$\max\left\{\overline{D}_r(\mu_1), \overline{D}_r(\mu_2)\right\} \leq \overline{D}_r(\mu_1 \otimes \mu_2) \leq \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2).$$

PROOF. For $n_1, n_2 \in \mathbb{N}$, let

$$\beta_i \in C_{n_i,r}(\mu_i), \ \operatorname{card}(\beta_i) = n_i.$$

Then by Fubini theorem we have

$$\begin{aligned} V_{n_1 n_2,r}(\mu_1 \otimes \mu_2) &\leq \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} \min_{a \in \beta_1 \times \beta_2} \|w - a\|^r \, d(\mu_1 \otimes \mu_2)(w) \\ &\leq 2^r \int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} (\min_{b \in \beta_1} \|x - b\|^r + \min_{c \in \beta_2} \|y - c\|^r) d\mu_1(x) d\mu_2(y) \\ &= 2^r \int_{\mathbb{R}^{d_1}} \min_{b \in \beta_1} \|x - b\|^r \, d\mu_1(x) + 2^r \int_{\mathbb{R}^{d_2}} \min_{c \in \beta_2} \|y - c\|^r \, d\mu_2(y) \\ &= 2^r V_{n_1,r}(\mu_1) + 2^r V_{n_2,r}(\mu_2). \end{aligned}$$

It follows that

$$n_{r,2^{(r+1)/r}\epsilon}(\mu_1 \otimes \mu_2)) \leq n_{r,\epsilon}(\mu_1) n_{r,\epsilon}(\mu_2).$$

By Theorem 1.1.2 we have

$$\begin{aligned}
\overline{D}_r(\mu_1 \otimes \mu_2) &= \limsup_{\epsilon \to 0} \frac{\log n_{r,\epsilon}(\mu_1 \otimes \mu_2)}{-\log \epsilon} \\
&= \limsup_{\epsilon \to 0} \frac{\log n_{r,2^{(r+1)/r}\epsilon}(\mu_1 \otimes \mu_2)}{-\log \epsilon - (r+1)\log 2/r} \\
&\leq \limsup_{\epsilon \to 0} \frac{\log n_{r,\epsilon}(\mu_1) + \log n_{r,\epsilon}(\mu_2)}{-\log \epsilon} \\
&\leq \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2).
\end{aligned}$$

We now show the left-hand-side inequality. Let $\alpha \in C_{n,r}(\mu_1 \otimes \mu_2)$ and let α_1, α_2 respectively denote the projections of α onto $\mathbb{R}^{d_1}, \mathbb{R}^{d_2}$. Then clearly,

$$\alpha \subset \alpha_1 \times \alpha_2, \;\; \mathrm{card}\,(\alpha_i) \leq n, \;\; i = 1, 2.$$

Using this and the fact that $(A+B)^r \geq A^r + B^r$ for any $A, B \geq 0$ and any $r \geq 1$, we have

$$\begin{aligned}
V_{n,r}(\mu_1 \otimes \mu_2) &= \int \min_{a \in \alpha} \|w - a\|^r \, d(\mu_1 \otimes \mu_2)(w) \\
&\geq \int \min_{a \in \alpha_1 \times \alpha_2} \|w - a\|^r \, d(\mu_1 \otimes \mu_2)(w) \\
&\geq \int \min_{(b,c) \in \alpha_1 \times \alpha_2} (\|x - b\| + \|y - c\|)^r \, d\mu_1(x)\, d\mu_2(y) \\
&\geq \int_{\mathbb{R}^{d_1}} \min_{b \in \alpha_1} \|x - b\|^r \, d\mu_1(x) + \int_{\mathbb{R}^{d_2}} \min_{c \in \alpha_2} \|y - c\|^r \, d\mu_2(y) \\
&\geq \max\{V_{n,r}(\mu_1), V_{n,r}(\mu_2)\}.
\end{aligned}$$

This implies the first inequality in the theorem and finishes the proof. □

REMARK 1.3.2. Let λ^d denote the d-dimensional Lebesgue measure. If $\mu_1 = \mu_2 = \lambda^1(\cdot|[0,1])$, then $\mu_1 \otimes \mu_2 = \lambda^2(\cdot|[0,1]^2)$. In this case we clearly have $2 = D_r(\mu_1 \otimes \mu_2) = D_r(\mu_1) + D_r(\mu_2) = 1 + 1$. This shows that the second inequality in Theorem 1.3.1 is sharp.

1.4. Essential covering rate and the upper quantization dimension

In this section, we give upper bounds for the quantization dimension of measures by means of the essential covering rate defined in (1.4.1). This upper bound is especially effective for measures with $\int \|x\|^{r+\delta} \, d\mu(x) = \infty$ for all $\delta > 0$. Before we give the main result of this section, let us first have some discussions on the range of the quantization dimension.

1.4.1. The range of quantization dimension.

This subsection studies the set of all possible values for the upper quantization dimension of order r of a given probability measure. The following lemma is immediate.

LEMMA 1.4.1. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int \|x\|^{r+\delta} d\mu(x) < \infty$ for some $\delta > 0$. Then we have*

$$0 \leq \underline{D}_r(\mu) \leq \overline{D}_r(\mu) \leq d.$$

PROOF. This follows immediately from [**9**, Theorem 6.2]. □

In general, the upper quantization dimension of a Borel probability measure on $\mathbb{R}^d$ is not bounded by d if its support is not compact. This is illustrated by [**9**, Example 6.4], where the lower quantization dimension equals infinity since the n-th quantization error of order r is comparable with $-\log n$. If $\int \|x\|^{r+\delta} d\mu(x) < \infty$ and the absolute continuous part with respect to λ^d doesnot vanish, we know that $D_r(\mu) = d$. Hence it remains to study the case when the $(r+\delta)$-moment condition fails for all $\delta > 0$, i.e., $\int \|x\|^{r+\delta} d\mu(x) = \infty$ for all $\delta > 0$. The following proposition shows that in this case we may construct a Borel probability measure μ with quantization dimension s of order r for any given $s \in (0, \infty)$, in other words, the range of the quantization dimension of order r function is $[0, \infty]$.

PROPOSITION 1.4.2. *Let $r \geq 1$. For any $s \in [0, \infty]$, there exists a Borel probability measure μ such that $D_r(\mu) = s$.*

PROOF. For $s = 0$, we take μ to be any Dirac measure, and for $s = \infty$, we take μ to be the discrete measure constructed in [**9**, Example 6.4]. In the following let $s \in (0, \infty)$. We give the proof of the proposition by constructing a concrete Borel probability measure.

Let $x_k = 3 \cdot 2^{k-1}$ and $P(\{X = x_k\}) = \frac{c}{2^{kr} k^{1+r/s}}, k \geq 2$, where

$$c := \left(\sum_{k=2}^{\infty} \frac{1}{2^{kr} k^{1+r/s}} \right)^{-1}.$$

Then it is easy to see that

$$E\|X\|^r = \frac{3^r c}{2^r} \sum_{k=2}^{\infty} \frac{1}{k^{1+r/s}} < \infty.$$

and for any $\delta > 0$, we have

$$E\|X\|^{r+\delta} = \frac{3^r c}{2^r} \sum_{k=2}^{\infty} \frac{2^{\delta(k-1)}}{k^{1+r/s}} = \infty.$$

The same arguments as in [**9**] show that

$$\begin{aligned} V_{n,r}(\mu) &\geq \frac{3^r c}{2^r}\sum_{k=n+2}^{\infty}\frac{1}{k^{1+r/s}} \geq \frac{3^r c}{2^r}\int_{n+2}^{\infty}\frac{1}{x^{1+r/s}}dx \\ &\geq \frac{3^r rc}{s\,2^r}(n+2)^{-r/s}, \\ V_{n,r}(\mu) &\leq \frac{3^r c}{2^r}\sum_{k=n+2}^{\infty}\frac{1}{k^{1+r/s}} \leq \frac{3^r c}{2^r}\int_{n+1}^{\infty}\frac{1}{x^{1+r/s}}dx \\ &\leq \frac{3^r rc}{s\,2^r}(n+1)^{-r/s}. \end{aligned}$$

It follows easily that $D_r(\mu) = s$. □

1.4.2. Upper bounds and the essential covering rate.

In this subsection, we introduce the notion of essential covering rate. By using this concept, we give an effective upper bound for the upper quantization dimension. Proposition 1.4.2 shows that a Borel probability measure with non-compact support may have finite upper quantization dimension, so it is significant to examine when the upper quantization dimension is finite. Define the ϵ-*essential covering radius* by

$$R_{r,\epsilon}(\mu) := \inf\left\{R : \int_{B(0,R)^C} ||x||^r d\mu\,(x) < \epsilon^r\right\}.$$

Let $m_{r,\epsilon}(\mu)$ denote the smallest number of closed balls with radii ϵ which cover $B\left(0, R_{r,\epsilon}(\mu)\right)$. We define the *upper* and *lower covering rate* by

$$\overline{\Delta}_r(\mu) := \limsup_{\epsilon\to 0}\frac{\log m_{r,\epsilon}(\mu)}{-\log\epsilon},\ \ \underline{\Delta}_r(\mu) := \liminf_{\epsilon\to 0}\frac{\log m_{r,\epsilon}(\mu)}{-\log\epsilon}. \tag{1.4.1}$$

Let $m'_{r,\epsilon}(\mu)$ denote the largest number of mutually disjoint closed balls of radii ϵ which are centered in $B\left(0, R_{r,\epsilon}(\mu)\right)$. Define

$$\overline{\Delta}'_r(\mu) := \limsup_{\epsilon\to 0}\frac{\log m'_{r,\epsilon}(\mu)}{-\log\epsilon},\ \ \underline{\Delta}'_r(\mu) := \liminf_{\epsilon\to 0}\frac{\log m'_{r,\epsilon}(\mu)}{-\log\epsilon}.$$

It is easy to check that $m'_{r,2\epsilon}(\mu) \leq m_{r,\epsilon}(\mu) \leq m'_{r,\epsilon/2}(\mu)$ implying

$$\overline{\Delta}'_r(\mu) = \overline{\Delta}_r(\mu),\ \ \underline{\Delta}'_r(\mu) = \underline{\Delta}_r(\mu).$$

We also define the following two quantities with respect to μ which is not the Dirac measure at 0 (denoted by δ_0) by

$$\overline{R}_r(\mu) := \limsup_{\epsilon\to 0}\frac{\log R_{r,\epsilon}(\mu)}{-\log\epsilon},\ \ \underline{R}_r(\mu) := \liminf_{\epsilon\to 0}\frac{\log R_{r,\epsilon}(\mu)}{-\log\epsilon}.$$

For $\mu = \delta_0$, we define $R_r(\mu) = 0$.

THEOREM 1.4.3. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with the moment condition $\int ||x||^r d\mu(x) < \infty$. Then we have*

$$\begin{aligned}\underline{D}_r(\mu) &\leq \underline{\Delta}_r(\mu) = d\left(\underline{R}_r(\mu)+1\right),\\ \overline{D}_r(\mu) &\leq \overline{\Delta}_r(\mu) = d\left(\overline{R}_r(\mu)+1\right).\end{aligned}$$

PROOF. By the moment condition, for any $\epsilon > 0$, there exists $R > 0$ such that

$$\int_{B(0,R)^C} ||x||^r d\mu(x) < \epsilon^r.$$

By the continuity of measures and the definition of $R_{r,\epsilon}(\mu)$, we have

$$\int_{B(0,R_{r,\epsilon}(\mu))^C} ||x||^r d\mu(x) \leq \epsilon^r.$$

Let $\{x_i : 1 \leq i \leq m_{r,\epsilon}(\mu)\}$ be the centers of the collection of balls covering $B(0, R_{r,\epsilon}(\mu))$. Then for any point $x \in B(0, R_{r,\epsilon}(\mu))$, we have

$$\min_{1\leq i\leq m_{r,\epsilon}(\mu)} ||x - x_i|| \leq \epsilon.$$

Setting $\delta := \{x_i : 1 \leq i \leq m_{r,\epsilon}(\mu)\} \cup \{0\}$, for any point x outisde $B(0, R)$, we have

$$\min_{b\in\delta} ||x - b|| \leq ||x - 0|| = ||x||. \tag{1.4.2}$$

It follows that for $R = R_{r,\epsilon}(\mu)$,

$$\begin{aligned}V_{m_{r,\epsilon}(\mu)+1,r}(\mu) &\leq \int \min_{b\in\delta} ||x-b||^r d\mu(x)\\ &= \int_{B(0,R)} \min_{b\in\delta} ||x-b||^r d\mu(x) + \int_{B(0,R)^c} \min_{b\in\delta} ||x-b||^r d\mu(x)\\ &\leq \int_{B(0,R)} \epsilon^r d\mu(x) + \int_{B(0,R)^c} ||x||^r d\mu(x)\\ &\leq \epsilon^r + \epsilon^r\\ &\leq 2^r \epsilon^r.\end{aligned}$$

By the definition of $n_{r,\epsilon}(\mu)$, we immediately have $n_{r,2\epsilon}(\mu) \leq m_{r,\epsilon}(\mu) + 1$. Now it follows from Theorem 1.1.2 that

$$\begin{aligned}\overline{D}_r(\mu) &= \limsup_{\epsilon\to 0} \frac{\log n_{r,\epsilon}(\mu)}{-\log\epsilon} = \limsup_{\epsilon\to 0} \frac{\log n_{r,2\epsilon}(\mu)}{-\log\epsilon - \log 2}\\ &\leq \limsup_{\epsilon\to 0} \frac{\log\left(m_{r,\epsilon}(\mu)+1\right)}{-\log\epsilon}\\ &= \overline{\Delta}_r(\mu).\end{aligned} \tag{1.4.3}$$

We denote by γ the set of the centers of $m'_{r,\epsilon}(\mu)$ mutually disjoint closed balls of radii ϵ which are centered in $B(0, R_{r,\epsilon}(\mu))$. Then for all $x \in B(0, R_{r,\epsilon}(\mu))$, we have

$$\min_{c\in\gamma} \|x - c\| \leq 2\epsilon. \tag{1.4.4}$$

It follows from (1.4.2), (1.4.4) and the same argument as above, we have $n_{r,2^{r+1}\epsilon}(\mu) \leq m'_{r,\epsilon}(\mu)$. This implies that

$$\underline{D}_r(\mu) \leq \underline{\Delta}'_r(\mu), \quad \overline{D}_r(\mu) \leq \overline{\Delta}'_r(\mu).$$

By estimating the volumes, for ϵ sufficiently small, we have

$$m'_{r,\epsilon}(\mu)\epsilon^d \leq 2^d R_{r,\epsilon}(\mu)^d \leq 4^d m'_{r,\epsilon}(\mu)\epsilon^d.$$

This, together with (1.4.3) yields that

$$\overline{D}_r(\mu) \leq \limsup_{\epsilon\to 0} \frac{\log m'_{r,\epsilon}(\mu)}{-\log\epsilon} = d\left(\overline{R}_r(\mu) + 1\right).$$

To get the inequalities for $\underline{D}_r(\mu)$, we need only use the same argument as above by replacing $\limsup$ with $\liminf$. The proof is now complete. □

To check the finiteness of the upper or lower quantization dimension, it suffices to check the finiteness of $\overline{R}_r(\mu)$ or $\underline{R}_r(\mu)$. A careful examination of the proof of Theorem 1.4.3 shows that we actually need only to consider the intersection of $B(0, R_{r,\epsilon}(\mu))$ with the support of μ. This is because integration with respect to the measure μ only depends on the support of μ. We therefore introduce the following.

DEFINITION 1.4.4. Let $L_{r,\epsilon}(\mu)$ denote the smallest number of balls with radius ϵ covering

$$\operatorname{supp}(\mu) \cap B(0, R_{r,\epsilon}(\mu)).$$

We define the *upper* and *lower essential covering rate* respectively by

$$\bar{l}_r(\mu) := \limsup_{\epsilon\to 0} \frac{\log L_{r,\epsilon}(\mu)}{-\log\epsilon}, \quad \underline{l}_r(\mu) := \liminf_{\epsilon\to 0} \frac{\log L_{r,\epsilon}(\mu)}{-\log\epsilon}. \tag{1.4.5}$$

Then Theorem 1.4.3 is easily refined by the following:

THEOREM 1.4.5. $\underline{D}_r(\mu) \leq \underline{l}_r(\mu)$, $\overline{D}_r(\mu) \leq \bar{l}_r(\mu)$.

It is known that if $E||x||^{r+\delta} < \infty$ for some $\delta > 0$, then we have $\overline{D}_r(\mu) \leq d$. The following proposition shows that $E||x||^{r+\delta} < \infty$ implies $\underline{R}_r(\mu) < r/\delta$.

PROPOSITION 1.4.6. *Suppose that* $E\|x\|^{r+\delta} < \infty$. *Then* $\underline{R}_r(\mu) < r/\delta$.

PROOF. By Hölder's inequality with $p := \frac{r+\delta}{r}$, $q := \frac{r+\delta}{\delta}$ we have

$$\int_{B(0,R)^C} \|x\|^r \, d\mu(x) \leq \left(\int_{B(0,R)^C} \|x\|^{r+\delta} \, d\mu(x) \right)^{\frac{r}{r+\delta}} \left(\int_{B(0,R)^C} 1 d\mu(x) \right)^{\frac{\delta}{r+\delta}}.$$

By Markov inequality, for $R > 0$, we have

$$\mu(B(0,R)^C) \leq R^{-(r+\delta)} \int_{B(0,R)^C} \|x\|^{r+\delta} \, d\mu(x).$$

Now we take $R := R_{r+\delta,\epsilon}(\mu)$, then we have

$$\int_{B(0,R)^C} \|x\|^r \, d\mu(x) \leq \epsilon^r R^{-\delta} \epsilon^\delta = \epsilon^{r+\delta} R^{-\delta} \leq \epsilon^r R^{-\delta}.$$

It follows that $R_{r,\epsilon R^{-\delta/r}}(\mu) \leq R_{r+\delta,\epsilon}(\mu)$. Hence

$$\begin{aligned} \underline{R}_r(\mu) &\leq \liminf_{\epsilon \to 0} \frac{\log R_{r,\epsilon R^{-\delta/r}}}{-\log \epsilon R^{-\delta/r}}. \\ &\leq \liminf_{\epsilon \to 0} \frac{\log R_{r+\delta,\epsilon}(\mu)}{-\log \epsilon + \delta \log R_{r+\delta,\epsilon}(\mu)/r} \\ &\leq r/\delta. \end{aligned}$$

The rest follows immediately from the above inequality and Theorem 1.4.3. □

The following example shows that the upper bound given in Theorem 1.4.5 is effective in many interesting cases when known results are not applicable.

EXAMPLE 1.4.7. Let C be the middle-third Cantor set on $[0,1]$ and ν the classical Cantor measure. Let μ_i, $i \in \mathbb{N}$ be the Cantor measure on the Cantor sets $(C + 2^i)$. Define the measure μ by $\mu =: \sum_{i=1}^\infty s_i \mu_i$, where $s_i = c(2^{ir} i^{11r})^{-1}$ and

$$c = \left(\sum_i (2^{ir} i^{11r})^{-1} \right)^{-1}.$$

Then we have $\overline{D}_r(\mu) < s + 1/10$, where $s = \log 2 / \log 3 = \dim_H C$, while $E\|x\|^r < \infty$ and $E\|x\|^{r+\delta} = \infty$.

PROOF. It is convenient to see that

$$\begin{aligned}
E\left\|x\right\|^{r} &= \sum_{i=1}^{\infty} s_i \int \left\|x\right\|^{r} d\mu_i\left(x\right) = \sum_{i=1}^{\infty} s_i \int \left\|x\right\|^{r} d\mu_i\left(x\right) \\
&= \sum_{i=1}^{\infty} s_i \int_{[0,1]} (x+2^{i})^{r} d\nu\left(x\right) \\
&\leq 2^{r} c \sum_{i=1}^{\infty} \frac{1}{i^{11r}} < \infty. \\
E\left\|x\right\|^{r+\delta} &= \sum_{i=1}^{\infty} s_i \int \left\|x\right\|^{r+\delta} d\mu_i\left(x\right) \\
&= \sum_{i=1}^{\infty} s_i \int_{[0,1]} (\left\|x\right\|+2^{i})^{r+\delta} d\nu\left(x\right) \\
&\geq c \sum_{i=1}^{\infty} \frac{2^{i\delta}}{i^{11r}} = \infty.
\end{aligned}$$

For any $\epsilon > 0$, take $k(\epsilon) := [\epsilon^{-r/(11r-1)}] + 1$, then

$$\int_{B(0,2^{k(\epsilon)})^{c}} \left\|x\right\|^{r} d\mu\left(x\right) \leq A^{r}\epsilon^{r},$$

where $A^{r} = 2^{r}c$. On the other hand, for the above ϵ, there exists some $k \geq 1$ such that

$$3^{-k} \leq A\epsilon < 3^{-k+1}.$$

Each Cantor set can be covered by 2^{k} balls of radii $A\epsilon$. Hence

$$L_{r,A\epsilon} \leq k\left(\epsilon\right) \cdot 2^{k} \leq k(\epsilon) 2^{\frac{-\log(A\epsilon)}{\log 3}+1}.$$

It follows that $\overline{D}_r(\mu) \leq \bar{l}_r(\mu) \leq s + 1/10 < 1$. □

REMARK. In the definition of the essential covering rate, we actually can consider balls $B(a,R)$ with arbitrary center a instead of the balls $B(0,R)$. We define

$$R_{r,\epsilon}(\mu,a) := \inf\left\{R : \int_{B(a,R)^{C}} \left\|x-a\right\|^{r} d\mu\left(x\right) < \epsilon^{r}\right\}.$$

Let $L_{r,\epsilon}(\mu,a)$ denote the smallest number of balls with radius ϵ covering the intersection of $\text{supp}(\mu)$ with $B(a, R_{r,\epsilon}(\mu))$, and define the upper and lower essential covering rate as in (1.4.5) by replacing $\overline{L}_{r,\epsilon}(\mu)$, $\underline{L}_{r,\epsilon}(\mu)$ respectively with $\overline{L}_{r,\epsilon}(\mu,a)$, $\underline{L}_{r,\epsilon}(\mu,a)$. Then Theorem 1.4.5 remains true.

Theorem 1.4.5 provides us with an upper bound for the quantization dimension by means of essential covering rate which is not difficult to calculate in many interesting cases. However, by a slight modification of the above proof, we can further refine the upper bound in terms of the quantization number. Let $\mu_{r,\epsilon}$ be the conditional probability measure $\mu_{r,\epsilon} = \frac{\mu(\cdot \cap B(0,R_{r,\epsilon}(\mu)))}{\mu(B(0,R_{r,\epsilon}(\mu)))}$ and write

$$\overline{d}_r(\mu) = \limsup_{\epsilon \to 0} \frac{\log n_{r,\epsilon}(\mu_{r,\epsilon})}{-\log \epsilon}, \ \underline{d}_r(\mu) = \liminf_{\epsilon \to 0} \frac{\log n_{r,\epsilon}(\mu_{r,\epsilon})}{-\log \epsilon},$$

where $n_{r,\epsilon}(\mu_{r,\epsilon})$ be defined as in Section 1. Since $\mu_{r,\epsilon}$ depends on ϵ, we emphasize that in general $\overline{d}_r(\mu), \underline{d}_r(\mu)$ do not coincide with the upper and lower quantization dimension of $\mu_{r,\epsilon}$ of order r.

THEOREM 1.4.8. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\int \|x\|^r \, d\mu(x) < \infty$. Then*

$$\overline{D}_r(\mu) \le \overline{d_r}(\mu) \le \overline{l}_r(\mu), \ \underline{D}_r(\mu) \le \underline{d}_r(\mu) \le \underline{l}_r(\mu).$$

PROOF. For any $\epsilon > 0$ and each n, let $\alpha_n \in C_{n,r}(\mu_{r,\epsilon})$. Then

$$\begin{aligned}\int_{B(0,R_{r,\epsilon}(\mu))} \min_{a \in \alpha_n} \|x-a\|^r \, d\mu(x) &= \mu(B(0,R_{r,\epsilon})) \int_{B(0,R_{r,\epsilon}(\mu))} \min_{a \in \alpha_n} \|x-a\|^r \, d\mu_{r,\epsilon}(x) \\ &\le \int_{B(0,R_{r,\epsilon}(\mu))} \min_{a \in \alpha_n} \|x-a\|^r \, d\mu_{r,\epsilon}(x). \qquad (1.4.6)\end{aligned}$$

For $n = n_{r,\epsilon}(\mu_{r,\epsilon})$ we have

$$\int_{B(0,R_{r,\epsilon}(\mu))} \min_{a \in \alpha_n} \|x-a\|^r \, d\mu_{r,\epsilon}(x) \le \epsilon^r. \qquad (1.4.7)$$

On the other hand, by the definition of $R_{r,\epsilon}(\mu)$, we have

$$\int_{B(0,R_{r,\epsilon}(\mu))^C} \|x\|^r \, d\mu(x) \le \epsilon^r. \qquad (1.4.8)$$

Combining (1.4.6), (1.4.7) and (1.4.8), for $\delta_n := \alpha_n \cup \{0\}$ and $n = n_{r,\epsilon}(\mu_{r,\epsilon})$, we have

$$\begin{aligned}V_{n+1,r}(\mu) &\le \int \min_{a \in \delta_n} \|x-a\|^r \, d\mu(x) \\ &\le \int_{B(0,R_{r,\epsilon}(\mu))} \min_{1 \le i \le n} \|x-a_i\|^r \, d\mu(x) + \int_{B(0,R_{r,\epsilon}(\mu))^C} \|x\|^r \, d\mu(x) \\ &\le V_{n,r}(\mu_{r,\epsilon}) + \epsilon^r. \\ &\le 2\epsilon^r.\end{aligned}$$

This implies that $n_{r,2\epsilon}(\mu) \le n_{r,\epsilon}(\mu_{r,\epsilon})+1$, hence the first inequality in the theorem follows from Theorem 1.1.2. We next show the second. By the definition, $B(0, R_{r,\epsilon}(\mu))$ can be covered by $L_{r,\epsilon}(\mu)$ closed balls with radii ϵ. Let γ denote the set of the centers of such a collection of closed balls. Then

$$V_{L_{r,\epsilon}(\mu),r}(\mu_{r,\epsilon}) \le \int \min_{a \in \gamma} \|x-a\|^r \, d\mu_{r,\epsilon}(x) \le \epsilon^r.$$

This implies that $n_{r,\epsilon}(\mu_{r,\epsilon}) \leq L_{r,\epsilon}(\mu)$. It follows by the definition that

$$\overline{d}_r(\mu) \leq \overline{l}_r(\mu).$$

The same argument gives the proof of the inequalities regarding $\underline{D}_r(\mu)$. □

1.5. Essential covering rate and limit quantization dimensions

In this section, we first make some further investigations into the essential covering rate and then study the limits of the quantization dimension as the order $r \to \infty$.

1.5.1. Essential covering rate.

We start with the comparison between the essential covering rate and the box dimension of the support. By this, we show that the upper bound in Theorem 1.4.5 remains valid when the support is compact. In fact, we have

PROPOSITION 1.5.1. *Let μ be a Borel probability measure with compact support. Then*

$$\underline{l}_r(\mu) \leq \underline{\dim}_B(\mathrm{supp}(\mu),) \quad \overline{l}_r(\mu) \leq \overline{\dim}_B(\mathrm{supp}(\mu)).$$

PROOF. Let $N_\epsilon(\mathrm{supp}(\mu))$ denote the smallest number of closed balls of radii ϵ which cover $\mathrm{supp}(\mu)$. For any $\epsilon > 0$, we have $B(0, R_{r,\epsilon}) \cap \mathrm{supp}(\mu) \subset \mathrm{supp}(\mu)$. Hence $L_{r,\epsilon}(\mu) \leq N_\epsilon(\mathrm{supp}(\mu))$. The proposition follows immediately by the definitions. □

PROPOSITION 1.5.2. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then $\lim_{r\to\infty} \overline{l}_r(\mu)$ exists. In particular, if $\mathrm{supp}(\mu)$ is compact, then this limit is finite. Moreover, if $\mathrm{supp}(\mu) \subset B(0,1)$, then we have $\overline{l}_s(\mu) \leq \frac{s}{r}\overline{l}_r(\mu)$.*

PROOF. We assume that $r < s$. By the Hölder's inequality, we have

$$\begin{aligned} \int_{B(0,R)^C} \|x\|^r d\mu(x) &\leq \left(\int_{B(0,R)^C} \|x\|^s d\mu(x)\right)^{r/s} \left(\int_{B(0,R)^C} 1 d\mu(x)\right)^{(s-r)/s} \\ &\leq \left(\int_{B(0,R)^C} \|x\|^s d\mu(x)\right)^{r/s}. \end{aligned}$$

Hence for any $R > R_{s,\epsilon}(\mu)$, we have

$$\int_{B(0,R)^C} \|x\|^r d\mu(x) \leq \left(\int_{B(0,R)^C} \|x\|^s d\mu(x)\right)^{r/s} < \epsilon^r.$$

It follows that $R_{r,\epsilon}(\mu) \le R$. By the arbitrariness of R, we have $R_{r,\epsilon}(\mu) \le R_{s,\epsilon}(\mu)$. Hence

$$\bar{l}_r(\mu) = \limsup_{\epsilon\to 0} \frac{\log L_{r,\epsilon}(\mu)}{-\log \epsilon} \le \limsup_{\epsilon\to 0} \frac{\log L_{s,\epsilon}(\mu)}{-\log \epsilon} = \bar{l}_s(\mu).$$

Thus $\bar{l}_r(\mu)$ is increasing with r, so $\lim_{r\to\infty} \bar{l}_r(\mu)$ exists. If, in addition, $\mathrm{supp}(\mu)$ is compact, then by Proposition 1.5.1, $\bar{l}_r(\mu)$ is bounded by d and the limit is finite.

Now we assume that $\mathrm{supp}(\mu) \subset B(0,1)$. Let $1 \le r \le s \le \infty$. Then for all $x \in \mathrm{supp}(\mu)$, we have $\|x\|^r \ge \|x\|^s$. Thus for any $\epsilon > 0$ and for any $R > R_{r,\epsilon}(\mu)$ we have

$$\int_{B(0,R)^C} \|x\|^s \, d\mu(x) \le \int_{B(0,R)^C} \|x\|^r \, d\mu(x) \le \epsilon^r.$$

This implies $R_{s,\epsilon^{r/s}}(\mu) \le R$. By the arbitrariness of R, we have $R_{s,\epsilon^{r/s}}(\mu) \le R_{r,\epsilon}(\mu)$. Hence $B\left(0, R_{s,\epsilon^{r/s}}(\mu)\right) \subset B(0, R_{r,\epsilon}(\mu))$. Since $\epsilon^{r/s} \ge \epsilon$ for all $0 < \epsilon < 1$, we have $L_{s,\epsilon^{r/s}}(\mu) \le L_{r,\epsilon}(\mu)$. Hence,

$$\bar{l}_s(\mu) = \limsup_{\epsilon\to 0} \frac{\log L_{s,\epsilon^{r/s}}(\mu)}{-r/s \log \epsilon} \le \limsup_{\epsilon\to 0} \frac{\log L_{r,\epsilon}(\mu)}{-r/s \log \epsilon} = \frac{s}{r} \bar{l}_r(\mu).$$

□

COROLLARY 1.5.3. *Let μ be a self-similar measure μ on $\mathbb{R}^d$ with open set condition. Then we have*

$$\lim_{r\to\infty} \underline{l}_r(\mu) = \lim_{r\to\infty} \bar{l}_r(\mu) = D_\infty(\mu).$$

PROOF. By [**10**], Theorem 1.4.5 and Proposition 1.5.1,

$$D_\infty(\mu) = \lim_{r\to\infty} D_r(\mu) \le \lim_{r\to\infty} \underline{l}_r(\mu) \le \lim_{r\to\infty} \bar{l}_r(\mu) \le D_\infty(\mu).$$

Hence the corollary follows. □

1.5.2. Limit quantization dimensions.

In this subsection, we study the quantization for measures satisfying the r-moment condition for all $r \in [1, \infty)$. For the sake of convenience, we propose the following definition.

DEFINITION 1.5.4. Let μ be a Borel probability measure. We say that μ staisfies *the complete moment condition* if

$$\int \|x\|^r \, d\mu(x) < \infty \ f\text{or all } r \in [1, \infty).$$

We denote by $\mathcal{M}_C$ the set of all Borel probability measures satisfying the complete moment condition and by $\mathcal{M}_\infty$ the set all Borel probability measures with compact support. Then clearly $\mathcal{M}_\infty \subset \mathcal{M}_C$ and $\mathcal{M}_C$ is a convex set, i.e., if $\mu_i \in \mathcal{M}_C, s_i \ge$

$0, i = 1, 2$ and $s_1 + s_2 = 1$, then $s\mu_1 + (1-s)\mu_2 \in \mathcal{M}_C$. We denote by $\mathcal{M}_{UC}$ the set of all measures in $\mathcal{M}_C$ with unbounded support. We now give a simple lemma for measures in $\mathcal{M}_C$.

LEMMA 1.5.5. *For* $\mu \in \mathcal{M}_C$*, the limits* $\lim_{r\to\infty} \overline{D}_r(\mu), \lim_{r\to\infty} \underline{D}_r(\mu)$ *exist and are not greater than* d*.*

PROOF. Since $e_{n,r}(\mu)$ increases as r tends to infinity, the upper and lower quantization dimension are both increasing when r increases. This implies the existence of the two limits in the lemma. By Definition 1.5.4 and [**9**, Theorem 6.2], $\overline{D}_r(\mu) \le d$ for all $r \in [1, \infty)$. The lemma now follows. □

It is known (cf. [**9**, Lemma 14.16]) that for a self-similar measure μ satisfying the open set condition, we have

$$\lim_{r\to\infty} D_r(\mu) = D_\infty(\mu).$$

The following example shows that for general measures this is not true.

EXAMPLE 1.5.6. Let $\beta > 1$ for $x_k = \frac{1}{2}\left(k^{-\beta} + (k+1)^{-\beta}\right)$, $k \ge 1$, we define the measure μ by

$$\mu(\{x_k\}) = 2^{-k} \ k \ge 1.$$

Then $D_r(\mu) = 0$ for all $r \in [1, \infty)$, in particular, $\lim_{r\to\infty} D_r(\mu) \ne D_\infty(\mu)$.

PROOF. For each $n \ge 1$, we take $\alpha_n := \{x_k : 1 \le k \le n\} \cup \{0\}$. Then we have

$$\begin{aligned} V_{n+1,r}(\mu) &\le \int \min_{a\in\alpha_n} \|x-a\|^r d\mu(x) \le \sum_{k=n+1}^{\infty} 2^{-k} \left(\frac{k^{-\beta} + (k+1)^{-\beta}}{2}\right)^r \\ &\le \sum_{k=n+1}^{\infty} 2^{-k} = 2^{-n}. \end{aligned}$$

This implies that $D_r(\mu) = 0$ for all $r \in [1, \infty)$. On the other hand, it is easy to show that $D_\infty(\mu) = 1/(\beta+1)$. Hence for the measure μ we have

$$\lim_{r\to\infty} D_r(\mu) \ne D_\infty(\mu).$$

□

CONJECTURE 1.5.7. *For* $\mu \in \mathcal{M}_C$*,*

$$\lim_{r\to\infty} \overline{D}_r(\mu) = \lim_{r\to\infty} \overline{l}_r(\mu), \ \lim_{r\to\infty} \underline{D}_r(\mu) = \lim_{r\to\infty} \underline{l}_r(\mu).$$

It is known that $\lim_{r\to\infty} e_{n,r}(\mu) = e_{n,\infty}(\mu)$ only depends on $\operatorname{supp}(\mu)$ if $\operatorname{supp}(\mu)$ is compact. However, Example 1.5.6 show that the limits

$$\lim_{r\to\infty} \overline{D}_r(\mu), \quad \lim_{r\to\infty} \underline{D}_r(\mu)$$

in general depend upon the underlying measure μ. This seems more reasonable since the dimension of a measure "should" not depend only on its support. Based on Lemma 1.5.5 and the above analysis, we propose the following definition of *limit quantization dimension*.

DEFINITION 1.5.8. Let μ be a Borel probability measure in $\mathcal{M}_C$. Define

$$\overline{D}(\mu) := \lim_{r\to\infty} \overline{D}_r(\mu), \quad \underline{D}(\mu) := \lim_{r\to\infty} \underline{D}_r(\mu). \tag{1.5.1}$$

We respectively call the above two quantities $\overline{D}(\mu)$, $\underline{D}(\mu)$ in (1.5.1) the *upper* and *lower limit quantization dimension* of μ. If they coincide, we then denote the common value the *limit quantization dimension* of μ and denote it by $D(\mu)$.

According to Lemma 1.5.5, we have

$$0 \leq \underline{D}(\mu) \leq \overline{D}(\mu) \leq d.$$

As we note in Definition 1.5.8, the complete moment condition is very strong, especially for measures with unbounded support. This condition intuitively implies that the mass of μ is widely scattered in $B(0,1)^C$ and nowhere concentrated. However, the distribution of μ has some freedom on a certain compact set, allowing the quantization dimension of μ to assume any value in $[0, d]$. The following theorem show that $\mathcal{M}_{UC}$ is actually rather large, and the range of the upper quantization dimension of measures in $\mathcal{M}_{UC}$ is $[0, d]$. Setting

$$A_k := B(0, k+1) \setminus B(0, k), \ k \geq 1,$$

we define

$$\underline{W}(\mu) := \liminf_{k\to\infty} -\frac{1}{k}\log^+ \mu(A_k), \ \overline{W}(\mu) := \limsup_{k\to\infty} -\frac{1}{k}\log^+ \mu(A_k),$$

where $\log^+ x := \max\{0, \log x\}$.

THEOREM 1.5.9. *Suppose that* $\underline{W}(\mu) > 0$. *Then* $\mu \in \mathcal{M}_C$, *and if in addition* $\overline{D}_r(\mu) < \infty$ *then we have*

$$\limsup_{n\to\infty} n^{r/s} V_{n,r}(\mu(\cdot|B(0,n))) = \begin{cases} 0 & if\ s > \overline{D}_r(\mu) \\ \infty & if\ s < \overline{D}_r(\mu) \end{cases}.$$

PROOF. We assume that $\underline{W}(\mu) = s > t > 0$. Then there exists $k_0 > 0$ such that $k \geq k_0$ implies that

$$\mu(A_k) \leq e^{-kt}.$$

It follows that for any $r \in [1, \infty)$,

$$\begin{aligned} \int \|x\|^r \, d\mu(x) &\leq \int_{B(0,k_0)} \|x\|^r \, d\mu(x) + \sum_{k=k_0}^{\infty} \mu(A_k)(k+1)^r \\ &\leq k_0^r \mu(B(0,k_0)) + \sum_{k=k_0}^{\infty} \mu(A_k)(k+1)^r \\ &\leq k_0^r + \sum_{k=k_0}^{\infty} e^{-kt}(k+1)^r < \infty. \end{aligned}$$

This implies that $\mu \in \mathcal{M}_C$. For n large enough, let $\alpha_n \in C_{n,r}(\mu(\cdot|B(0,n)))$. Setting $\beta_n = \alpha_n \cup \{0\}$, we have

$$\begin{aligned} V_{n+1,r}(\mu) &\leq \int_{B(0,n)} \min_{b\in\beta_n} \|x-b\|^r \, d\mu(x) + \int_{B(0,n)^C} \min_{b\in\beta_n} \|x-b\|^r \, d\mu(x) \\ &\leq \int_{B(0,n)} \min_{a\in\alpha_n} \|x-a\|^r \, d\mu(x) + \int_{B(0,n)^C} \min_{b\in\beta_n} \|x-b\|^r \, d\mu(x) \qquad (1.5.2) \\ &\leq \mu(B(0,n)) V_{n,r}(\mu(\cdot|B(0,n))) + \sum_{k=n}^{\infty} (k+1)^r \mu(A_k) \\ &\leq V_{n,r}(\mu(\cdot|B(0,n))) + \sum_{k=n}^{\infty} (k+1)^r e^{-kt}. \end{aligned}$$

For any $n \geq 1$, we have

$$\begin{aligned} n^{r/s} \sum_{k=n}^{\infty} (k+1)^r e^{-kt} &\leq \sum_{k=n}^{\infty} (k+1)^{r(1+1/s)} e^{-kt} \\ &\leq \sum_{k=1}^{\infty} (k+1)^{r(1+1/s)} e^{-kt} < \infty. \end{aligned}$$

On the other hand, for any $s < \overline{D}_r(\mu)$, we have $\overline{Q}_r^s(\mu) = \infty$. It follows that

$$\limsup_{n\to\infty} n^{r/s} V_{n,r}(\mu(\cdot|B(0,n))) = \infty.$$

Now for n large enough such that $\mu(B(0,n)) \geq 1/2$, let $\alpha \in C_{n,r}(\mu)$. Then

$$\begin{aligned} V_{n,r}(\mu) &\geq \int_{B(0,n)} \min_{a\in\alpha} \|x-a\|^r \, d\mu(x) \\ &= \mu(B(0,n)) \int_{B(0,n)} \min_{a\in\alpha} \|x-a\|^r \, d(\mu(\cdot|B(0,n)))(x) \\ &\geq \frac{1}{2} V_{n,r}(\mu(\cdot|B(0,n))). \end{aligned}$$

Thus for any $s > \overline{D}_r(\mu)$, we have $\overline{Q}_r^s(\mu) = 0$. Thus,

$$\limsup_{n\to\infty} n^{r/s} V_{n,r}(\mu(\cdot|B(0,n))) = 0.$$

This finishes the proof of the theorem. □

LEMMA 1.5.10. *Let $\beta > 1$, $x_k = \left(k^\beta + (k+1)^\beta\right)/2, k \geq 1$. Let δ_{x_k} be the Dirac measure at the point x_k. We Define*

$$\mu = \sum_{k=1}^{\infty} 2^{-k} \delta_{\{x_k\}}.$$

Then we have $\mu \in \mathcal{M}_{UC}, \quad D(\mu) = 0$.

PROOF. It is clear that $|x_k| \leq (k+1)^\beta$. Hence for any $r \geq 1$ we have

$$\mathbb{E}\|X\|^r = \sum_{i=1}^{\infty} (k+1)^{\beta r} 2^{-k} < \infty.$$

This implies that $\mu \in \mathcal{M}_{UC}$. On the other hand,

$$\begin{aligned} V_{n,r}(\mu) &\leq \int \min_{1\leq k\leq n} \|x - x_k\|^r d\mu(x) = \sum_{k=n+1}^{\infty} (k+1)^{\beta r} 2^{-k} \\ &= \sum_{i=1}^{\infty} (i+n+1)^{\beta r} 2^{-(n+i)} = 2^{-n} \sum_{i=1}^{\infty} 2^{-i} (i+n+1)^{\beta r} \\ &\leq 2^{-n} \left(\sum_{i=1}^{\infty} 2^{-i} 2^{\beta r} \left(i^{\beta r} + (n+1)^{\beta r} \right) \right) \\ &= 2^{-n} \left(2^{\beta r} \sum_{i=1}^{\infty} 2^{-i} i^{\beta r} + 2^{\beta r} (n+1)^{\beta r} \right). \end{aligned}$$

It follows that $\overline{D}_r(\mu) = 0$. □

In the proof of the following proposition, we will apply the finite stability of the upper quantization dimension which will be proved in Theorem 2.3.2.

PROPOSITION 1.5.11. *For any $s \in [0, d]$, there exists a Borel probability measure $\nu \in \mathcal{M}_{UC}$ with $\overline{D}(\nu) = s$.*

PROOF. Let μ be as in Lemma 1.5.10. For $s \in [0, d]$, choose an arbitrary Borel probability measure τ with compact support $K \subset B(0, M)$ and $\overline{D}(\tau) = s$. Taking $\nu := \frac{1}{2}\mu + \frac{1}{2}\tau$, the proposition follows by Theorem 2.3.2. □

CHAPTER 2

Stability and stabilization of the upper quantization dimension

In this chapter, we introduce the notions of stability and stabilization for dimensions of measures. In particular, we show that the upper quantization dimension is finitely stable but not countably stable. Then we give a stabilization of the upper quantization dimension and prove that the stabilized upper quantization dimension coincides with the packing dimension; for measures with compact support, they also coincide with the stabilized upper box-counting dimension. In Section 2.5, we study in detail the quantization for homogeneous Cantor measures. Using the results obtained there, we construct examples to show that there exists a Borel probability measure with its upper quantization dimension strictly greater than the lower one and that the lower quantization dimension is not finitely stable.

2.1. Preliminary concepts and facts

We first recall the definitions of Hausdorff measures and Hausdorff dimension. Let A be a subset of $\mathbb{R}^d$. For any $\delta > 0$, we call a countable family $\left\{U_i \subset \mathbb{R}^d : \ i \in \Lambda\right\}$ of sets a δ-cover of A if

$$A \subset \bigcup_i U_i, \quad \operatorname{diam}(U_i) \leq \delta, \ i \in \Lambda.$$

We denote by $\mathcal{C}_\delta(A)$ the collection of all δ-covers of A. We call a countable family $\{B_i : \ i \in \Lambda\}$ of closed balls a centered δ-packing of A if all the balls are mutually disjoint and centered in A and with radii not greater than δ. We denote by $\mathcal{K}_\delta(\mathcal{A})$ the collection of all centered δ-packings of A. The s-dimensional Hausdorff measure $H^s(A)$ is defined by

$$H^s(A) := \lim_{\delta \to 0} \inf \left\{ \sum_{i \in \Lambda} (\operatorname{diam}(U_i))^s : \ \{U_i : \ i \in \Lambda\} \in \mathcal{C}_\delta(A) \right\}.$$

The Hausdorff dimension $\dim_H A$ of A is defined to the critical point at which $H^t(A)$ jumps from ∞ to zero, i.e.,

$$\dim_H A := \inf\{s : \ H^s(A) = 0\} = \sup\{s : \ H^s(A) = \infty\}.$$

We define a pre-measure $\mathcal{P}_0^s$ by

$$\mathcal{P}_0^s(A) = \sup\left\{\sum_{i\in\Lambda}(\mathrm{diam}(B_i))^s : \{B_i : i \in \Lambda\} \in \mathcal{K}_\delta(A)\right\}.$$

Then the s-dimensional packing measure $\mathcal{P}^s(A)$ is defined to be

$$\mathcal{P}^s(A) = \inf\left\{\sum_{i\in\Lambda}\mathcal{P}_0^s(F_i) : A \subset \bigcup_{i\in\Lambda} F_i\right\}.$$

And the packing dimension of A is defined by

$$\dim_p A := \inf\{s : \mathcal{P}^s(A) = 0\} = \sup\{s : \mathcal{P}^s(A) = \infty\}.$$

It is well known that both Hausdorff and packing dimension are countably stable, i.e., for any sequence (E_n) of subsets of $\mathbb{R}^d$, we have

$$\dim_H\left(\bigcup_n E_n\right) = \sup_n \dim_H E_n, \quad \dim_p\left(\bigcup_n E_n\right) = \sup_n \dim_p E_n.$$

Next we recall the definition and some properties of box-counting dimension.

Let $N_\delta(A)$ denote the smallest number of sets which covers A and have diameters not greater than δ. Then the upper and lower box-counting dimension of A are respectively defined by

$$\overline{\dim}_B A := \limsup_{\delta\to 0}\frac{\log N_\delta(A)}{-\log\delta}, \quad \underline{\dim}_B A := \liminf_{\delta\to 0}\frac{\log N_\delta(A)}{-\log\delta}.$$

The upper box-counting dimension is finitely stable in the sense that for a finite family $E_n, 1 \le n \le m$, we have

$$\overline{\dim}_B\left(\bigcup_{1\le n\le m} E_n\right) = \max_{1\le n\le m}\overline{\dim}_B E_n.$$

However, the lower box-counting dimension does not enjoy this finite stability. We remark here that there are many equivalent definitions for box-counting dimensions and one can choose the most suitable definition according to the circumstances. For more details, we refer to [**4**].

We end this section by pointing out the fact that the upper and lower quantization dimension are respectively the critical points for the upper and lower quantization coefficient. This fact leads to an equivalent definition for quantization dimension which appeared in [**22**]. The equivalence can also be seen from [**9**, Proposition 11.3]. The proof is simple, so we give it here for convenience.

LEMMA 2.1.1. *The upper and lower quantization dimension* $\overline{D}_r(\mu), \underline{D}_r(\mu)$ *respectively coincide with* $\overline{s}_0, \underline{s}_0$, *where* $\overline{s}_0, \underline{s}_0$ *satisfy*

$$\overline{Q}_r^s(\mu) = \begin{cases} 0 & \text{if } s > \overline{s}_0 \\ \infty & \text{if } s < \overline{s}_0 \end{cases}, \quad \underline{Q}_r^s(\mu) = \begin{cases} 0 & \text{if } s > \underline{s}_0 \\ \infty & \text{if } s < \underline{s}_0 \end{cases}. \tag{2.1.1}$$

PROOF. We only give the proof for the first equality. For the second, one can use a similar argument. By (0.0.1), for any $t > \overline{D}_r(\mu)$, there exists a positive real number $q > 0$, and $N > 0$ such that $n \geq N$ implies

$$t - q > \log n/(-\log e_{n,r}).$$

Hence $ne_{n,r}^t < e_{n,r}^q$ and $n^{r/t}V_{n,r} < V_{n,r}^q$. It follows that $\overline{Q}_r^t(\mu) = 0$. On the other hand, if $t < \overline{D}_r(\mu)$, there exists a real number $C > 0$, and a sequence n_i of positive integers with $n_i \to \infty (i \to \infty)$ such that

$$t + C < \log n_i/(-\log e_{n_i,r}).$$

Hence $n_i^{r/t}V_{n_i,r} > V_{n_i,r}^{-C}$. It follows that $\overline{Q}_r^t(\mu) = \infty$. □

REMARK 2.1.2. For Lemma 2.1.1, we would like to make the following two remarks.

- The above equivalent definition for the quantization dimension looks like the definition of fractal dimensions. As we know, Hausdorff, packing, box-counting dimension of sets are all critical points of some set functions.
- If $\overline{D}_r(\mu) = \infty$, we have that $\overline{Q}_r^s(\mu) = \infty$ whenever $s < \infty$; if $\underline{D}_r(\mu) = 0$, we have $\underline{Q}_r^s(\mu) = 0$ whenever $s > 0$.

2.2. Stability and stabilization of dimensions for measures

Various types of dimensions, including Hausdorff and packing dimension and the lower and upper box-counting dimension are important to characterize the complexity of highly irregular sets. In the past decades, a lot of research has been done aiming at the calculations of these dimensions or establishing significant properties. TRICOT introduced in [**24**] a stabilization for the upper box-counting dimension and proved that the stabilized upper box-counting dimension coincides with packing dimension. In recent years, paralleling methods have been adopted to study the dimensions of measures (c.f. [**5**]). Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}^d$ and $\mathcal{M}$ denote the set of all Borel probability measures on $\left(\mathbb{R}^d, \mathcal{B}\right)$. We denote by $\dim$ any one of the dimensions mentioned above, then the corresponding dimension $\dim^*$ for Borel measures is defined by

$$\dim^* \mu := \inf\{\dim E : \ \mu(E) = 1, \ E \in \mathcal{B}\}. \tag{2.2.1}$$

Analogous to the stability for dimensions of sets, we next define the stability for dimensions of measures. Let $\mu \in \mathcal{M}$ and let $(E_i)_{i\in\Lambda}$ be Borel subsets of $\mathbb{R}^d$ with

$$E_i \cap E_j = \emptyset, i \neq j,\ \mu(E_i) > 0, \sum_{i\in\Lambda} \mu(E_i) = 1,$$

where Λ is a finite or infinite subset of $\mathbb{N}$. Then for any Borel set A, we have

$$\mu(A) = \sum_{i\in\Lambda} \mu(A \cap E_i) = \sum_{i\in\Lambda} \mu(E_i) \frac{\mu(A \cap E_i)}{\mu(E_i)} =: \sum_{i\in\Lambda} \mu(E_i)\mu(A|E_i).$$

Setting $s_i = \mu(E_i)$ and $\mu_i = \mu(\cdot|E_i),\ i \in \Lambda$, we have

$$s_i > 0, \mu_i \in \mathcal{M},\ i \in \Lambda,\ \sum_{i\in\Lambda} s_i = 1,\ \mu = \sum_{i\in\Lambda} s_i\mu_i. \tag{2.2.2}$$

Based on the above observation, we propose the following definition of the stability of dimensions for measures.

DEFINITION 2.2.1. If for $\mu \in \mathcal{M}$ there exists $\mathcal{G} \subset \mathcal{M}$ and $\Lambda \subset \mathbb{N}, \mu_i \in \mathcal{G}, s_i > 0, i \in \Lambda$ such that

$$\sum_{i\in\Lambda} s_i = 1 \text{ and } \mu = \sum_{i\in\Lambda} s_i\mu_i. \tag{2.2.3}$$

Then we call (2.2.3) a *decomposition* of μ in $\mathcal{G}$ and denote it by $(\mu_i, s_i)_{i\in\Lambda}$. If Λ is finite we then call $(\mu_i, s_i)_{i\in\Lambda}$ a *finite decomposition*, otherwise an *infinite decomposition*.

For the above definition, we make the following remarks.

(1) For any measure $\mu \in \mathcal{M}$ which is not Dirac, there exists a non-trivial decomposition $(\mu_i, s_i)_{i\in\Lambda}$ of μ.
(2) It is possible that a measure μ has not only finite decompositions but also infinite ones.
(3) The decompositions of a measure do not necessarily come from conditional probabilities. For example, given a sequence $(\mu_i)_{i=1}^\infty$ of measures in $\mathcal{B}$ and a sequence $(s_i)_{i=1}^\infty$ with $s_i > 0, \sum_{i=1}^\infty s_i = 1$, we define $\mu = \sum_{i=1}^\infty s_i\mu_i$, then $(\mu_i, s_i)_{i=1}^\infty$ is a decomposition of μ, but the intersection of the support of some μ_i and that of some μ_j may have positive μ-measure.

DEFINITION 2.2.2. Let $\dim$ be an arbitrary dimension for measures. We say that $\dim$ is *finitely stable* on $\mathcal{G}$, if for any $\mu \in \mathcal{G}$ and any finite decomposition $(\mu_i, s_i)_{i\in\Lambda}$ in $\mathcal{G}$ of μ, we have

$$\dim \mu = \sup_{i\in\Lambda} \dim \mu_i. \tag{2.2.4}$$

We say that $\dim$ is *countably stable on* $\mathcal{G}$ if (2.2.4) holds for any $\mu \in \mathcal{G}$ and any finite or infinite decomposition $(\mu_i, s_i)_{i\in\Lambda}$in $\mathcal{G}$ of μ.

If some dimension $\dim$ is finitely stable but not countably stable, we then give a stabilization of this dimension. We denote by $\mathcal{T}(\mu, \mathcal{G})$ the collection of all decompositions of μ in $\mathcal{G}$.

DEFINITION 2.2.3. Let $\dim$ denote an arbitrary dimension function for measures. For any measure $\mu \in \mathcal{G} \subset \mathcal{M}$, we define the *stabilized dimension* of μ by

$$\text{s-}\dim(\mu) := \inf \left\{ \sup_{i\in\Lambda} \dim \mu_i : \ (\mu_i, s_i)_{i\in\Lambda} \in \mathcal{T}(\mu, \mathcal{G}) \right\}. \tag{2.2.5}$$

Since μ is a Borel probability measure, we know that the upper and lower local dimension $\overline{\dim}_{\rm loc}\mu(x)$, $\underline{\dim}_{\rm loc}\mu(x)$ are both Borel measurable. Hence by [**5**] we have

$$\begin{aligned} \dim_H^* \mu &= \inf \{ s : \ \underline{\dim}_{\rm loc}\mu(x) \le s \ \mu - a.e \}. \\ \dim_p^* \mu &= \inf \left\{ s : \ \overline{\dim}_{\rm loc}\mu(x) \le s \ \mu - a.e \right\}. \end{aligned}$$

The following lemma is very simple and might be well known to experts.

LEMMA 2.2.4. *Let μ_i be Borel probability measures respectively on $\left(\mathbb{R}^{d_i}, \mathcal{B}_i\right)$, $i = 1, 2$. Then we have*

$$\begin{aligned} \overline{\dim}_B^*(\mu_1 \otimes \mu_2) &\le \overline{\dim}_B^*\mu_1 + \overline{\dim}_B^*\mu_2. \\ \dim_H^*(\mu_1 \otimes \mu_2) &\ge \dim_H^* \mu_1 + \dim_H^* \mu_2. \end{aligned}$$

PROOF. By the definition of $\overline{\dim}_B^*$, there exist Borel sets $E_i, i = 1, 2$ such that

$$\mu_i(E_i) = 1, \ \overline{\dim}_B^*\mu_i = \overline{\dim}_B E_i, \ \ i = 1, 2.$$

Since $(\mu_1 \otimes \mu_2)(E_1 \times E_2) = \mu_1(E_1)\mu_2(E_2) = 1$, we know that

$$\overline{\dim}_B^*(\mu_1 \otimes \mu_2) \le \overline{\dim}_B(E_1 \times E_2) \le \overline{\dim}_B E_1 + \overline{\dim}_B E_2.$$

Hence the first inequality follows. Now we show the second. For simplicity we only consider the case when both $\mu_i, i = 1, 2$ are on $\mathbb{R}$. For any $X := (x, y) \in \mathbb{R}^2$, we have $B(X, t) \supset B(x, t/2) \times B(y, t/2)$. Hence

$$\underline{\dim}_{\rm loc}(\mu_1 \otimes \mu_2)(x, y) \ge \underline{\dim}_{\rm loc}\mu_1(x) + \underline{\dim}\mu_2(y).$$

Now for any $t_1 < \dim_H^* \mu_1, t_2 < \dim_H^* \mu_2$, there exist Borel sets $F_i, i = 1, 2$ with $\mu_i(F_i) > 0$ and

$$\underline{\dim}_{\rm loc}\mu_1(x) > t_1, \ \ \underline{\dim}\mu_2(y) > t_2 \text{ for all } x \in F_1, y \in F_2.$$

Hence $\underline{\dim}_{\rm loc}(\mu_1 \otimes \mu_2)(x, y) > t_1 + t_2$ for all $(x, y) \in F_1 \times F_2$. Note that

$$(\mu_1 \otimes \mu_2)(F_1 \times F_2) = \mu_1(F_1)\mu_2(F_2) > 0,$$

we have $\dim_H^*(\mu_1 \otimes \mu_2) > t_1 + t_2$. Now the second inequality follows from the arbitrariness of t_1, t_2. □

PROPOSITION 2.2.5. $\dim_H^*, \ \dim_p^*$ *are both countably stable.*

PROOF. We only give the proof for $\dim_H^*$. First let $s > \dim_H^* \mu$ and let $(\mu_i, s_i)_{i\in\Lambda}$ be a at most countable decomposition of μ. Then there exists a Borel set E with

$$\mu(E) = 1, \ \dim_H E < s.$$

Note that for all $i \in \Lambda$, we have $\mu_i(E) = 1$, we know that

$$\dim_H^* \mu_i \le \dim_H E < s.$$

It follows that $\sup_{i\in\Lambda} \dim_H^* \mu_i \le s$. Then by the arbitrariness of s, we get

$$\sup_{i\in\Lambda} \dim_H^* \mu_i \le \dim_H^* \mu.$$

Now we show the reverse. For any $t > \sup_{i\in\Lambda} \dim_H^* \mu_i$ and $i \in \Lambda$, there exist Borel sets E_i with $\mu_i(E_i) = 1$ and $\dim_H E_i < t$. Since $\mu(\cup_{i\in\Lambda} E_i) = 1$, we have

$$\dim_H^* \mu \le \dim_H(\bigcup_{i\in\Lambda} E_i) = \sup_{i\in\Lambda} \dim_H E_i \le t.$$

So the proof is complete by the arbitrariness of t. □

PROPOSITION 2.2.6. $\overline{\dim}_B^*$ *is finitely stable.*

PROOF. It suffices to follow the same argument as in the proof of Proposition 2.2.5 by using the finite stability of the upper box-counting dimension of sets. □

2.3. Finite stability of the upper quantization dimension

We denote by $\mathcal{M}_r$ the set of all Borel probability measures satisfying the r-moment condition $\int \|x\|^r \, d\mu < \infty$. Recall that $\mathcal{M}_\infty$ denotes the set all Borel probability measures with compact support. Then it is clear that $\mathcal{M}_\infty \subset \mathcal{M}_r$ for all $r \in [1, \infty)$.

LEMMA 2.3.1. *(*[**9**]*) Let* $\mu, \mu_i, 1 \le i \le m$ *be Borel probability measures on* $\mathbb{R}^d$ *and* $\mu = \sum_{i=1}^m s_i \mu_i, s_i > 0, \sum_{i=1}^m s_i = 1$. *Then*

(i) $V_{n,r}(\mu) \ge \sum_{i=1}^m s_i V_{n,r}(\mu_i)$,
(ii) *If* $\sum_{i=1}^m n_i \le n, n_i \ge 1$, *then* $V_{n,r}(\mu) \le \sum_{i=1}^m s_i V_{n_i,r}(\mu_i)$.

Now we apply the above lemma to show the finite stability of the upper quantization dimension.

THEOREM 2.3.2. *The upper quantization dimension is finitely stable. For the lower quantization dimension we have , if* $(\mu_i, s_i)_{i=1}^m$ *is a finite decomposition of* μ*, then*

$$\underline{D}_r(\mu) \ge \max_{1\le i\le m} \underline{D}_r(\mu_i).$$

PROOF. We first give the proof for $r \in [1, \infty)$. Let $\mu \in \mathcal{M}_r$ and $(\mu_i, s_i)_{i=1}^m$ is a finite decomposition of μ. By Lemma 2.3.1 (i), we have

$$s_i V_{n,r}(\mu_i) \leq V_{n,r}(\mu), \quad 1 \leq i \leq m.$$

Hence

$$\max_{1\leq i\leq m} \overline{D}_r(\mu_i) \leq \overline{D}_r(\mu), \quad \max_{1\leq i\leq m} \underline{D}_r(\mu_i) \leq \underline{D}_r(\mu). \tag{2.3.1}$$

On the other hand, by taking $k(n) = [n/m]$ and using Lemma 2.3.1 (ii), we have

$$V_{n,r}(\mu) \leq \sum_{i=1}^m s_i V_{k(n),r}(\mu_i) \leq \max_{1\leq i\leq m} V_{k(n),r}(\mu_i), \tag{2.3.2}$$

where $[x]$ denotes the largest integer not greater than x. Note that for any fixed $k \geq 1$,

$$\limsup_{n\to\infty} \frac{\log n}{-\log e_{n,r}(\mu)} = \limsup_{n\to\infty} \frac{\log [n/k]}{-\log e_{[n/k],r}(\mu)}. \tag{2.3.3}$$

Since $n/(2m) \leq k(n) \leq n/m$ for all $n \geq 2m$, by (2.3.2) and (2.3.3), we have

$$\overline{D}_r(\mu) \leq \limsup_{n\to\infty} \frac{\log n}{-\log \max_{1\leq i\leq m} e_{k(n),r}(\mu_i)} \leq \max_{1\leq i\leq m} \overline{D}_r(\mu_i).$$

This together with (2.3.1) completes the proof for the case $r \in [1, \infty)$.

To prove the case $r = \infty$, let $\mu \in \mathcal{M}_\infty$ and let $(\mu_i, s_i)_{i\in\Lambda}$ be an arbitrary finite decomposition of μ. We note that

$$\operatorname{supp}\left(\sum_{i\in\Lambda} s_i \mu_i\right) = \bigcup_{i\in\Lambda} \operatorname{supp}(\mu_i).$$

This, together with the finite stability of the upper box-counting dimension of sets finishes the proof.

REMARK 2.3.3. (1). After this thesis is almost finished, we learned that LINDSAY has got the following result in his PhD thesis (cf. [**16**]): if $D_r(\mu_1), D_r(\mu_2)$ both exist, then for $s \in (0,1)$

$$D_r(s\mu_1 + (1-s)\mu_2) = \max(D_r(\mu_1), D_r(\mu_2)).$$

His proof seems by means of the definition of the quantization dimension. Here our proof is by means of GRAF and LUSCHGY's result and we particularly distinguish the upper and lower quantization dimension, more exactly, we show that the upper quantization dimension enjoy the above finite stability and we will prove later that the lower quantization dimension does not have this property.

(2). The upper quantization dimension of order r is not countably stable on $\mathcal{M}_r$. This can be seen from the proof of [**9**, Example 6.4]. It is not countably stable even on $\mathcal{M}_\infty$, for this one can see [**14**] or Lemma 3.3.1.

□

COROLLARY 2.3.4. *If* $\overline{D}_r(\mu_{i_0}) = \max_{1\leq i\leq m} \overline{D}_r(\mu_i)$ *and* $\overline{D}_r(\mu_{i_0}) = \underline{D}_r(\mu_{i_0})$*. Then we have* $D_r(\mu) = D_r(\mu_{i_0})$*.*

PROOF. By the hypothesis and Theorem 2.3.2, we have

$$\overline{D}_r(\mu) = \overline{D}_r(\mu_{i_0}) = \underline{D}_r(\mu_{i_0}) \leq \max_{1\leq i\leq m} \underline{D}_r(\mu_i) \leq \underline{D}_r(\mu).$$

□

In general, one can not expect the lower quantization dimension of order r to have finite stability (see Section 2.3). The problem is, that we cannot guarantee the following inequality

$$\underline{D}_r\left(\mu\right) \leq \max\left\{\underline{D}_r\left(\mu_1\right), \underline{D}_r\left(\mu_2\right)\right\}.$$

However, the above inequality holds in some special cases.

Let μ be a Borel probability measure on $\mathbb{R}^d$. Let f be a Borel measurable function on $\mathbb{R}^d$. Then $\mu\circ f^{-1}$ defines a Borel probability measure which we call the image measure of μ with respect to f. The quantization dimension of μ and that of its image measure coincide provided that the corresponding function is bi-Lipschitz. By $\alpha \asymp \beta$ we mean that there exist two positive constants C_1, C_2 such that

$$C_1\alpha \leq \beta \leq C_2\alpha.$$

Then we have

PROPOSITION 2.3.5. *If f is a Lipschitz function on $\mathbb{R}^d$. Then*

$$V_{n,r}(\mu\circ f^{-1}) \leq C^r V_{n,r}\left(\mu\right),$$

where C is the Lipschitz constant of f. In particular,

$$\overline{D}_r\left(\mu\circ f^{-1}\right) \leq \overline{D}_r\left(\mu\right), \ \ \underline{D}_r\left(\mu\circ f^{-1}\right) \leq \underline{D}_r\left(\mu\right).$$

PROOF. Let $\alpha = \{a_i\}_1^n \in C_{n,r}(\mu)$. Then

$$\begin{aligned} V_{n,r}(\mu\circ f^{-1}) &\leq \int \min_{1\leq i\leq n} \|x - f(a_i)\|^r \, d\mu\circ f^{-1}\left(x\right) \\ &= \int \min_{1\leq i\leq n} \|f(x) - f(a_i)\|^r \, d\mu\left(x\right) \\ &\leq C^r \int \min_{1\leq i\leq n} \|x - a_i\|^r \, d\mu\left(x\right) \\ &= C^r V_{n,r}(\mu). \end{aligned}$$

The rest is clear from the definitions. □

Let $\mu_1,\ \mu_2$ be Borel probability measures on $\mathbb{R}^d$. Write

$$g(x,y) = x + y, \ \ x, y \in \mathbb{R}^d.$$

Then $g : \mathbb{R}^{2d} \to \mathbb{R}^d$ is a Borel measurable mapping. The convolution of $\mu_1,\ \mu_2$ is defined to be the image measure

$$\mu_1 \star \mu_2 = (\mu_1 \otimes \mu_2) \circ g^{-1}.$$

COROLLARY 2.3.6. *Let $\mu_1,\ \mu_2$ be Borel probability measures on $\mathbb{R}^d$. Then*

$$\overline{D}_r(\mu_1 \star \mu_2) \le \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2).$$

PROOF. Using the norms in (1.3.1), we see that g is Lipschitz. In fact,

$$\begin{aligned} \|g(x_1,y_1) - g(x_2,y_2)\| &= \|(x_1 - x_2) + (y_1 - y_2)\| \\ &\le \|x_1 - x_2\| + \|y_1 - y_2\| \\ &= \|(x_1,y_1) - (x_2,y_2)\|. \end{aligned}$$

Hence by Proposition 2.3.5, we have

$$\overline{D}_r(\mu_1 \star \mu_2) \le \overline{D}_r(\mu_1 \otimes \mu_2) \le \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2).$$

The last inequality follows from Theorem 1.3.1. □

COROLLARY 2.3.7. *Let ν be a Borel probability measure on $\mathbb{R}^d$ and let f be a Lipschitz function on $\mathbb{R}^d$. Then for the probability measure $\mu = s\nu + (1-s)\,\nu \circ f^{-1}$, $s \in [0,1]$, we have*

$$\underline{D}_r(\mu) = \max\left\{\underline{D}_r(\nu), \underline{D}_r\left(\nu \circ f^{-1}\right)\right\} = \underline{D}_r(\nu).$$

PROOF. By Proposition 2.3.5, we have $\underline{D}_r\left(\nu \circ f^{-1}\right) \le \underline{D}_r(\nu)$. So observing Theorem 2.3.2 it remains to show that $\underline{D}_r(\mu) \le \underline{D}_r(\nu)$. In fact, By taking $n_1 = n_2 = [n/2]$ and using Lemma 2.3.1 and Proposition 2.3.5, we have

$$\begin{aligned} V_{n,r}(\mu) &\le sV_{n_1,r}(\nu) + (1-s)\,V_{n_2,r}\left(\nu \circ f^{-1}\right) \\ &\le sV_{n_1,r}(\nu) + C(1-s)\,V_{n_2,r}(\nu) \\ &=: C_1 V_{n_1,r}(\nu). \end{aligned}$$

It follows that

$$\begin{aligned} \underline{D}_r(\mu) &\le \liminf_{n\to\infty} \frac{\log n}{-\log C_1 - \log V_{n_1,r}(\nu)} \\ &= \liminf_{n\to\infty} \frac{\log n}{-\log V_{n_1,r}(\nu)} \\ &= \underline{D}_r(\nu), \end{aligned}$$

where the last equality follows from (2.3.3). □

The following Corollary will be useful when we construct examples showing that the lower quantization dimension is not finitely stable.

COROLLARY 2.3.8. *Let μ_1, μ_2 be two Borel probability measures on $\mathbb{R}^d$ and let $\mu = s\mu_1 + (1-s)\,\mu_2, s \in (0,1)$. Suppose that $\overline{D}_r(\mu_2) < \underline{D}_r(\mu_1)$. Then*

$$\underline{D}_r(\mu) = \max\left\{\underline{D}_r(\mu_1), \underline{D}_r(\mu_2)\right\}.$$

PROOF. By the hypothesis, for all large n, we have

$$\frac{\log n}{-\log e_{n,r}(\mu_2)} < \frac{\log n}{-\log e_{n,r}(\mu_1)}.$$

Thus $V_{n,r}(\mu_1) > V_{n,r}(\mu_2)$. It follows from Lemma 2.3.1 (ii) that

$$V_{n,r}(\mu) \leq sV_{[n/2],r}(\mu_1) + (1-s)\,V_{[n/2],r}(\mu_2) < V_{[n/2],r}(\mu_1).$$

Hence $e_{n,r}(\mu) < e_{[n/2],r}(\mu_1)$. The corollary now follows from (2.3.3). □

LEMMA 2.3.9. *Let $(\mu_i, s_i)_{i\in\Lambda}$ be a countable decomposition of μ. Then*

$$\overline{D}_r(\mu) \geq \sup_{i\in\Lambda} \overline{D}_r(\mu_i), \quad \underline{D}_r(\mu) \geq \sup_{i\in\Lambda} \underline{D}_r(\mu_i).$$

PROOF. Let $i \in \Lambda$ and $n \geq 1$. Assume that $\alpha \in C_{n,r}(\mu)$. We have

$$V_{n,r}(\mu) \geq s_i \int \min_{a\in\alpha} \|x-a\|^r \, d\mu_i \geq s_i V_{n,r}(\mu_i).$$

The lemma follows immediately from (0.0.1). □

2.4. Stabilization of the upper quantization dimension and coefficient

In this section we first treat the stabilization of the upper quantization dimension and then consider the stabilization of the upper quantization coefficient.

2.4.1. Stabilization of the upper quantization dimension.

We start with the following simple lemma showing that the upper box-counting dimension actually coincides with the upper quantization dimension of order infinity.

LEMMA 2.4.1. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. Then for any Borel set $E \subset \operatorname{supp}(\mu)$ with $\mu(E) = 1$, we have $\overline{\dim}_B E = \overline{\dim}_B^* \mu$. In particular, we have $\overline{\dim}_B^* \mu = \overline{D}_\infty(\mu)$.*

PROOF. For any Borel set $E \subset \operatorname{supp}(\mu)$ with $\mu(E) = 1$, by the definition of $\overline{\dim}_B^*$, it is clear that $\overline{\dim}_B E \geq \overline{\dim}_B^* \mu$. On the other hand, for any Borel set $B \subset \operatorname{supp}(\mu)$ with $\mu(B) = 1$, we have $\overline{B} = \operatorname{supp}(\mu)$, otherwise $\mu(B) < 1$. It follows that

$$\overline{\dim}_B B = \overline{\dim}_B \overline{B} = \overline{\dim}_B \left(\operatorname{supp}(\mu)\right).$$

It follows that $\overline{\dim}_B^* \mu = \overline{\dim}_B \left(\operatorname{supp}(\mu)\right) = \overline{\dim}_B E$. In addition, it is proved in [**9**] that $\overline{D}_\infty(\mu) = \overline{\dim}_B \left(\operatorname{supp}(\mu)\right)$. So the final conclusion follows. □

LEMMA 2.4.2. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. Then we have*

$$\dim_p^* \mu \leq \overline{D}_r(\mu) \leq \overline{\dim}_B^* \mu.$$

PROOF. The right-hand inequality follows immediately from Lemma 2.4.1 and the fact that $\overline{D}_r(\mu) \leq \overline{D}_\infty(\mu)$. For the left-hand side, we may use an equivalent definition of the upper box-counting dimension. We denote by $N(E, \delta)$ the largest number of mutually disjoint closed balls with centers in E and with radius δ. Let $\{B_j\}$ be such balls. Then by estimating the volumes we know that for any $a \in \mathbb{R}^d$, there are at most 3^d balls B_i for which $\operatorname{dist}(a, B_i) \leq \delta$. Now it suffices to follow [**22**] and make minor modifications. □

COROLLARY 2.4.3. *For any Borel probability measure μ on $\mathbb{R}^d$, we have*

$$\overline{D}_r(\mu) \geq \dim_p^* \mu.$$

PROOF. Let $(\mu_i, s_i)_{i \in \Lambda}$ be a decomposition of μ with $\mu_i \in \mathcal{M}_\infty$ for all $i \in \Lambda$. By Lemma 2.3.9 and Lemma 2.4.2, we have

$$\overline{D}_r(\mu) \geq \sup_{i \in \Lambda} \overline{D}_r(\mu_i) \geq \sup_{i \in \Lambda} \dim_p^* \mu_i.$$

This and Proposition 2.2.5 finishes the proof. □

REMARK. By the countable stability of $\dim_H^*$, one can use the same argument as above to extend Pötzelberger's result $\dim_H^* \mu \leq \underline{D}_2(\mu)$ for compactly supported measures to the general case, which was proved by GRAF and LUSCHGY independently.

COROLLARY 2.4.4. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then*

$$\dim_H^* \mu_1 + \dim_H^* \mu_2 \leq \overline{D}_r(\mu_1 \otimes \mu_2) \leq \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2).$$

In particular, if $\dim_H^* \mu_i = \overline{D}_r(\mu_i),\ i = 1, 2$, then

$$\overline{D}_r(\mu_1 \otimes \mu_2) = \overline{D}_r(\mu_1) + \overline{D}_r(\mu_2).$$

PROOF. By [**9**, Corollary 12.16], we know that $\underline{D}_r(\mu) \geq \dim_H^* \mu$. Hence by Lemma 2.2.4,

$$\overline{D}_r(\mu_1 \otimes \mu_2) \geq \dim_H^*(\mu_1 \otimes \mu_2) \geq \dim_H^* \mu_1 + \dim_H^* \mu_2.$$

The second inequality is proved in Theorem 1.3.1. □

COROLLARY 2.4.5. *Suppose that μ_1, μ_2 are both Ahlfors-David regular, i.e., there exist constants $D_i, C_i, t_{i,}\ i = 1, 2$ such that*

$$C_i^{-1} t^{D_i} \leq \mu_i(B(x,t)) \leq C_i t^{D_i},\ \forall x \in \operatorname{supp}(\mu_i),\ \forall t \in (0, t_i).$$

Then we have $D_r(\mu_1 \otimes \mu_2) = D_r(\mu_1) + D_r(\mu_2)$.

PROOF. By [**18**, Theorem 5.7], we know that

$$\dim_H^* \mu_i = \overline{\dim}_B \mu_i = D_i.$$

It follows from Lemma 2.4.2 and Lemma 2.4.1 that $D_r(\mu_i) = D_i$. Now by Corollary 2.4.4, the desired result follows. □

THEOREM 2.4.6. *Let $\text{s-}\overline{\dim}_{\mathrm{B}}^*(\mu)$ be as defined in (2.2.5). Then we have*

$$\text{s-}\overline{\dim}_{\mathrm{B}}^*(\mu) = \dim_p^* \mu.$$

PROOF. Let $s > \text{s-}\overline{\dim}_{\mathrm{B}}^*(\mu)$. Then there exists a decomposition $(\mu_i, s_i)_{i\in\Lambda}$ such that

$$s > \sup_{i\in\Lambda} \overline{\dim}_B^* \mu_i.$$

Hence there exists a sequence $E_i,\ i \in \Lambda$ such that

$$\mu_i(E_i) = 1,\ s > \overline{\dim}_B E_i.$$

It follows that $\mu(\bigcup_i E_i) = 1$ and

$$\begin{aligned}\dim_p^* \mu &\leq \dim_p(\bigcup_i E_i) = \sup_{i\in\Lambda} \dim_p E_i \\ &\leq \sup_{i\in\Lambda} \overline{\dim}_B E_i \leq s.\end{aligned}$$

Thus from the arbitrariness of s, we know that $\dim_p^* \mu \leq \text{s-}\overline{\dim}_{\mathrm{B}}^*(\mu)$. Now we assume that $t > \dim_p^* \mu$. Then there exists a Borel set E with $\mu(E) = 1$ and $\dim_p E < t$. By [**4**], we know that $\dim_p E = \overline{\dim}_{\mathrm{MB}} E$, where $\overline{\dim}_{\mathrm{MB}} E$ is the revised (stabilized) upper box-counting dimension for sets (cf. [**24, 4**]), i.e.,

$$\overline{\dim}_{\mathrm{MB}} E = \inf\left\{\sup_i \overline{\dim}_B E_i :\ E = \bigcup_i E_i\right\}.$$

So we have $\overline{\dim}_{\rm MB} E < t$. It is not difficult to check that

$$\overline{\dim}_{\rm MB} E = \inf\left\{\sup_i \overline{\dim}_B E_i : \ E = \bigcup_i E_i, E_i \cap E_j = \emptyset, i \neq j, E_i \in B\right\}.$$

Therefore there exist Borel sets $E_i, i \in \Gamma$ such that $\sup_{i\in\Gamma} \overline{\dim}_B E_i < t$. Write

$$\Lambda = \{i \in \Gamma : \ \mu(E_i) > 0\}, \ \mu_i = \mu(\cdot|E_i), \ s_i = \mu(E_i), \ i \in \Lambda.$$

Then $(\mu_i, s_i)_{i\in\Lambda}$ is a decomposition of μ. Note that

$$\begin{aligned} \text{s-}\overline{\dim}^*_{\rm B}(\mu) &\leq \sup_{i\in\Lambda} \overline{\dim}_B \mu_i \leq \sup_{i\in\Lambda} \overline{\dim}_B E_i \\ &\leq \sup_{i\in\Gamma} \overline{\dim}_B E_i < t. \end{aligned}$$

The arbitrariness of t finishes the proof of the theorem. □

THEOREM 2.4.7. *Let μ be Borel probability measure on $\mathbb{R}^d$ with compact support. Then we have*

$$\text{s-}\overline{D}_r(\mu) = \text{s-}\overline{\dim}^*_{\rm B}(\mu) = \dim^*_p \mu = \text{s-}\overline{D}_\infty(\mu).$$

PROOF. The second equality is proved in the above theorem. Next we show the rest. Let $(\mu_i, s_i)_{i\in\Lambda}$ be an arbitrary decomposition of μ. By Lemma 2.4.2, we have

$$\dim^*_p \mu_i \leq \overline{D}_r(\mu_i) \leq \overline{\dim}^*_B \mu_i, \ \ i \in \Lambda.$$

By Lemma 2.2.5 and Lemma 2.4.2, we have

$$\dim^*_p \mu = \sup_{i\in\Lambda} \dim^*_p \mu_i \leq \sup_{i\in\Lambda} \overline{D}_r(\mu_i) \leq \sup_{i\in\Lambda} \overline{\dim}^*_B \mu_i.$$

It follows from the definition of $\text{s-}\overline{D}_r(\mu)$ and $\text{s-}\overline{\dim}^*_{\rm B}(\mu)$ that

$$\dim^*_p \mu \leq \text{s-}\overline{D}_r(\mu) \leq \text{s-}\overline{\dim}^*_{\rm B}(\mu).$$

This implies the first and third equality. For the last equality, it suffices to apply Lemma 2.4.1. □

For measures with unbounded support, the upper and lower box-counting dimension are not well defined. So one may ask the following .

PROBLEM 2.4.8. Does the equality $\text{s-}\overline{D}_r(\mu) = \dim^*_p \mu$ hold true for Borel probability measures with unbounded support?

The following theorem gives a positive answer to the above problem.

THEOREM 2.4.9. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then for all $r \in [1, \infty)$, we have*

$$\text{s-}\overline{D}_r(\mu) = \dim^*_p \mu.$$

PROOF. By Corollary 2.4.3, we have $\dim_p^* \mu \leq \text{s-}\overline{D}_r(\mu)$. For the reverse inequality, let $t > \dim_p^* \mu$. Then by (2.2.1), there exists a Borel set $E \subset \mathbb{R}^d$ such that

$$\mu(E) = 1, \ \dim_p E < t.$$

Let $\{E_i : i \in \Lambda_0\}$ be a countable partition of E such that E_i, $i \in \Lambda_0$ are all bounded sets. Then we have $\dim_p E_i < t$ for all $i \in \Lambda_0$. Set

$$\Lambda := \{i \in \Lambda_0 : \ \mu(E_i) > 0\}.$$

For each $i \in \Lambda$, we have $\dim_p E_i = \overline{\dim}_{\text{MB}} E_i < t$. As in the proof of Theorem 2.4.6, we may choose a sequence $\{E_{ij} : \ j \in \Lambda_i\}$ such that $\sup_{j\in\Lambda_i} \overline{\dim}_B E_{ij} \leq t$ and

$$E_i = \bigcup_{j\in\Lambda_i} E_{ij}, \ , \mu(E_{ij}) > 0, \ E_{ij_1} \cap E_{ij_2} = \emptyset, \ j_1 \neq j_2.$$

Setting $s_{ij} = \mu(E_{ij})$, $\mu_{ij} = \mu(\cdot|E_{ij})$, then $(\mu_{ij}, s_{ij} : \ i \in \Lambda, \ j \in \Lambda_i)$ is a decomposition of μ. It follows by Lemma 2.4.2 and (0.0.3) that

$$\text{s-}\overline{D}_r(\mu) \leq \sup_{i\in\Lambda} \sup_{j\in\Lambda_i} \overline{D}_r(\mu_{ij}) \leq \sup_{i\in\Lambda} \sup_{j\in\Lambda_i} \overline{\dim}_B E_{ij} \leq t.$$

Hence by the arbitrariness of t, we have $\text{s-}\overline{D}_r(\mu) \leq \dim_p^* \mu$. The proof is now complete. □

COROLLARY 2.4.10. *Let μ be any Borel probability measure on $\mathbb{R}^d$ with $\overline{D}_r(\mu) > \dim_p^* \mu$. Then there exists some decomposition $(\mu_i, s_i)_{i\in\Lambda}$ such that* $\sup_{i\in\Lambda} \overline{D}_r(\mu_i) < \overline{D}_r(\mu)$.

PROOF. By Theorem 2.4.9, for any $t \in \left(\dim_p^* \mu, \overline{D}_r(\mu)\right)$, there is some decomposition $(\mu_i, s_i)_{i\in\Lambda}$ such that

$$\sup_{i\in\Lambda} \overline{D}_r(\mu_i) < t < \overline{D}_r(\mu).$$

□

2.4.2. Stabilization of the upper quantization coefficient.

In the following, we give a stabilization of the upper s-dimensional quantization coefficient and show that the critical point is equal to the stabilized upper quantization dimension. Recall that the upper and lower s-dimensional quantization coefficient respectively by

$$\overline{Q}_r^s(\mu) = \limsup_{n\to\infty} n^{r/s} V_{n,r}(\mu) \quad \underline{Q}_r^s(\mu) = \liminf_{n\to\infty} n^{r/s} V_{n,r}(\mu).$$

Now we first stabilize the upper quantization coefficient by

$$\text{s-}\overline{\mathrm{Q}}_r^{\mathrm{s}}(\mu) := \inf\left\{\sum_{i\in\Lambda}\overline{Q}_r^s(\mu_i) : \ (\mu_i, s_i)_{i\in\Lambda} \in \mathcal{T}\left(\mu, \mathcal{M}_r\right)\right\}.$$

Then we have a critical point at which $\text{s-}\overline{\mathrm{Q}}_r^{\mathrm{s}}(\mu)$ jumps from zero to infinity. Now we show that this critical point is exactly the stabilized quantization dimension of order r.

THEOREM 2.4.11. *Let* $\text{s-}\overline{\mathrm{Q}}_r^{\mathrm{s}}(\mu)$ *be defined as above. Then*

$$\text{s-}\overline{\mathrm{Q}}_r^{\mathrm{s}}(\mu) = \begin{cases} 0 & \textit{if}\, s > \text{s-}\overline{D}_r(\mu) \\ \infty & \textit{if}\, s < \text{s-}\overline{D}_r(\mu) \end{cases}.$$

PROOF. For any $t > \text{s-}\overline{D}_r(\mu)$, there exists a decomposition $(\mu_i, s_i)_{i\in\Lambda}$ such that $\sup_{i\in\Lambda}\overline{D}_r(\mu_i) < t$. Hence $Q_r^t(\mu_i) = 0$ for all $i \in \Lambda$. It follows that $\overline{\mathrm{Q}}_r^{\mathrm{t}}(\mu) = 0$. Now let $t < \text{s-}\overline{D}_r(\mu)$. Then for any decomposition $(\mu_i, s_i)_{i\in\Lambda}$ of μ, we have $t < \sup_i \overline{D}_r(\mu_i)$. Hence $\overline{D}_r(\mu_i) > t$ for some i. It follows that $Q_r^t(\mu_i) = \infty$. Thus $\overline{\mathrm{Q}}_r^{\mathrm{t}}(\mu) = \infty$. □

2.5. Quantization for homogeneous Cantor measures

In this section, we study the quantization problems for the uniform probability measures on the homogeneous cantor sets which we call *homogeneous Cantor measures*. We divide this section into five subsections. In the first subsection, we recall the definition of the homogeneous Cantor sets and homogeneous Cantor measures. In the second subsection, we prove a general result connecting the quantization dimension with the quantities $\overline{\xi}$ and $\underline{\xi}$ defined in (2.5.1). In the third subsection, we determine the n-optimal sets of a dyadic homogeneous Cantor measure with *unbounded distortion*, i.e.

$$n_k = 2, \quad c_k \leq 1/4, \quad k \in \mathbb{N}.$$

We emphasize that in such cases $\inf_{k\geq 1} c_k = 0$ is admissible. As an application, we construct in the fifth subsection an example to show that the lower quantization dimension is not finitely stable.

2.5.1. Homogeneous Cantor sets and homogeneous Cantor measures.

The definitions of homogeneous Moran sets are proposed by Prof. Zhiying Wen (cf. [**25**]). Homogeneous Cantor sets are special cases of homogeneous Moran sets. Some dimensional results for homogeneous Moran sets are given in [**6**].

Let $(n_k)_1^\infty$ be a sequence of positive integers and $(c_k)_1^\infty$ a sequence of positive real numbers with $n_k \geq 2,\ n_k c_k \leq 1$. Then the homogeneous Cantor set is constructed by induction.

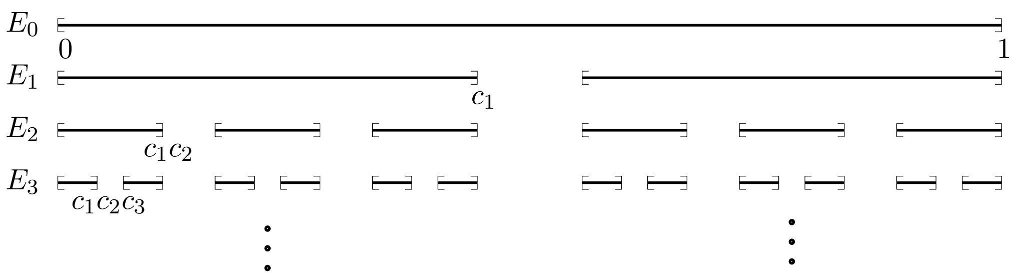

FIGURE 2.5.1. The first four levels of the construction of a homogeneous Cantor set with $n_1 = n_3 = 2,\ n_2 = 3$

step 1: Let $E_0 = [0, 1]$. We remove $(n_1 - 1)$ open intervals of equal length from E_0 such that the n_1 closed intervals which are left satisfy

(i) the left endpoints of the leftmost interval is the same as that of E_0,while the right endpoint of the rightmost interval is the same as that of E_0;

(ii) the length of each closed interval left is equal to c_1.

Then we get n_1 closed intervals which we call the basic intervals of order 1. Denote respectively by E_1,$\mathcal{D}_1$ the union and the collection of all basic intervals of order 1. In addition, we call the removed open intervals the complementary intervals of order 1.

step 2: Now suppose that we have constructed $n_1 \cdots n_k$ basic intervals of order k of length $c_1 \cdots c_k$. Denote by E_k the union of all basic intervals of order k and by $\mathcal{D}_k$ the collection of these intervals. We remove from each basic interval F of order k exactly $(n_{k+1} - 1)$ open intervals of equal length such that

(i) the left endpoint of the leftmost closed interval is the same as that of F and the right endpoint of the rightmost closed interval is the same as that of F.

(ii) all the closed intervals which are left are of the same length $c_1 \cdots c_k c_{k+1}$.

Then we have $n_1 \cdots n_{k+1}$ basic intervals of order $(k + 1)$. We respectively denote by $E_{k+1}, \mathcal{D}_{k+1}$ the union and the collection of these intervals. We call the removed open intervals complementary intervals of order $(k + 1)$. Write

$$E = C([0, 1], (n_k), (c_k)) := \bigcap_{i \geq 1} E_k,\ \mathcal{D} = \bigcup_{k \geq 1} \mathcal{D}_k.$$

The set E is called homogeneous cantor set on $[0, 1]$ determined by the sequences $(n_k), (c_k)$. The classical Cantor set is a simplest example of a homogeneous Cantor set.

Define two quantities $\overline{\xi}$ and $\underline{\xi}$ by

$$\overline{\xi} := \liminf_{k\to\infty} \frac{\log (n_1 \cdots n_k)}{-\log (c_1 \cdots c_k)}, \quad \underline{\xi} := \liminf_{k\to\infty} \frac{\log (n_1 \cdots n_k)}{-\log (c_1 \cdots c_k)}. \tag{2.5.1}$$

It is proved in [**6**] that

$$\dim_H E = \underline{\xi}, \ \overline{\dim}_B E = \limsup_{n\to\infty} \frac{\log(n_1 \cdots n_k) + \log n_{k+1}}{-\log(c_1 \cdots c_k) + \log n_{k+1}}. \tag{2.5.2}$$

REMARK 2.5.1. Homogeneous Cantor sets are of Cantor-like constructions. However, the dimensional and measure-theoretic properties can be quite different from those for classical Cantor sets. Namely, their Hausdorff, packing, lower and upper box-counting dimension may not coincide and their Hausdorff measures need not to be positive and finite. It is known that self-similar measures and self-conformal measures determined by finitely many maps have their upper and lower quantization dimensions coincide, but we are going to show that this is in general not the case even for homogeneous Cantor measures.

The uniform probability measure is the Borel probability measure induced by the natural mass distribution μ on E which assigns to each of the basic intervals of order k a weight $(n_1 \cdots n_k)^{-1}$. That is,

$$\mu(I) = (n_1 \cdots n_k)^{-1}, \ \forall I \in \mathcal{D}_k,$$

$$\mu(A) = \inf\left\{ \sum_i \mu(U_i) : A \cap E \subset \bigcup_i U_i, U_i \in D \right\}.$$

We call μ the *homogeneous Cantor measure* on E. First we note that the measure μ is a Borel probability measure. The proof is a standard argument by considering two positively separated sets.

2.5.2. Quantization dimension formula for homogeneous Cantor measures.

We begin this subsection with the following theorem. Let $\overline{\xi}, \underline{\xi}$ be as defined in (2.5.2). We have

THEOREM 2.5.2. *Let μ be the homogeneous Cantor measure on E determined by (n_k) and (c_k). Then*

$$\underline{D}_r(\mu) \le \underline{\xi} \le \overline{\xi} \le \overline{D}_r(\mu).$$

In particular, $\overline{D}_r(\mu) > \underline{D}_r(\mu)$ if $\overline{\xi} > \underline{\xi}$.

PROOF. For any $x \in E$, the ball $B(x, c_1 \cdots c_k)$ contains a basic interval of order k and $B(x, c_1 \cdots c_k) \cap E$ is contained in three such intervals. Hence

$$(n_1 \cdots n_k)^{-1} \le \mu(B(x, c_1 \cdots c_k)) \le 3(n_1 \cdots n_k)^{-1}.$$

It follows that for any $x \in E$,

$$\liminf_{\epsilon\to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon} \leq \liminf_{k\to\infty} \frac{\log \mu(B(x,c_1\cdots c_k))}{\log(c_1\cdots c_k)} = \underline{\xi},$$
$$\limsup_{\epsilon\to 0} \frac{\log \mu(B(x,\epsilon))}{\log \epsilon} \geq \limsup_{k\to\infty} \frac{\log \mu(B(x,c_1\cdots c_k))}{\log(c_1\cdots c_k)} = \overline{\xi}.$$

Since $\mu(E) = 1$, by [**5**, (10.12), (10.13)] we have $\dim_H^* \mu \leq \underline{\xi}$ and $\dim_p^* \mu \geq \overline{\xi}$. Thus by Lemma 2.4.2,

$$\overline{D}_r(\mu) \geq \dim_p^* \mu \geq \overline{\xi}.$$

On the other hand, it is clear that E can be covered by $n_1 \cdots n_k$ basic intervals of order k, so

$$\underline{D}_r(\mu) \leq \underline{\dim}_B^* \mu = \underline{\dim}_B E \leq \underline{\xi}.$$

Since $\dim_H E = \underline{\xi}$, we actually have $\underline{\dim}_B E = \underline{\xi}$. Now the theorem follows. □

Next we need the following example to show that $\overline{\xi} > \underline{\xi}$ is indeed possible. This example is motivated by [**3, 28**].

EXAMPLE 2.5.3. Let $(m_i)_1^\infty$, $(n_i)_1^\infty$ be two sequences of positive integers with

$$n_i = 2m_i,\ m_{i+1} \geq 3m_i + 1.$$

Let $a \leq 1/4$ be a positive real number. Define

$$c_k := \begin{cases} a & \text{if } 1 \leq k \leq m_1 - 1 \\ a^{n_i - m_i + 1} & \text{if } k = m_i \\ a^{\frac{m_{i+1} - n_i - 1}{m_{i+1} - m_i - 1}} & \text{if } m_i < k < m_{i+1} \end{cases}.$$

Let $E = C([0,1], (2), (c_k))$ and μ the uniform probability measure on E. Then $\underline{D}_r(\mu) < \overline{D}_r(\mu)$.

PROOF. Since $n_i = 2m_i$, $m_{i+1} \geq 3m_i + 1$, we have

$$\frac{m_{i+1} - n_i - 1}{m_{i+1} - m_i - 1} \geq \frac{(3m_i + 1) - 2m_i - 1}{(3m_i + 1) - m_i - 1} = \frac{1}{2}.$$

Hence $c_k \leq a^{1/2} \leq 1/2$. Note that for $k \in \{m_i\}$and $k \in \{m_i - 1\}$, we have

$$\lim_{k\to\infty} \frac{m_i \log 2}{-n_i \log a} = -\frac{\log 2}{2\log a},\ \lim_{k\to\infty} \frac{(m_i - 1)\log 2}{-(m_i - 1)\log a} = -\frac{\log 2}{\log a}.$$

It follows that $\underline{\xi} \leq -\frac{\log 2}{2\log a} < -\frac{\log 2}{\log a} \leq \overline{\xi}$. Now Theorem 2.5.2 applies. □

REMARK. (1). In the above example, we have $\inf_k c_k = 0$. We will see later in Example 2.5.16 that even if $\inf_k c_k > 0$, $\overline{\xi} > \underline{\xi}$ is still possible. (2). We have just learned from Prof. S. GRAF that LINDSAY also gave an example in his PhD thesis (cf. [**16**]) showing that $\overline{D}_r(\mu) > \underline{D}_r(\mu)$.

THEOREM 2.5.4. *If* $\inf_{k\geq 1} c_k > 0$*, then* $\overline{D}_r(\mu) = \overline{\xi},\ \underline{D}_r(\mu) = \underline{\xi}$.

PROOF. We may assume that $\inf_{k\geq 1} c_k = a > 0$. For any $t < \underline{\xi}$, there exists $N > 0$ such that $k \geq N$ implies

$$t < \frac{\log(n_1 \cdots n_k)}{-\log(c_1 \cdots c_k)}.$$

Now let $x \in E$ and $0 < \epsilon < c_1$, there exists $k \geq 1$ such that

$$c_1 \cdots c_k \leq \epsilon < c_1 \cdots c_{k-1}.$$

So the ball $B(x, \epsilon)$ contains a basic interval of order k and intersects at most three basic intervals of order $(k-1)$. For sufficiently small ϵ,we have

$$\begin{aligned}(n_1 \cdots n_k)^{-1} &\leq \mu(B(x,\epsilon)) \leq 3(n_1 \cdots n_{k-1})^{-1} \\ &\leq 3(c_1 \cdots c_{k-1})^{\frac{\log(n_1 \cdots n_{k-1})}{-\log(c_1 \cdots c_{k-1})}} \\ &\leq 3(c_1 \cdots c_{k-1})t \\ &\leq 3a^{-t}\epsilon^t.\end{aligned}$$

It follows that $\dim_{\text{loc}} \mu(x) \geq t$ for all $x \in E$. Hence by [**5**, (10.12)] and [**9**, Corollary 12.16], we have $\underline{D}_r(\mu) \geq \dim_H^* \mu \geq \underline{\xi}$. This, together with Theorem 2.5.2 yields that $\underline{D}_r(\mu) = \underline{\xi}$. Now we show the equality for $\overline{D}_r(\mu)$. By [**6**], we have

$$\overline{\dim}_B E = \limsup_{k\to\infty} \frac{\log(n_1 \cdots n_k) + \log n_{k+1}}{-\log(c_1 \cdots c_k) + \log n_{k+1}} = \overline{\xi}. \tag{2.5.3}$$

since $n_k c_k \leq 1$ and $c_k \geq a > 0$ for all $k \geq 1$. Using (2.5.3), Lemma 2.4.2 and Lemma 0.0.3, we have

$$\overline{D}_r(\mu) \leq \overline{\dim}_B^* \mu = \overline{\dim}_B E = \overline{\xi}.$$

This and Theorem 2.5.2 completes the proof. □

REMARK 2.5.5. If we drop the condition $\inf_k c_k > 0$, the situation is much more difficult to handle. However, if the sequence $\{n_k\}$ is bounded from above, then the formula for the upper quantization dimension remains true.

THEOREM 2.5.6. *Let* μ *be the homogeneous Cantor measure on* $C([0,1], \{n_k\}, \{c_k\})$ *with* $\sup_k n_k \leq M < \infty$*. Then* $\overline{D}_r(\mu) = \overline{\xi}$.

PROOF. By Theorem 2.5.2 we have that $\overline{D}_r(\mu) \geq \overline{\xi}$. On the other hand, using (2.5.3) and Lemma 2.4.2, Lemma 2.4.1 we get

$$\overline{D}_r(\mu) \leq \limsup_{k\to\infty} \frac{\log(n_1 \cdots n_k) + \log n_{k+1}}{-\log(c_1 \cdots c_k) + \log n_{k+1}} = \overline{\xi},$$

where the fact that $\sup_k n_k \leq M < \infty$ is used. This finishes the proof. □

2.5.3. Optimal sets for homogeneous Cantor measures.

Next we give a careful analysis of the n-optimal sets for the homogeneous Cantor measure μ so that we can relax the condition $\inf_k c_k > 0$. We need several lemmas on the analysis of the n-optimal sets for μ. GRAF and LUCHGY have determined the n-optimal set of order 2 for the classical Cantor measure. In there they made use of the self-similarity of Cantor measure. For the measures on general homogeneous Cantor sets, we in general have no self-similarity formula. Also, for this reason, we generally can not get the exact value of the quantization error.

From now on, we assume that

$$n_k = 2,\ c_k \leq 1/4,\ \ k \geq 1.$$

We denote by μ the homogeneous Cantor measure on $C\left([0,1],(2),(c_k)\right)$. If $F \in \mathcal{D}_k$, we denote by a the midpoint of F, by F_1, F_2 the two basic intervals of order $(k+1)$ contained in F with F_1 on the left and F_2 on the right, a_1, a_2 the midpoint of F_1, F_2. For $i = 1, 2$, we denote by F_{i1}, F_{i2} the basic intervals of order $(k+2)$ contained in F_i with F_{i1} on the left and F_{i2} on the right, and by a_{i1}, a_{i2} the midpoint of F_{i1}, F_{i2}.

LEMMA 2.5.7. *Let $k \geq 1$ and $F \in \mathcal{D}_k$. Then for $r > 1,\ b \in \mathbb{R} \setminus \{a\}$,*

$$\int_F |x-b|^r \, d\mu(x) > \int_F |x-a|^r \, d\mu(x).$$

PROOF. According to the position of b we have the following three cases.

Case 1: $b \in F \setminus E_{k+1}$. Without loss of generality, we may assume that b is closer to the right endpoint of F. Then for $x \in F_1$, we have $|x-b| > |x-a|$ and for $y \in F_2, |y-a| > |y-b|$. Setting $\Delta = b - a$, $t(x) := |x-a|, x \in F$, we have

$$\begin{aligned}
\int_{F_1} |x-b|^r \, d\mu(x) - \int_{F_1} |x-a|^r \, d\mu(x) &= \int_{F_1} \left(|x-b|^r - |x-a|^r\right) d\mu(x) \\
&= \int_{F_1} (t(x)+\Delta)^r - t^r(x) \, d\mu(x) := I_1 \\
\int_{F_2} |y-b|^r \, d\mu(y) - \int_{F_2} |y-a|^r \, d\mu(y) &= \int_{F_2} \left(|x-b|^r - |x-a|^r\right) d\mu(x) \\
&= \int_{F_2} (t(y)-\Delta)^r - t^r(y) \, d\mu(y) =: I_2
\end{aligned}$$

We compare each point $x \in F_1$ with the point $y \in F_2$ which is symmetric to x with respect to the midpoint a of C. Then we have that $t(x) = t(y)$. Since C is symmetric with respect to a and μ is uniform, we have

$$\int_{F_2} |y-b|^r \, d\mu(y) - \int_{F_2} |y-a|^r \, d\mu(y) = \int_{F_1} (t(x)-\Delta)^r - t^r(x) \, d\mu(x).$$

For $r > 1$, then by the mean-value theorem,

$$(t(x) + \Delta)^r - t^r(x) > t^r(x) - (t(x) - \Delta)^r .$$

Since $t(x)$ is continuous, it follows that $I_1 > I_2$. Hence

$$\int_F |y - b|^r \, d\mu(y) > \int_F |y - a|^r \, d\mu(y) . \tag{2.5.4}$$

Case 2: $b \in F_1 \cup F_2$. We may assume that $b \in F_2$. Then for $x \in F_1$, we have $|x - b| = |x - a| + \Delta$; for $y \in F_2 \cap [0, b]$, we have $||y - b| - |y - a|| \le \Delta$ and for $y \in F_2 \cap [b, 1]$, $|y - b| - |y - a| = \Delta$. Similar as in Case 1 we consider symmetric points with respect to the midpoint a of C and it is not difficult to show that (2.5.4) holds.

Case 3: $b \notin F$. We may assume that b is on the right side of the right-end point of F. Then for all $x \in F_1$, we have $|x - b| \ge 2|x - a|$, so

$$\int_F |x - b|^r \, d\mu(x) > \int_{F_1} |x - b|^r \, d\mu(x) \ge 2^r \int_{F_2} |x - a|^r \, d\mu(x) \ge \int_F |x - a|^r \, d\mu(x) .$$

The above three cases completes the proof of the lemma. □

LEMMA 2.5.8. *Let* $F \in \mathcal{D}_k$ *and* $r > 1$. *Then for* $b_1, b_2 \in \mathbb{R}$ *with* $\{b_1, b_2\} \ne \{a_1, a_2\}$,

$$\int_F \min_{i=1,2} |x - b_i|^r \, d\mu(x) > \sum_{i=1}^{2} \int_{F_i} |x - a_i|^r \, d\mu(x) .$$

PROOF. Without loss of generality, we may assume that $b_1 < b_2$. According to the positions of b_1, b_2 we have three cases.

Case 1: $b_1, b_2 \notin F \setminus E_{k+1}$. We have the following four subcases.

(a) b_1, b_2 respectively lie in F_1, F_2. without loss of generality, we assume that $b_i \in C_i$, $i = 1, 2$. Since $c_k(1 - 2c_{k+1}) \ge 2c_{k+1}$, we have

$$\begin{aligned} \int_F \min_{i=1,2} |x - b_i|^r \, d\mu(x) &= \int_{F_1} |x - b_1|^r \, d\mu(x) + \int_{F_2} |x - b_2|^r \, d\mu(x) \\ &> \int_{F_1} |x - a|^r \, d\mu(x) + \int_{F_2} |x - a_2|^r \, d\mu(x) . \end{aligned} \tag{2.5.5}$$

where the inequality follows from Lemma 2.5.7.

(b). $b_1 \in F_1$ and b_2 is on the right side of the right endpoint of F_2, or, $b_2 \in F_2$ and b_1 is on the left side of the left endpoint of F_1. We may assume the former. If $\operatorname{dist}(F_2, b_2) \le \operatorname{diam}(F_2)$, then clearly (2.5.5) holds. Otherwise, $\operatorname{dist}(F_2, b_2) > \operatorname{diam}(F_2)$. Then we have

$$\begin{aligned}\int_F \min_{i=1,2} |x-b_i|^r \, d\mu(x) &= 4^r \int_{F_{21}} |x-b_1|^r \, d\mu(x) + 2^r \int_{F_{22}} |x-b_2|^r \, d\mu(x) \\ &\geq \int_{F_1} |x-a_1|^r \, d\mu(x) + \int_{F_2} |x-a_2|^r \, d\mu(x). \end{aligned} \tag{2.5.6}$$

(c). $b_1(b_2)$ is on the left (right) side of the left (right) endpoint of $F_1(F_2)$. If either of $\mathrm{dist}(F_i, b_i), i = 1, 2$ is greater than $\mathrm{diam}(F_i)$, then (2.5.6) holds. Otherwise, both of them are less than or equal to $\mathrm{diam}(F_i)$, then we have (2.5.5) holds.

(d). b_1, b_2 are both on the left side of the right endpoint of F_1, or, both on the right side of the left endpoint of F_2. We may assume the former. Then

$$\begin{aligned}\int \min_{i=1,2} |x-b_i|^r \, d\mu &= 4^r \int_{F_{21}} |x-a_2|^r \, d\mu(x) + 6^r \int_{F_{22}} |x-a_2|^r \, d\mu(x) \\ &\geq \int_{F_1} |x-a_1|^r \, d\mu(x) + \int_{F_2} |x-a_2|^r \, d\mu(x). \end{aligned}$$

Case 2: one of b_1, b_2 lies in $F \in F \setminus E_{k+1}$. We may assume that $b_2 \in F \setminus E_{k+1}$ and b_1 is on the left side of right endpoint of F_1. Then we have the following three subcases.

(a). $b_1 \in F_1$ and $b_2 = a$ or b_2 is closer to F_2. Then (2.5.5) holds.

(b). $b_1 \in \mathbb{R} \setminus F$, and $b_2 = a$ or b_2 is closer to F_2. Then we have

$$\begin{aligned}\int_F \min_{i=1,2} |x-b_i|^r \, d\mu(x) &= \int_{F_{12}} \min_{i=1,2} |x-b_i|^r \, d\mu(x) + \int_{F_2} |x-b_2|^r \, d\mu(x) \\ &> 2^r \int_{F_{12}} |x-a_1|^r \, d\mu(x) + \int_{F_2} |x-a_2|^r \, d\mu(x) \\ &\geq \int_{F_1} |x-a_1|^r \, d\mu(x) + \int_{F_2} |x-a_2|^r \, d\mu(x). \end{aligned}$$

(c). b_2 is closer to F_1. Then

$$\begin{aligned}\int_F \min_{i=1,2} |x-b_i|^r \, d\mu &> 2^r \int_{F_{21}} |x-a_2|^r \, d\mu(x) + 4^r \int_{F_{22}} |x-a_2|^r \, d\mu(x) \\ &\geq \int_{F_1} |x-a_1|^r \, d\mu(x) + \int_{F_2} |x-a_2|^r \, d\mu(x). \end{aligned}$$

Case 3: $b_1, b_2 \in F \setminus E_{k+1}$. In this case, (2.5.5) clearly holds.

The above three cases completes the proof of the lemma. □

LEMMA 2.5.9. *Let $r > 1$ and $\beta \subset E_k$ be a finite set. Suppose that for some $F, G \in \mathcal{D}_k$ we have*

$$\mathrm{card}(F \cap \beta) \geq 3, \;\; G \cap \beta = \emptyset.$$

Then there exists $\gamma \subset E_k$ *with* $\operatorname{card}(\gamma) = \operatorname{card}(\beta)$ *such that*

$$\int \min_{b\in\beta} \|x-b\|^r \, d\mu(x) > \int \min_{c\in\gamma} \|x-c\|^r \, d\mu(x).$$

PROOF. By the hypothesis, we assume that $b_1, b_2, b_3 \in F \cap \beta$ with $b_1 < b_2 < b_3$. Since $\beta \subset E_k$ and the distance between any two different basic intervals of order k is not less than $2\operatorname{diam}(G)$, we have $\operatorname{dist}(\beta, G) \geq 2\operatorname{diam}(G)$. Hence

(2.5.7)
$$\int_{F\cup G} \min_{b\in\beta} \|x-b\|^r \, d\mu(x) \geq \int_G \min_{b\in\beta} \|x-b\|^r \, d\mu(x) \geq (2\operatorname{diam}(G))^r \mu(G).$$

Let a_G denote the midpoint of G. We take $\gamma := (\beta \setminus \{b_2\}) \cup \{a_G\}$. Then

$$\begin{aligned}
\int_{F\cup G} \min_{c\in\gamma} \|x-c\|^r \, d\mu(x) &\leq (\operatorname{diam}(F))^r \mu(F) + \int_G \|x-a_G\|^r \, d\mu(x) \\
&\leq \left(1+2^{-r}\right) (\operatorname{diam}(G))^r \mu(G) \\
&< (2\operatorname{diam}(G))^r \mu(G). \qquad (2.5.8)
\end{aligned}$$

Combining (2.5.7) and (2.5.8) we have

$$\int_{F\cup G} \min_{b\in\beta} \|x-b\|^r \, d\mu(x) > \int_{F\cup G} \min_{c\in\gamma} \|x-c\|^r \, d\mu(x). \qquad (2.5.9)$$

On the other hand, for any $x \in E_k \setminus (F \cup G)$, we have

$$\min_{b\in\beta} \|x-b\| = \min_{c\in\gamma\setminus G} \|x-c\| \geq \min_{b\in\gamma} \|x-c\|.$$

This implies that

$$\int_{E\setminus(F\cup G)} \min_{b\in\beta} \|x-b\|^r \, d\mu(x) \geq \int_{E\setminus(F\cup G)} \min_{c\in\gamma} \|x-c\|^r \, d\mu(x). \qquad (2.5.10)$$

The lemma now follows from (2.5.9) and (2.5.10). □

LEMMA 2.5.10. *Let* $r > 1$ *and* $\beta \subset E_k$ *be a finite set with* $\beta \cap I \neq \emptyset$ *for any* $I \in \mathcal{D}_k$. *Suppose that for some* $F, G \in \mathcal{D}_k$ *we have*

$$\operatorname{card}(F\cap\beta) \geq 3, \quad \operatorname{card}(G\cap\beta) = 1.$$

Then there exists $\gamma \subset E_k$ *with* $\operatorname{card}(\gamma) = \operatorname{card}(\beta)$ *such that*

$$\int \min_{b\in\beta} \|x-b\|^r \, d\mu(x) > \int \min_{c\in\gamma} \|x-c\|^r \, d\mu(x).$$

PROOF. We assume that $b_1, b_2, b_3 \in F \cap \beta$ and $b_4 \in G \cap \beta$. Let a_1, a_2 respectively denote the midpoints of F_1, F_2 and let a_3, a_4 respectively denote the midpoints of G_1, G_2. Define

$$\gamma := (\beta \setminus \{b_1, b_2, b_3, b_4\}) \cup \{a_1, a_2, a_3, a_4\}.$$

Since $\beta, \gamma \subset E_k$ and β, γ both intersect all basic intervals of order k, for any $x \in F \in \mathcal{D}_k$, we have

$$\min_{b\in\beta} \|x-b\| = \min_{b\in\beta\cap F} \|x-b\|, \quad \min_{c\in\gamma} \|x-c\| = \min_{c\in\gamma\cap F} \|x-c\|. \tag{2.5.11}$$

It follows that

$$\begin{aligned}
\int_{E\setminus(F\cup G)} \min_{b\in\beta} \|x-b\|^r \, d\mu(x) &= \sum_{I\in\mathcal{D}_k\setminus\{F,G\}} \int_I \min_{b\in\beta\cap I} \|x-b\|^r \, d\mu(x) \\
&= \sum_{I\in\mathcal{D}_k\setminus\{F,G\}} \int_I \min_{c\in\gamma\cap I} \|x-c\|^r \, d\mu(x) \\
&= \int_{E\setminus(F\cup G)} \min_{c\in\gamma} \|x-c\|^r \, d\mu(x). \qquad (2.5.12)
\end{aligned}$$

Similar to (2.5.11), we have

$$\int_G \min_{c\in\gamma} \|x-c\|^r \, d\mu(x) = \int \min_{c\in\{a_3,a_4\}} \|x-c\|^r \, d\mu(x).$$

On the other hand, using (2.5.11) and Lemma 2.5.7, we have

$$\begin{aligned}
\int_{F\cup G} \min_{b\in\beta} \|x-b\|^r \, d\mu(x) &\geq \int_G \min_{b\in\beta} \|x-b\|^r \, d\mu(x) = \int_G \|x-b_4\|^r \, d\mu(x) \\
&\geq \int_G \|x-a_G\|^r \, d\mu(x) \geq 2^r \int \min_{c\in\{a_3,a_4\}} \|x-c\|^r \, d\mu(x) \\
&> \int_{F\cup G} \min_{c\in\gamma} \|x-c\|^r \, d\mu(x). \qquad (2.5.13)
\end{aligned}$$

The lemma now follows from (2.5.12) and (2.5.13). □

Next we determine the n-optimal sets of μ of order r. We remark that in [**8**] GRAF and LUSCHGY used mathematical induction to determine the n-optimal set for the classical Cantor measure. In there the self-similarity seems to be crucial for the induction. Since self-similarity is not guaranteed for general homogeneous Cantor measures, our method here is based on direct analysis according to the structure of homogeneous Cantor sets.

LEMMA 2.5.11. *Let* $r > 1, n \geq 2^k$ *and* $\beta \in C_{n,r}(\mu)$. *Then we have* $\beta \subset E_k$ *and for each* $F \in \mathcal{D}_k$, $\operatorname{card}(\beta \cap F) \geq 1$.

PROOF. The proof is divided into two parts.

ad (1). We first show that $\beta \subset E_1$. We denote by F_1, F_2 the two basic intervals of order 1. It is clear that $\beta \subset E_0$. We only need to show that β does not intersect the complementary interval of order 1. In fact, suppose that there are more than two points of β in complementary interval of order 1. We denote these points by $p_i, 1 \leq i \leq m$

with $m > 2$ and $p_i < p_{i+1}$. Then for any $x \in E_1$,

$$\min_{b\in\beta} \|x-b\| = \min_{b\in\beta\setminus\{p_2,\cdots,p_{m-1}\}} \|x-b\|.$$

This contradicts the optimality of β since $e_{n,r}(\mu) < e_{n-1,r}(\mu)$ by [**9**, Theorem 4.1, Theorem 4.12]. Now we suppose that there are exactly two points $p_1 < p_2$ of β in the complementary interval of order 1. If the right (left) endpoint of F_1 (F_2) belongs to β, then we again get a contradiction by [**9**, Theorem 4.1, Theorem 4.12]. Otherwise, we consider the nearest sub-basic-intervals I_1, I_2 of F_1, F_2 of large enough order such that

$$\operatorname{diam}(I_i) < \min\{\operatorname{dist}(\beta\cap F_i), \operatorname{dist}(p_i, I_i)\}.$$

Let q_1, q_2 respectively denote the midpoint of I_1, I_2. We take $\gamma = (\beta\setminus\{p_1,p_2\})\cup\{q_1,q_2\}$, we have

$$\begin{aligned}\min_{c\in\gamma}\|x-c\| &\le \min_{b\in\beta}\|x-b\|,\ x\in E_1\setminus(I_1\cup I_2),\\ \min_{c\in\gamma}\|x-c\| &< \frac{1}{2}\min_{b\in\beta}\|x-b\|,\ x\in(I_1\cup I_2).\end{aligned}$$

This implies that

$$\int \min_{c\in\gamma}\|x-c\|^r d\mu(x) < \int \min_{b\in\gamma}\|x-b\|^r d\mu(x), \tag{2.5.14}$$

which contradicts the optimality of β. We finally suppose that there is only one point p of β in the complementary interval of order 1. If p is the midpoint of E_0, then this contradicts the fact that $e_{n,r}(\mu) < e_{n-1,r}(\mu)$. Otherwise we have two cases.

Case 1: $F_i\cap\beta\neq\emptyset, i=1,2$. In this case we consider a sub-basic interval of large enough order as above and get a contradiction.

Case 2: $F_i\cap\beta=\emptyset$ for $i=1$ or 2. Without loss of generality we assume the latter. If $\operatorname{dist}(p,F_2) < \operatorname{dist}(p,F_1)$. Then as in Case 1 , we get a contradiction, or else, $\operatorname{dist}(p,F_2) > \operatorname{dist}(p,F_1)$. Then we take $\gamma := (\beta\setminus\{p,q\})\cup\{a_1,a_2\}$, where q is an arbitary point of $\beta\cap F_1$ and a_i is the midpoint of $F_i, i=1,2$. Then

$$\begin{aligned}\int \min_{b\in\beta}\|x-b\|^r d\mu(x) &\ge \int_{F_2}\min_{b\in\gamma}\|x-b\|^r d\mu(x) \ge (\operatorname{diam}(F_2))^r\mu(F_2)\\ &\ge 2\cdot 2^{-r}(\operatorname{diam}(F_2))^r\mu(F_2)\\ &\ge \int\min_{c\in\gamma}\|x-c\|^r d\mu(x).\end{aligned} \tag{2.5.15}$$

This contradicts the optimality of β. By the above analysis, there is no point of β in the complementary interval of order 1, i.e., $\beta\subset E_1$. Moreover, similar to the argument in (2.5.15), we have $\operatorname{card}(\beta\cap F_i)\ge 1, i=1,2$.

ad (2). Now we assume that $\beta \subset E_j$. We are going to show that $\beta \subset E_{j+1}$. Suppose that some $F \in \mathcal{D}_j$ does not contain any point of β, then there exists some $G \in \mathcal{D}_j$ such that $\operatorname{card}(G \cap \beta) \geq 3$. Otherwise,

$$n = \sum_{I \in \mathcal{D}_j} \operatorname{card}(\beta \cap I) \leq 2\left(2^j - 1\right) < 2^{j+1} \leq n, \tag{2.5.16}$$

which is a contradiction. By Lemma 2.5.9, we may choose some $\gamma \subset E_j$ with $\operatorname{card}(\gamma) = \operatorname{card}(\beta)$ such that (2.5.14) holds. This contradicts the optimality of β. Thus for any $F \in \mathcal{D}_j$ we have $\operatorname{card}(\beta \cap F) \geq 1$, which implies that

$$\min_{b \in \beta} \|x - b\| = \min_{b \in \beta \cap F} \|x - b\|, \text{ for all } x \in F \in \mathcal{D}_j. \tag{2.5.17}$$

Suppose that some $F \in \mathcal{D}_j$ contains only one point of β then as in (2.5.16) we deduce that there exists some $G \in \mathcal{D}_j$ such that $\operatorname{card}(G \cap \beta) \geq 3$. By Lemma 2.5.10, we may choose some $\gamma \subset E_k$ with $\operatorname{card}(\gamma) = \operatorname{card}(\beta)$ such that (2.5.14) holds. This contradicts the optimality of β. Therefore , for any $F \in \mathcal{D}_j$, we have $\operatorname{card}(F \cap \beta) \geq 2$. Using this and (2.5.17), one can use the same argument as in step (1) to show that there is no point of β in the complementary interval of order $(j+1)$. This implies that $\beta \subset E_{j+1}$. Moreover, by Lemma 2.5.10, for any $F \in \mathcal{D}_{j+1}$, we have $\operatorname{card}(F \cap \beta) \geq 1$.

Since the above procedure continues till $j = k - 1$, the lemma follows. □

THEOREM 2.5.12. *Let $n = 2^k + j, 0 \leq j < 2^k$. Then $C_{n,r}(\mu)$ consists of the midpoints of some $\left(2^k - j\right)$ basic intervals of order k and the midpoints of the sub-basic intervals of order $(k+1)$ of the other j basic intervals of order k.*

PROOF. According to Lemma 2.5.11, $\beta \subset E_k$ and we have $\operatorname{card}(F \cap \beta) \geq 1$ for any $F \in \mathcal{D}_k$. Suppose that some $F \in \mathcal{D}_k$ contains more than two points of β. Then there exists $G \in \mathcal{D}_k$ such that $\operatorname{card}(G \cap \beta) = 1$. By Lemma 2.5.10, we may choose some $\gamma \subset E_k$ with $\operatorname{card}(\gamma) = \operatorname{card}(\beta)$ such that (2.5.14) holds. This contradicts the optimality of β. Hence for all $F \in \mathcal{D}_k$ we have

$$1 \leq \operatorname{card}(F \cap \beta) \leq 2.$$

It is clear that for all $F \in \mathcal{D}_k$ we have

$$\min_{b \in \beta} \|x - b\| = \min_{b \in \beta \cap F} \|x - b\|.$$

If $F \in \mathcal{D}_k$ contains exactly two points of β, then by Lemma 2.5.8, $F \cap \beta$ consists exactly of the midpoints of F_1, F_2; if $F \in \mathcal{D}_k$ contains only one point of β, then by Lemma 2.5.7 $F \cap \beta$ exactly consists of the midpoint of F. Hence the theorem follows. □

REMARK 2.5.13. (1). If neither $\inf_k c_k > 0$ nor $\sup_k c_k \leq 1/4$, then the situation would be much more difficult to handle. In such cases, Theorem 2.5.2 and Theorem 2.5.12 both may fail.

(2). A modified part of this chapter is to appear in *Math. Nachr.* (c.f. [**14**]). The referee has given us valuable suggestions which led to much improvement of the proof for Theorem 2.5.12. I would once more like to thank him (her).

In the following, we give a simple example to show that if $\sup_k c_k \leq 1/4$ doesnot hold, then Theorem 2.5.12 may fail.

EXAMPLE 2.5.14. Let $n_k \equiv 2$ and $c_k \equiv 1/2$. Then the 3rd optimal set of order-1 is $\{1/6, 1/2, 5/6\}$.

PROOF. Let $a_1, a_2, a_3 \in [0, 1]$ with $a_1 < a_2 < a_3$. Note that

$$\begin{aligned} I &= \int_0^{a_1} tdt + \int_0^{\frac{a_2-a_1}{2}} tdt + \int_0^{\frac{a_3-a_2}{2}} tdt + \int_0^{1-a_3} tdt \\ &= \frac{3}{4}\left(a_1 - \frac{a_2}{3}\right)^2 + \frac{3}{4}\left(a_2 - \frac{3}{5}a_3\right)^2 + \frac{1}{4}\left(a_3 - \frac{5}{6}\right)^2 + C. \end{aligned}$$

So for the integral to be minimal, we have $a_1 = 1/6, a_2 = 1/2, a_3 = 5/6$. □

2.5.4. Dimension formula for homogeneous Cantor measures.

We now state and prove the quantization dimension formula for dyadic homogeneous Cantor measures.

THEOREM 2.5.15. *Let* $c_k \leq 1/4$ *for all* $k \geq 1$. *Then for all* $r \in [1, \infty]$, *we have*

$$\overline{D}_r(\mu) = \overline{\xi}, \quad \underline{D}_r(\mu) = \underline{\xi}.$$

PROOF. By Theorem 2.5.2, for any homogeneous Cantor measure μ, we have

$$\overline{D}_r(\mu) \geq \overline{\xi}, \quad \underline{D}_r(\mu) \leq \underline{\xi}.$$

By Theorem 2.5.6, under the condition in Theorem 2.5.15, we always have $\overline{D}_r(\mu) = \overline{\xi}$. So it remains to show that $\underline{D}_r(\mu) \geq \underline{\xi}$. If $\underline{\xi} = 0$ then things are trivial. So we assume that $\underline{\xi} > 0$. Let $n \geq 2$. Then $n = 2^k + j$ for some $k \geq 1$ and some $0 \leq j < 2^k$. By Proposition 2.5.12, the n-optimal set consists of the midpoints of some $\left(2^k - j\right)$ order-k basic intervals and the midpoints of the $2j$ order-$(k+1)$ basic intervals respectively contained in the rest j order-k basic intervals. According to the value of j, we have two cases.

Case 1: $0 \leq j < 2^{k-1}$. Then $2^k - j > 2^{k-1}$. Hence the integral I_1 on the order-k basic intervals which have only one optimal point satisfies

$$\begin{aligned} I_1 &\geq \left(2^k - j\right)\left(4^{-1}c_1 \cdots c_k\right)^r 2^{-1} \cdot 2^{-k} \\ &\geq 2^{k-1}\left(\frac{1}{4}c_1 \cdots c_k\right)^r 2^{-1} \cdot 2^{-k} \\ &\geq 4^{-1}4^{-r}\left(c_1 \cdots c_k\right)^r. \end{aligned}$$

Case 2: $2^{k-1} \leq j < 2^k$. Then $2j \geq 2^k$. Hence the integral I_2 on those order-k basic intervals which have two optimal points satisfies

$$\begin{aligned} I_2 &\geq 2j \cdot \left(4^{-1}c_1 \cdots c_{k+1}\right)^r 2^{-1}2^{-(k+1)} \\ &\geq 2^k\left(4^{-1}c_1 \cdots c_{k+1}\right)^r 2^{-1}2^{-(k+1)} \\ &= 4^{-(1+r)}\left(c_1 \cdots c_{k+1}\right)^r. \end{aligned}$$

It follows that for $2^k \leq n < 2^{k+1}$,

(2.5.18) $$e_{n,r}(\mu) \geq 4^{-\frac{1+r}{r}} c_1 \cdots c_{k+1}.$$

The theorem follows now by the definition of the lower quantization dimension. □

2.5.5. Non-finite-stability of the lower quantization dimension.

In the following we show that the lower quantization dimension is not finitely stable. In the calculations of the lower limit, we shall make use of the following elementary inequality. Suppose that $\alpha_1, \alpha_2, \beta_1, \beta_2$ are positive real numbers and $\alpha_1\beta_2 > \alpha_2\beta_1$. Then

$$\frac{\beta_1}{\beta_2} < \frac{\alpha_1 + \beta_1}{\alpha_2 + \beta_2} < \frac{\alpha_1}{\alpha_2}.$$

THEOREM 2.5.16. *The lower quantization dimension is not finitely stable.*

PROOF. We give the proof by constructing two homogeneous Cantor measures μ_1, μ_2 such that

$$\underline{D}_r\left(\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2\right) > \max\left\{\underline{D}_r(\mu_1), \underline{D}_r(\mu_2)\right\}.$$

Let $n_k \equiv 2, k \geq 1$ and $c_k, d_k, k \geq 1$ are defined by

$$c_k = \begin{cases} \frac{1}{9} & if \;\; k = 2\sum_{i=1}^{m} 2^i + j, 1 \leq j \leq 2^{m+1},\ m \geq 0, \\ \frac{1}{4} & if \;\; k = 2\sum_{i=1}^{m} 2^i + 2^{m+1} + j, 1 \leq j \leq 2^{m+1},\ m \geq 0. \end{cases},$$

$$d_k = \begin{cases} \frac{1}{9} & if \;\; k = \sum_{i=2}^{2m+1} 2^i + j, 1 \leq j \leq 2^{2m+2},\ m \geq 0, \\ \frac{1}{4} & if \;\; k = \sum_{i=2}^{2m} 2^i + j, 1 \leq j \leq 2^{2m+1},\ m \geq 1. \end{cases}.$$

We denote respectively by E_1, E_2 the homogeneous Cantor sets $C([0,1],\{n_k\},\{c_k\})$ and $C([0,1],\{n_k\},\{d_k\})$. Let μ_1,μ_2 be the homogeneous Cantor measures respectively on E_1, E_2 and $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_1$. Then we will prove that

$$\underline{D}_r(\mu) > \max\{\underline{D}_r(\mu_1), \underline{D}_r(\mu_2)\}.$$

Let $\underline{\xi}_i, \overline{\xi_i}, i = 1,2$ be as defined in (2.5.1) for $\{c_k\}$and $\{d_k\}$ respectively. By some calculations, we have

$$\overline{\xi}_1 = \frac{2\log 2}{\log 9 + \log 4},\ \underline{\xi}_1 = \underline{\xi}_2 = \frac{3\log 2}{2\log 9 + \log 4},\ \overline{\xi}_2 = \frac{3\log 2}{\log 9 + 2\log 4}.$$

Thus by Theorem 2.5.4 or Theorem 2.5.15, we have

$$\underline{D}_r(\mu_1) = \underline{\xi}_1,\ \overline{D}_r(\mu_1) = \overline{\xi}_1,\ \underline{D}_r(\mu_2) = \underline{\xi}_2, \overline{D}_r(\mu_2) = \overline{\xi}_2.$$

On the other hand, by (2.5.18), for $n \geq 2^k$ we have

$$V_{n,r}(\mu_1) \geq 4^{-(1+r)} c_1 \cdots c_{k+1},\ V_{n,r}(\mu_2) \geq 4^{-(1+r)} d_1 \cdots d_{k+1}.$$

Hence Lemma 2.3.1 yields that

$$\begin{aligned} V_{n,r}(\mu) &\geq \frac{1}{2}\max\{V_{n,r}(\mu_1), V_{n,r}(\mu_2)\} \\ &\geq \frac{1}{2}\cdot 4^{-(1+r)}\max\left\{c_1^r \cdots c_{k+1}^r, d_1^r \cdots d_{k+1}^r\right\} \\ &\geq \frac{1}{2}\cdot 9^{-r}4^{-(1+r)}\max\{c_1^r \cdots c_k^r, d_1^r \cdots d_k^r\} \\ &:= C^r \max\{c_1^r \cdots c_k^r, d_1^r \cdots d_k^r\}, \end{aligned}$$

where $C^r = \frac{1}{2}\cdot 9^{-r}4^{-(1+r)}$. It follows that

$$\begin{aligned} \underline{D}_r(\mu) &\geq \liminf_{k\to\infty} \frac{k\log 2}{-\log C\max\{c_1\cdots c_k, d_1\cdots d_k\}} \\ &= \liminf_{k\to\infty} \frac{k\log 2}{-\log\max\{c_1\cdots c_k, d_1\cdots d_k\}}. \end{aligned} \tag{2.5.19}$$

Now let (k_i) be an arbitrary sub-sequence for which the limit of the sequence in (2.5.19) exists. Then we either have

(2.5.20)

$$\limsup_{i\to\infty} \frac{k_i \log 2}{-\log(c_1\cdots c_k)} > \underline{\xi}_1 \text{ implying } \lim_{i\to\infty} \frac{k_i\log 2}{-\log\max\{c_1\cdots c_{k_i}, d_1\cdots d_{k_i}\}} > \underline{\xi}_1,$$

or the above inequality fails. Then by the definition of $\underline{\xi}_1$ we have

$$\lim_{i\to\infty} \frac{k_i\log 2}{-\log(c_1\cdots c_k)} = \underline{\xi}_1. \tag{2.5.21}$$

However, for any sequence $\{k_i\}$ fulfilling (2.5.21), we have

$$\limsup_{i\to\infty} \frac{k_i\log 2}{-\log(d_1\cdots d_k)} > \underline{\xi}_1. \tag{2.5.22}$$

Combining (2.5.20) and (2.5.22) we have

$$\lim_{i\to\infty} \frac{k_i \log 2}{-\log \max\{c_1 \cdots c_{k_i}, d_1 \cdots d_{k_i}\}} > \underline{\xi}_1.$$

Since the $\liminf$ in (2.5.19) is attained at some sub-sequence, we have $\underline{D}_r(\mu) > \underline{\xi}_1$.

Now we consider the lower quantization dimension of order infinity. Taking

$$\delta_k = \max\{c_1 \cdots c_k, d_1 \cdots, d_k\},$$

we have

$$\lim_{k\to\infty} \delta_k = 0, \quad \delta_{k+1} \geq \frac{1}{9}\delta_k, \quad N_{\delta_k}(E_1 \cup E_2) \geq 2^k,$$

where $N_\delta(A)$ denotes the smallest number of sets covering A with diameter not greater than δ. Hence by [**4**], we have

$$\underline{\dim}_B(E_1 \cup E_2) = \liminf_{n\to\infty} \frac{k \log 2}{-\log \max\{c_1 \cdots c_k, d_1 \cdots, d_k\}}.$$

This together with the above analysis finishes the proof. □

2.6. Stability and stabilization of limit quantization dimension

We end this chapter with some further discussions on the limit of quantization dimensions. As a straightforward consequence of Theorem 2.3.2, we have

PROPOSITION 2.6.1. *$\overline{D}(\mu)$ is finitely stable, $\underline{D}(\mu)$ is not finitely stable.*

PROOF. Let $(\mu_i, s_i)_1^m$ be a finite decomposition of μ. By Definition 1.5.1 and Theorem 2.3.2,

$$\begin{aligned} \overline{D}(\mu) &= \lim_{r\to\infty} \overline{D}_r(\mu) = \lim_{r\to\infty} \overline{D}_r(\sum_i^m s_i\mu_i) \\ &= \lim_{r\to\infty} \max_{1\leq i\leq m} \overline{D}_r(\mu_i) = \max_{1\leq i\leq m} \lim_{r\to\infty} \overline{D}_r(\mu_i) \\ &= \max_{1\leq i\leq m} \overline{D}(\mu_i). \end{aligned}$$

This proves the finite stability of μ. To show that $\underline{D}(\mu)$ is not finitely stable, it suffices to consider the measure μ defined as in Theorem 2.5.16. Then we know that for all $r \in [1, \infty]$, we have

$$\underline{D}_r(\mu_i) = \underline{\xi}_1 = \underline{\xi}_2, i = 1, 2, \ \underline{D}_r(\mu) > \underline{\xi}.$$

This implies that

$$\underline{D}(\mu) = \underline{D}_\infty(\mu) > \underline{D}_\infty(\mu_i) = \underline{D}(\mu_i), \ i = 1, 2,$$

which finishes the proof of the Proposition. □

REMARK 2.6.2. In Lemma 3.3.1, we will construct a measure μ such that

$$\mu = \sum_{i=1}^{\infty} \frac{a}{k^{\lambda}} \delta_{x_k}, \quad \overline{D}\,(\mu) = \frac{1}{1+\beta}.$$

This implies that $\overline{D}$ is not countably stable.

We also give a stabilization s-$\overline{D}(\mu)$ for $\overline{D}(\mu)$ as in (2.2.5), i.e.,

$$\text{s-}\overline{D}(\mu) := \inf \left\{ \sup_{i\in\Lambda} \overline{D}\,(\mu_i) : \ (\mu_i, s_i) \in \mathcal{T}\,(\mu, \mathcal{M}) \right\}.$$

We have

THEOREM 2.6.3. *For all* $\mu \in \mathcal{M}_\infty$*, we have* s-$\overline{D}\,(\mu) = \dim_p^* \mu$.

PROOF. By Lemma 2.4.2 and definition 1.5.8, we have for any $r \in [1, \infty)$ and for all $\nu \in \mathcal{M}_\infty$,

$$\overline{D}\,(\nu) \geq \overline{D}_r\,(\nu) \geq \dim_p^* \nu. \tag{2.6.1}$$

It follows that s-$\overline{D}\,(\mu) \geq \dim_p^* \mu$. On the other hand, we always have for $\mu \in \mathcal{M}_\infty$ that

$$\overline{D}\,(\mu) \leq \overline{\dim}_B^* \mu = \overline{D}_\infty\,(\mu)\,.$$

This, together with Theorem 2.4.7 implies that

$$\text{s-}\overline{D}\,(\mu) \leq \text{s-}\overline{\dim}_{\mathrm{B}}^*\,(\mu) = \dim_p^* \mu. \tag{2.6.2}$$

The proof is complete by (2.6.1) and (2.6.2) . □

REMARK 2.6.4. Since $D(\mu) \geq \dim_p^* \mu$ for any Borel probability measure μ, by Lemma 2.2.5 we know that s-$\overline{D}(\mu) \geq \dim_p^* \mu$ for any $\mu \in \mathcal{M}$. It is natural to ask whether s-$\overline{D}(\mu) = \dim_p^* \mu$ remains true when the support of μ is unbounded.

CHAPTER 3

Quantization and absolute continuity of measures

In this chapter, we study the relationship between the quantization and absolute continuity of probability measures. Let ν, μ be two Borel measures on $\mathbb{R}^d$. We say that μ is absolutely continuous with respect to μ (denoted by $\nu \ll \mu$) if for any Borel set A we have

$$\mu(A) = 0 \Rightarrow \nu(A) = 0.$$

3.1. Preliminary facts

Quantization for absolute continuous measures with respect to d-dimensional Lebesgue measure have been well studied. ZADOR, and BUCKLEW and WISE proved the following theorem for measures with compact support:

THEOREM. *(cf.* [**27**] [**2**]*) Let μ be a Borel probability measure on $\mathbb{R}^d$ with compact support. Then the d-dimensional quantization coefficient of μ of order r exists and for some constant $C > 0$ independent of μ we have*

$$\lim_{n\to\infty} n^{r/d} V_{n,r}(\mu) = C \left\| d\mu/d\lambda^d \right\|_{\frac{d}{r+d}}.$$

This theorem is then proved valid for all Borel probability measures (cf. [**9**]) satisfying $(r+\delta)$-moment condition $E\|X\|^{r+\delta} < \infty$ by means of Pierce's Lemma and its generalization to arbitrary dimensions.

By the above theorem, we know that if a measure fulfills the $(r+\delta)$-moment condition and its absolutely continuous part does not vanish, then its quantization properties are completely determined. Under the $(r+\delta)$-moment condition, the quantization dimension for any singular measure must be bounded by d from above. It is natural to ask whether we have some similar results if we replace the Lebesgue measure in the theorem with some other measures which are singular with respect to Lebesgue measure. Recall that

$$\begin{aligned} \dim_H \mu &:= \sup\{s : \underline{\dim}_{\text{loc}}\mu(x) \geq s \ \mu - a.e.\}, \\ \dim_p \mu &:= \sup\left\{s : \overline{\dim}_{\text{loc}}\mu(x) \geq s \ \mu - a.e.\right\}. \end{aligned}$$

LEMMA 3.1.1. *Let ν, μ be two Borel probabilty measures with $\nu \ll \mu$. Then*

$$\begin{aligned}\dim_H^* \nu &\le \dim_H^* \mu,\ \dim_p^* \nu \le \dim_p^* \mu,\ \overline{\dim}_B^* \nu \le \overline{\dim}_B^* \mu,\\ \dim_H \nu &\ge \dim_H \mu,\ \dim_p \nu \ge \dim_p \mu,\ \overline{\dim}_B \nu \ge \overline{\dim}_B \mu.\end{aligned}$$

Let $\dim$ *denote the Hausdorff or packing dimension. If* $\dim \mu = \dim^* \mu$*, then*

$$\dim \nu = \dim^* \nu = \dim \mu = \dim^* \mu,$$

PROOF. Let $E \subset \mathbb{R}^d$ be any Borel set with $\mu(E) = 1$. Then $\nu \ll \mu$ implies that $\nu(E) = 1$. By the definitions, we have

$$\begin{aligned}\dim_H^* \mu &= \inf\{\dim_H F :\ F \in \mathcal{B},\ \mu(F) = 1\}\\ &\ge \inf\{\dim_H F :\ F \in \mathcal{B},\ \nu(F) = 1\}\\ &= \dim_H^* \nu.\end{aligned}$$

Similarly, we have $\dim_p^* \nu \le \dim^* \mu,\ \overline{\dim}_B^* \nu \le \overline{\dim}_B^* \mu$. To show the other inequalities in the lemma, it suffices to note that $\nu \ll \mu$ and $E \in \mathcal{B}, \nu(E) > 0$ implies that $\mu(E) > 0$ and use the corresponding definitions. □

Recall that a dimension $\dim$ of measures is said to satisfy a *variational law* with respect to the corresponding dimension of sets if

$$\dim(\mu) = \inf\{\dim E :\ E \in \mathcal{B},\ \mu(E) = 1\}. \tag{3.1.1}$$

On the relationship between quantization and absolute continuity, one may address the following questions:

(AC1) Is $\overline{D}_r(\cdot)$ monotone, i.e., $\nu \ll \mu \Rightarrow \overline{D}_r(\nu) \le \overline{D}_r(\mu)$? If not, under what conditions do we have $\overline{D}_r(\nu) \le \overline{D}_r(\mu)$ if $\nu \ll \mu$?
(AC2) Do the absolute continuity $\nu \ll \mu$ and the existence of $D_r(\mu)$ imply the existence of $D_r(\nu)$?
(AC3) What is the relationship between the quantization of an s-regular measure μ and that of probability measures ν with $\nu \ll \mu$?
(AC4) What about the relationship between the quantization for a self-similar measure μ and that for probability measures ν with $\nu \ll \mu$?
(AC5) Does there exist a dimension-like set function $\overline{\mathbf{D}}_r$ which satisfy the variational law with respect to the upper quantization dimension of order r?

In Section 2, we give some general results by means of box-counting dimension and upper and lower vanishing rates. In Section 3, we construct some examples to show that generally the absolute continuity between two measures does not imply the inequality between quantization dimensions. As an application, we deduce that there does not exist any dimension-like quantity which satisfies the variational law with respect to the upper quantization dimension. Section 4 deals with the quantization for measures absolutely continuous with respect to an s-regular measure. In the last section we treat the quantization for measures which are absolutely continuous with respect to self-similar measures.

3.2. Box-counting dimension, vanishing rates and quantization

In this section, we prove some general results by means of the upper and lower box-counting dimension and the upper and lower vanishing rate of the Radon-Nikodym derivative (cf. (3.2.2), (3.2.1)).

LEMMA 3.2.1. *Let A be a bounded subset of $\mathbb{R}^d$ with $\overline{\dim}_B A = \overline{s}, \underline{\dim}_B A = \underline{s}$. Then for any $\epsilon > 0$,*

(1) *for sufficiently large n, there exist a finite subset γ of $\mathbb{R}^d$ such that*

$$\mathrm{card}\,(\gamma) \leq n,\ \min_{c\in\gamma} \|x - c\| \leq C_1 n^{-\frac{1}{\overline{s}+\epsilon}},\ x \in A.$$

(2) *there exist a sequence (n_k) of natural numbers and a sequence (γ_k) of subsets of $\mathbb{R}^d$ with*

$$\mathrm{card}\,(\gamma_k) \leq n_k \to \infty\,(k \to \infty),\ \min_{c\in\gamma_k} \|x - c\| \leq C_2 n_k^{\frac{-1}{\underline{s}+\epsilon}},\ x \in A.$$

PROOF. *ad* (1). Let $N_t\,(A)$ denote the smallest number of balls of radii t which cover A. By the definition of the upper box-counting dimension, we have for any $\epsilon > 0$, there exists $0 < t_1 < 1$ such that $t \leq t_1$ implies that

$$N_t\,(A) \leq t^{-(\overline{s}+\epsilon)}.$$

For $n \geq t_1^{-(\overline{s}+\epsilon)}$, there exists $0 < t \leq t_1$ such that

$$t^{-(\overline{s}+\epsilon)} \leq n < 2t^{-(\overline{s}+\epsilon)}.$$

Denote by γ the set of the centers of $N_t\,(A)$ balls of radii t which cover A. Then $N_t\,(A) \leq n$ and for any $x \in A$, we have

$$\min_{c\in\gamma} \|x - c\| \leq t < 2^{\frac{1}{\overline{s}+\epsilon}} \cdot n^{\frac{-1}{\overline{s}+\epsilon}} \leq 2^{\frac{1}{\overline{s}}} \cdot n^{\frac{-1}{\overline{s}+\epsilon}}.$$

By taking $C_1 := 2^{1/\overline{s}}$, (1) follows.

ad (2). By the definition of the lower box-counting dimension, for any $\epsilon > 0$, there exists a sequence (t_k) with $t_k \to 0$ as $k \to \infty$ such that

$$N_{t_k}\,(A) \leq t_k^{-(\underline{s}+\epsilon)}.$$

Take $n_k = \left[t_k^{-(\underline{s}+\epsilon)}\right] + 1$, denote by γ_k the centers of the $N_{t_k}(A)$ balls of radii t_k which cover A. Then we have $\operatorname{card}(\gamma_k) \le n_k$, and for all $x \in A$, we have

$$\min_{c\in\gamma} \|x - c\| \le 2^{\frac{1}{\underline{s}}} n_k^{\frac{-1}{\underline{s}+\epsilon}}.$$

Taking $C_2 := 2^{1/\underline{s}}$ completes the proof. □

For a measurable function h, we define

$$B_k := \{x : \ h(x) > k\}.$$

For a Borel set $A \subset \mathbb{R}^d$ with $\mu(A) > 0$, we set

$$\mu_A := \mu(\cdot|A).$$

THEOREM 3.2.2. *Let $\nu \ll \mu$ be two Borel probability measures with compact support and h be the corresponding Radon-Nikodym derivative. Then*

(1) *If* $\lim_k \overline{\dim}_B B_k \le \overline{D}_r(\mu)$, *then* $\overline{D}_r(\nu) \le \overline{D}_r(\mu)$.
(2) *If $D_r(\mu)$ exists and* $\lim_k \underline{\dim}_B B_k \le D_r(\mu)$ *then* $\underline{D}_r(\nu) \le D_r(\mu)$.
(3) *Suppose that $h(x) \le M$, $x \in B_0$. If $\overline{D}_r(\mu_{B_0}) < \overline{D}_r(\mu)$, then $\overline{D}_r(\nu) < \overline{D}_r(\mu)$.*

PROOF. *ad* (1). If $\mu(B_k) = 0$ for some k, then $h(x) \le k$ for μ-a.e. x. Hence $V_{n,r}(\nu) \le kV_{n,r}(\mu)$ and (1) follows immediately from the definition. Now we assume that $\mu(B_k) > 0$ for all $k \ge 1$. By the hypothesis in (1), for any $\epsilon > 0$, there exists $k_0 \ge 1$ such that for $k \ge k_0$, we have $\overline{\dim}_B B_k \le \overline{D}_r(\mu) + \epsilon$. Thus by Lemma 2.4.2

$$\overline{D}_r(\nu(\cdot|B_{k_0})) \le \overline{\dim}_B B_{k_0} \le \overline{D}_r(\mu) + \epsilon.$$

On the other hand, $e_{n,r}\left(\nu\left(\cdot|B_k^C\right)\right) \le k_0 e_{n,r}(\mu)$ implies that

$$\overline{D}_r\left(\nu\left(\cdot|B_{k_0}^C\right)\right) \le \overline{D}_r(\mu).$$

Since $\nu = \nu(B_k)\nu(\cdot|B_k) + \nu\left(B_k^C\right)\nu\left(\cdot|B_k^C\right)$, by Theorem 2.3.2,

$$\overline{D}_r(\nu) = \max\left\{\overline{D}_r(\nu(\cdot|B_k)), \overline{D}_r\left(\nu\left(\cdot|B_k^C\right),\right)\right\} \le \overline{D}_r(\mu) + \epsilon.$$

By the arbitrariness of ϵ, (1) follows.

ad (2). Without loss of generality, we again assume that $\mu(B_k) > 0$ for all $k \ge 1$. For any $\epsilon > 0$, there exists $k_0 > 0$, such that $k \ge k_0$ implies that $\underline{\dim}_B B_k < D_r(\mu) + \epsilon$. by Lemma 3.2.1 (2), we may choose (n_j) and a set γ_j of n_j points such that

$$\min_{a\in\gamma_j} \|x - a\| \le C_2 n_j^{\frac{-1}{D_r(\mu)+\epsilon}}, \ \forall x \in B_k.$$

Let $\alpha \in C_{n_j,r}(\mu)$and $\delta_j = \alpha \cup \gamma_j$. Then

$$\begin{aligned} V_{2n_j,r}(\nu) &\leq \int_{B_{k_0}^C} \min_{b\in\delta_j} \|x-b\|^r h(x)\,d\mu(x) + \int_{B_{k_0}} \min_{b\in\delta_j} \|x-b\|^r h(x)\,d\mu(x) \\ &\leq k_0 \int_{B_{k_0}^C} \min_{b\in\alpha} \|x-b\|^r d\mu(x) + n_j^{-\frac{r}{D_r(\mu)+\epsilon}} \int_{B_{k_0}} h(x)\,d\mu(x) \\ &\leq k_0 V_{n_j,r}(\mu) + n_j^{-\frac{r}{D_r(\mu)+\epsilon}}. \end{aligned}$$

It follows that

$$\begin{aligned} \liminf_{n\to\infty} n^{\frac{r}{D_r(\mu)+\epsilon}} V_{n,r}(\nu) &\leq \liminf_{j\to\infty} (2n_j)^{\frac{r}{D_r(\mu)+\epsilon}} V_{2n_j,r}(\nu) \\ &\leq \liminf_{j\to\infty} (2n_j)^{\frac{r}{D_r(\mu)+\epsilon}} V_{n_j,r}(\mu) + 2^{1/D_r(\mu)} \\ &\leq 2^{1/D_r(\mu)} < \infty. \end{aligned}$$

This implies that $\underline{D}_r(\nu) \leq D_r(\mu) + \epsilon$.

ad (3). It is clear that $\nu = \nu_{B_0}$ and $\nu_{B_0} \leq M\mu(B_0)\,\mu_{B_0}$. Hence,

$$\overline{D}_r(\nu) \leq \overline{D}_r(\mu_{B_0}) < \overline{D}_r(\mu).$$

□

REMARK 3.2.3. If $\nu \ll \mu$ are two Borel probability measures on $\mathbb{R}^d$ with $d\nu/d\mu \geq m > 0$, then we easily have

$$\underline{D}_r(\nu) \geq \underline{D}_r(\mu), \quad \overline{D}_r(\nu) \geq \overline{D}_r(\mu).$$

However, the following example shows that the condition $h(x) \geq m,\ x \in B_0$ does-not imply the inequality $D_r(\nu) \geq D_r(\mu)$. Let μ_1, μ_2 be two self-similar measures respectively on $[0,1], [1,2]$ with $D_r(\mu_1) < D_r(\mu_2)$. Define $\mu := \frac{1}{2}(\mu_1 + \mu_2)$ and let h be some positive measurable function bounded from above with $\int h(x)\,d\mu(x) = 1$ and $h(x) = 0,\ x \in [1,2]$. Then for $d\nu := h\,d\mu$ we have by Proposition 3.2.2 (3) that $D_r(\nu) < D_r(\mu)$.

In the following, we introduce the notions of the upper and lower vanishing rates of a Borel function.

Let h be a measurable function. We define its *upper* and *lower vanishing rate* $\overline{R}(h), \underline{R}(h)$ with respect to μ by

$$\overline{R}(h) := \limsup_{k\to\infty} -\frac{1}{k} \log \int_{B_k} h(x)\,d\mu(x), \tag{3.2.1}$$

$$\underline{R}(h) := \liminf_{k\to\infty} -\frac{1}{k} \log \int_{B_k} h(x)\,d\mu(x). \tag{3.2.2}$$

Our next theorem shows that the upper vanishing rate is connected with the lower quantization dimension, while the lower vanishing rate is connected with the upper quantization dimension.

THEOREM 3.2.4. *Suppose that μ is compactly supported. We have*

(1) *If $\underline{R}(h) \geq r/\overline{D}_r(\mu)$, then $\overline{D}_r(\nu) \leq \overline{D}_r(\mu)$;*
(2) *If $D_r(\mu)$ exists and $\overline{R}(h) \geq r/D_r(\mu)$, then $\underline{D}_r(\nu) \leq D_r(\mu)$.*

PROOF. *ad* (1). By the hypothesis, for any $\epsilon > 0$ there exist $k_0 \geq 1$ such that $k \geq k_0$ implies that

$$\int_{B_k} h(x)\, d\mu \leq e^{-\frac{kr}{\overline{D}_r(1+\epsilon)}}.$$

By the definition of the upper quantization dimension, for the above $\epsilon > 0$, there exists $N \geq 1$ such that $n \geq N$ implies

$$V_{n,r}(\mu) \leq n^{-\frac{r}{\overline{D}_r(1+\epsilon)}}.$$

For each $n \geq \max\{e^{k_0}, N\}$, there is some $k \geq k_0$ such that $e^k \leq n < e^{k+1}$. Let $\alpha \in C_{n,r}(\mu)$. Since μ has compact support, by [**9**], there exists $M > 0$ such that

$$\min_{a\in\alpha} \|x-a\| \leq M, \text{ for all } x \in \operatorname{supp}(\mu). \tag{3.2.3}$$

Hence we have

$$\begin{aligned} V_{n,r}(\nu) &\leq \int_{\{h(x)\leq k\}} \min_{a\in\alpha} \|x-a\|^r h(x)\, d\mu(x) + \int_{B_k} \min_{a\in\alpha} \|x-a\|^r h(x)\, d\mu(x) \\ &\leq kV_{n,r}(\mu) + M^r e^{-\frac{kr}{\overline{D}_r(1+\epsilon)}}. \\ &\leq \log n \cdot V_{n,r}(\mu) + CM^r n^{-\frac{r}{\overline{D}_r(1+\epsilon)}} \\ &\leq C(\log n + M^r)\, n^{-\frac{r}{\overline{D}_r(1+\epsilon)}}, \end{aligned}$$

where $C = e^{r/\overline{D}_r}$. It follows that

$$\begin{aligned} \overline{D}_r(\nu) &\leq \limsup_{n\to\infty} \frac{r\log n}{-\log(\log n + M^r) + \log C - \log n \cdot r/\left(\overline{D}_r(1+\epsilon)\right)} \\ &= \overline{D}_r(1+\epsilon). \end{aligned}$$

By the arbitrariness of ϵ, we immediately have $\overline{D}_r(\nu) \leq \overline{D}_r(\mu)$.

ad (2). By the condition in (2), for any $\epsilon > 0$, there exists a sequence (k_i)such that

$$\int_{B_{k_i}} h(x)\, d\mu < e^{\frac{-k_i r}{D_r(\mu)(1+\epsilon)}}.$$

We take $n_i = \left[e^{k_i}\right] + 1$. For any $\alpha_i \in C_{n_i,r}(\mu)$, using (3.2.3) we have

$$\begin{aligned} V_{n_i,r}(\nu) &\leq \int_{\{h \leq k_i\}} \min_{a\in\alpha_i} \|x-a\|^r h(x)\, d\mu(x) + \int_{B_{k_i}} \min_{a\in\alpha_i} \|x-a\|^r h(x)\, d\mu(x) \\ &\leq k_i \int_{\{h\leq k_i\}} \min_{a\in\alpha_i} \|x-a\|^r d\mu(x) + M^r C \int_{B_{k_i}} h(x)\, d\mu(x) \\ &\leq \log n_i V_{n_i,r}(\mu) + M^r C n_i^{-\frac{-r}{\underline{D}_r(\mu)(1+\epsilon)}}. \end{aligned} \tag{3.2.4}$$

On the other hand, by the definition of quantization dimension, for the above $\epsilon > 0$, there corresponds an integer N such that $n \geq N$ implies that $V_{n,r}(\mu) < n^{-\frac{1}{D_r(\mu)(1+\epsilon)}}$. This together with (3.2.4) yields that for i large enough

$$V_{n_i,r}(\nu) \leq n_i^{-\frac{-r}{D_r(\mu)(1+\epsilon)}} \left(\log n_i + M^r C\right) \leq 2\log n_i \cdot n_i^{-\frac{-r}{D_r(\mu)(1+\epsilon)}}.$$

It follows that $\underline{D}_r(\nu) \leq D_r(\mu)(1+\epsilon)$. Thus by the arbitrariness of ϵ, (2) follows. □

The above proposition can be improved by using Lemma 3.2.1.

THEOREM 3.2.5. *Let μ, ν be two Borel probability measures on $\mathbb{R}^d$ with compact support and $\nu \ll \mu$. Let s_0 be the box-counting dimension of* $\operatorname{supp}(\mu)$. *If $R(h) \geq \frac{r(s_0 - s_r)}{s_0 s_r}$. Then $\overline{D}_r(\nu) \leq \overline{D}_r(\mu)$.*

In [**11**], GRAF and LUSCHGY introduced the following two quantities for the quantization dimension of μ of order 0:

$$\begin{aligned} \overline{D}^*(\mu) &:= \sup\left\{\overline{D}_0(\mu(\cdot|E)) : \ \mu(E) > 0\right\}, \\ \underline{D}^*(\mu) &:= \sup\left\{\underline{D}_0(\mu(\cdot|E)) : \ \mu(E) > 0\right\}. \end{aligned}$$

We now replace the quantization dimension of order 0 in the above definitions with quantization dimension of order $r \in [1,\infty]$. Then by Lemma 2.3.9 one easily sees that for $r \in [1,\infty]$,

$$\overline{D}_r^*(\mu) = \overline{D}_r(\mu), \ \ \underline{D}_r^*(\mu) = \underline{D}_r(\mu).$$

The key is that the quantization dimension of order 0 does not enjoy the monotonicity as showed in [**11**] while the quantization dimension of order $r \in [1,\infty]$ has this property. Analogously, we define

$$\begin{aligned} \overline{D}_{r*}(\mu) &:= \inf\left\{\overline{D}_r(\mu(\cdot|E)) : \ \mu(E) > 0\right\}, \\ \underline{D}_{r*}(\mu) &:= \inf\left\{\underline{D}_r(\mu(\cdot|E)) : \ \mu(E) > 0\right\}. \end{aligned}$$

If $\overline{D}_{r*}(\mu) = \overline{D}_r(\mu)$, then $\overline{D}_r(\mu(\cdot|E)) = \overline{D}_r(\mu)$ for any Borel set E with $\mu(E) > 0$. This implies μ in this case is rather "uniform" and exhibits some regularity. Moreover, we define

$$\overline{D}_r^{\flat}(\mu) := \inf\left\{\overline{D}_r(\nu) : \nu \in \mathcal{M},\ \nu \ll \mu\right\},$$
$$\underline{D}_r^{\flat}(\mu) := \inf\left\{\underline{D}_r(\nu) : \nu \in \mathcal{M},\ \nu \ll \mu\right\}.$$

If the two quantities coincide, we then denote the common value by $\overline{D}_r^{\flat}(\mu)$. It is easy to see that

$$\overline{D}_r^{\flat}(\mu) \geq \dim_p \mu,\ \underline{D}_r^{\flat}(\mu) \geq \dim_H \mu.$$

One might expect that the condition $\overline{D}_r^{\flat}(\mu) = \overline{D}_r(\mu)$ would imply a little more than $\overline{D}_{r*}(\mu) = \overline{D}_r(\mu)$. However, our next proposition shows that this is not the case. For the proof of this proposition, we will need the following theorem (cf. [**1**, Theorem 27.4]) as a lemma.

LEMMA 3.2.6. *Let μ, ν be two Borel probability measures on $\mathbb{R}^d$ with $\nu \ll \mu$. Then there exists $c > 0$ and some $A \in \mathcal{B}$ with $\mu(A) > 0$ such that*

$$c\mu(B) \leq \nu(B) \ \text{ for all } B \in \mathcal{B} \text{ with } B \subset A.$$

PROOF. This is a special case of [**1**, Theorem 27.4]. □

PROPOSITION 3.2.7. *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then*

$$\overline{D}_r^{\flat}(\mu) = \overline{D}_{r*}(\mu),\quad \overline{D}_r^{\flat}(\mu) = \underline{D}_{r*}(\mu).$$

PROOF. By the definitions, we immediately have

$$\overline{D}_r^{\flat}(\mu) \leq \overline{D}_{r*}(\mu),\quad \overline{D}_r^{\flat}(\mu) \leq \underline{D}_{r*}(\mu). \tag{3.2.5}$$

It remains to show the reverse. For any $\epsilon > 0$, let $\nu \ll \mu$ satisfy that

$$\overline{D}_r(\nu) \leq \overline{D}_r^{\flat}(\mu) + \epsilon. \tag{3.2.6}$$

By Lemma 3.2.6, there exist a constant $c > 0$ and a Borel set A with

$$\mu(A) > 0,\ \nu(B) \geq c\mu(B),\ A \supset B \in \mathcal{B}.$$

In particular, $\nu(A) > 0$. We consider the two measures $\mu(\cdot|A)$ and $\nu(\cdot|A)$. We have for $F \in \mathcal{B}$ that

$$\mu(F|A) = \frac{\mu(F \cap A)}{\mu(A)} \leq \frac{c^{-1}\nu(F \cap A)}{\mu(A)} = c^{-1}\frac{\nu(A)}{\mu(A)}\nu(F|A).$$

It follows that

$$\overline{D}_r(\mu(\cdot|A)) \leq \overline{D}_r(\nu(\cdot|A)) \leq \overline{D}_r(\nu).$$

This together with (3.2.6) implies that $\overline{D}_{r*}(\mu) \leq \overline{D}_r^{\flat}(\mu) + \epsilon$. By the arbitrariness of ϵ, we get $\overline{D}_{r*}(\mu) \leq \overline{D}_r^{\flat}(\mu)$. The same argument shows that $\underline{D}_{r*}(\mu) \leq \underline{D}_r^{\flat}(\mu)$. Combining these two inequalities with (3.2.5) completes the proof.

THEOREM 3.2.8. *Let μ be a Borel probability measure on $\mathbb{R}^d$.*

(1) Suppose that $\overline{D}_r^\flat(\mu) = \overline{D}_r(\mu)$. Then we have $\dim_p \mu = \overline{D}_r(\mu)$;
(2) If $\dim_p \mu = \dim_p^* \mu$, then $\overline{D}_r^\flat(\mu) = \dim_p \mu$.

□

The proof of this theorem will depend on the following lemma.

LEMMA 3.2.9. *Let μ be a Borel probability measure on $\mathbb{R}^d$ with $\overline{D}_r(\mu) > \dim_p \mu$. Then there exist $0 < s < \overline{D}_r(\mu)$ and a measurable set $A \subset \operatorname{supp}(\mu)$ such that $\mu(A) > 0$ and $\overline{\dim}_B A < s$.*

PROOF. By the hypothesis, there exist a Borel set F with $\mu(F) > 0$ and $\dim_p \mu < s < \overline{D}_r(\mu)$ such that $\overline{\dim}_{\text{loc}}\mu(x) < s$ for all $x \in F$. Hence for each $x \in F$ there exists $t_x > 0$ such that $t < t_x$ implies $\mu(B(x,t)) > t^s$. Setting

$$F_m := \{x \in F : \ t_x \geq 1/m\}.$$

Then F_m increases to F. Thus we may choose m large enough such that $\mu(F_m) > 0$. For all $x \in F_m$, $t < 1/m$ implies that $\mu(B(x,t)) > t^s$. Let $N_t(F_m)$ denote the largest number of mutually disjoint balls centered in F_m and of radii t. We have

$$N_t(F_m)\, t^s \leq N_t(F_m)\, \mu(B(x,t)) \leq 1.$$

This implies $N_t(F_m) \leq t^{-s}$. It follows that

$$\overline{\dim}_B(F_m) \leq s < \overline{D}_r(\mu).$$

Taking $A = \overline{F_m} \cap \operatorname{supp}(\mu)$, the proof is complete. □

Proof of Theorem 3.2.8.

(1) Suppose that $\dim_p \mu < \overline{D}_r(\mu)$. Then by Lemma 3.2.9, there exist $t < \overline{D}_r(\mu)$ and a Borel set $A \subset \operatorname{supp}(\mu)$ with $\mu(A) > 0$ and $\overline{\dim}_B A \leq t$. Hence by Lemma 2.4.2, we have

$$\overline{D}_r(\mu(\cdot|A)) \leq \overline{\dim}_B^* \mu(\cdot|A) = \overline{\dim}_B A \leq t.$$

It follows that $\overline{D}_r^\flat(\mu) \leq t < \overline{D}_r(\mu)$, which contradicts the hypothesis.
(2) By the condition $\dim_p \mu = \dim_p^* \mu$, for $\nu \ll \mu$, we have

$$\dim_p \mu \leq \dim_p \nu \leq \dim_p^* \nu \leq \dim_p^* \mu.$$

It follows by Lemma 2.4.2 that

$$\overline{D}_r(\nu) \geq \dim_p^* \nu = \dim_p \mu.$$

This implies that $\overline{D}_{r*}(\mu) \geq \dim_p^* \mu$. On the other hand, by Lemma 3.2.9, for any $s > \dim_p^* \mu$, there exists $F \in \mathcal{B}$ with

$$\mu(F) > 0, \ \overline{D}_r(\mu(\cdot|F)) < s.$$

Hence by definition, we have $\overline{D}_{r*}(\mu) \leq \dim_p^* \mu$. The proof is now complete.

REMARK. Assume that $\nu \ll \mu$. One may ask what conditions imply $\overline{D}_r(\nu) > \overline{D}_r(\mu)$? This question seems hard because it is very difficult to get a lower bound for the quantization dimension. It is clear that if $\overline{D}_r(\mu) = \overline{\dim}_B^* \mu$, then $\overline{D}_r(\nu) > \overline{D}_r(\mu)$ can not happen. Indeed, in this case, we have

$$\overline{D}_r(\nu) \leq \overline{\dim}_B^* \nu \leq \overline{\dim}_B^* \mu = \overline{D}_r(\mu).$$

Now suppose that $\overline{D}_r(\mu) < \overline{\dim}_B^* \mu$. Does there necessarily exist a probability measure ν such that $\nu \ll \mu$ and $\overline{D}_r(\nu) > \overline{D}_r(\mu)$?

We end this section by the following proposition answering question (AC2).

PROPOSITION 3.2.10. *There exists Borel probability measures μ, ν with $\nu \ll \mu$ and such that $D_r(\mu)$ exists but $\overline{D}_r(\nu) > \underline{D}_r(\nu)$.*

PROOF. Let ν be a homogeneous Cantor measure with $\overline{D}_r(\nu) > \underline{D}_r(\nu)$ and let τ be a Cantor measure on $\mathbb{R}$ such that $D_r(\tau)$ exists and $D_r(\tau) > \overline{D}_r(\nu)$. Define

$$\mu := \frac{1}{2}\nu + \frac{1}{2}\tau.$$

Then clearly $\nu \ll \mu$. However, by Corollary 2.3.4, we know that $D_r(\mu)$ exists and equals $D_r(\tau)$. This finishes the proof. □

3.3. $\overline{D}_r(\cdot)$ is not monotone and doesnot obey a variational law

In this section we show that in general absolute continuity of two measures doesnot imply any definite inequality between their quantization dimensions. Here we remark that [**9**, Example 6.4] shows for a singular measure μ (with respect to the d-dimensional Lebesgue measure), its quantization coefficient $\overline{Q}_r^d(\mu)$ is not necessarily finite if the $(r+\delta)$-moment condition fails. By the way, it shows that the quantization dimension $\overline{D}_r(\mu)$ is not necessarily bounded by d. But in this section, we only consider the relationship between absolute continuity and quantization dimension. So we are only concerned with those pairs of measures one of which is absolutely continuous with respect to the other. The method here is developed from GRAF and LUSCHGY's.

LEMMA 3.3.1. *Let $\lambda > 1, \beta > 1$. Define $c = \left(\sum_{k=1}^{\infty} k^{-\lambda}\right)^{-1}$ and*

$$x_k = \frac{1}{2}\left(k^{-\beta} + (k+1)^{-\beta}\right), \ \mu(\{x_k\}) = \frac{c}{k^\lambda}.$$

Then we have

$$\underline{D}_r(\mu) = \overline{D}_r(\mu) = \frac{1}{\beta + 1 + (\lambda - 1)/r}.$$

In particular, $\overline{D}(\mu) = 1/(\beta+1)$.

PROOF. For any $\alpha \subset \mathbb{R}$ with $\mathrm{card}(\alpha) = n$, we write

$$\mathcal{I} = \left\{ k \geq 1 : \ \alpha \cap \left[(k+1)^{-\beta}, k^{-\beta} \right] = \emptyset \right\}.$$

Then for any $k \in \mathcal{I}$, we have

$$\begin{aligned} \min_{a\in\alpha} |x_k - a|^r &\geq \left(\frac{k^{-\beta} - (k+1)^{-\beta}}{2} \right)^r = \frac{\left((k+1)^{\beta} - k^{\beta} \right)^r}{2^r k^{\beta r} (k+1)^{\beta r}} \\ &\geq \frac{k^{(\beta-1)r}}{2^r k^{\beta r} (k+1)^{\beta r}} = \frac{1}{2^r k^r (k+1)^{\beta r}} \end{aligned}$$

where we used the fact that $(k+1)^\beta - k^\beta \geq k^{\beta - 1}$ for $\beta > 1$. It follows that

$$\begin{aligned} E \min_{a\in\alpha} |x - a|^r &\geq \sum_{k\in\mathcal{I}} \min_{a\in\alpha} |x_k - a|^r \frac{c}{k^\lambda} \\ &\geq \sum_{k\in\mathcal{I}} \frac{1}{2^r k^r (k+1)^{\beta r}} \frac{c}{k^\lambda} \\ &\geq \frac{c}{2^r} \sum_{k=n+2}^{\infty} \frac{1}{(k+1)^{\beta r + r + \lambda}} \\ &\geq \frac{c}{2^r} \int_{n+2}^{\infty} \frac{1}{x^{(\beta+1)r+\lambda}} dx \\ &\geq C (n+2)^{1-(\beta+1)r-\lambda}. \end{aligned}$$

Since α is arbitrary, we have

$$e_{n,r}(\mu) \geq C_2(r, \beta, \lambda) (n+2)^{-(\beta+1)+(1-\lambda)/r}.$$

It follows from the definition that

$$\underline{D}_r(\mu) \geq 1/\left((\beta+1) + (\lambda - 1)/r \right). \tag{3.3.1}$$

On the other hand, for any $s > 1/(1+\beta)$, by Lemma 3.2.1, we may choose a set γ of n points such that

$$\min_{a\in\gamma} |x - a| \leq C_1 n^{-1/s}.$$

Taking $\delta := \gamma \cup \{x_k : 1 \le k \le n\}$, we have

$$\begin{aligned} V_{2n,r}(\mu) &\le \int \min_{b\in\delta} |x-b|^r \, d\mu(x) = \sum_{k=1}^{\infty} \min_{b\in\delta} |x_k - b|^r \frac{c}{k^\lambda} \\ &\le C_1 n^{-r/s} \sum_{k=n+1}^{\infty} \frac{c}{k^\lambda} \le C_1 a n^{-r/s} \int_n^\infty x^{-\lambda} dx \\ &= C n^{-r/s-\lambda+1}, \end{aligned}$$

where $C = \frac{C_1 c}{\lambda - 1}$. This implies that

$$e_{2n,r}(\mu) \le C^{1/r} n^{-1/s-(\lambda-1)/r}.$$

Hence we have $\overline{D}_r(\mu) \le 1/\left(1/s + (\lambda-1)/r\right)$. It follows from the arbitrariness of s that

$$\overline{D}_r(\mu) \le 1/\left(\beta + 1 + (\lambda - 1)/r\right). \tag{3.3.2}$$

Combining (3.3.1) and (3.3.2) we have

$$\underline{D}_r(\mu) = \overline{D}_r(\mu) = 1/\left(\beta + 1 + (\lambda-1)/r\right). \tag{3.3.3}$$

Recall that $\overline{D}(\mu) = \lim_{r\to\infty} \overline{D}_r(\mu)$, by (3.3.3) we immediately have

$$\lim_{r\to\infty} D_r(\mu) = D_\infty(\mu) = \frac{1}{\beta+1}.$$

□

REMARK 3.3.2. In fact, in the second part of the proof of the above lemma, if we take

$$\delta_n := \{x_k : \ 1 \le k \le n\} \cup \{0\},$$

then we have $\min_{b\in\delta} |x_k - b| \le |x_k|$. Hence we have

$$\begin{aligned} V_{n+1,r}(\mu) &\le \sum_{k=n+1}^{\infty} \left(\frac{k^{-\beta} + (k+1)^{-\beta}}{2} \right)^r \frac{a}{k^\lambda} \\ &\le \sum_{k=n+1}^{\infty} 2^{-r} k^{-\beta} \frac{a}{k^\lambda} \\ &\le \frac{a 2^{-r}}{\beta + \lambda - 1} n^{-\beta-\lambda+1}. \end{aligned}$$

This implies that $\overline{D}_r(\mu) \le 1/\left(\beta + (\lambda-1)/r\right)$.

The following theorem answers question (AC3) and the question (AC5).

THEOREM 3.3.3. *There exists two Borel probability measures μ, ν with compact supports and $\nu \ll \mu$ such that $\overline{D}_r(\nu) > \overline{D}_r(\mu)$. In particular, there doesnot exist a set function $\overline{\mathbf{D}}_r$ such that the variational law (3.1.1) holds with respect to the upper quantization dimension for measures.*

PROOF. According to the above lemma, we define two probability measures μ_1, μ_2 by respectively choosing λ to be λ_1 and λ_2 satisfying $\lambda_1 > \lambda_2 > 1$. Then clearly,

$$\frac{1}{\beta + 1 + (\lambda_2 - 1)/r} > \frac{1}{\beta + 1 + (\lambda_1 - 1)/r}.$$

Then By Lemma 1.5.6, we have $\overline{D}_r(\mu_1) < \underline{D}_r(\mu_2)$.

Now suppose that $\overline{\mathbf{D}}_r$ is a dimension-like set function satisfying the variational law. Let (ν, μ) be an arbitary pair of measures with $\nu \ll \mu$. It is clear that $\mu(E) = 1$ implies that $\nu(E) = 1$. Hence by the variational law, we immediately have $\overline{D}_r(\nu) \leq \overline{D}_r(\mu)$. This contradicts the first part of the theorem. □

The following proposition provides us with a method to construct other pairs (ν, μ) with $\nu \ll \mu$ but $\overline{D}_r(\nu) > \overline{D}_r(\mu)$.

PROPOSITION 3.3.4. *Let $\mu_i, \nu_i \in \mathcal{M}_r$ with $\nu_i \ll \mu_i, i = 1, 2$ and*

$$\overline{D}_r(\nu_1) \leq \overline{D}_r(\nu_2), \overline{D}_r(\mu_1) \leq \overline{D}_r(\mu_2), \overline{D}_r(\nu_2) > \overline{D}_r(\mu_2).$$

Let μ, ν be defined by

$$\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2, \quad \nu = \frac{1}{2}\nu_1 + \frac{1}{2}\nu_2.$$

Then we have $\nu \ll \mu$ and $\overline{D}_r(\nu) > \overline{D}_r(\mu)$.

PROOF. Since $\nu_i \ll \mu_i, i = 1, 2$, we immediately have $\nu \ll \mu$. Now by the finite stability of the upper quantization dimension (cf. Theorem 2.3.2) we have,

$$\overline{D}_r(\nu) = \overline{D}_r(\nu_2) > \overline{D}_r(\mu_2) = \overline{D}_r(\mu).$$

This finishes the proof of the proposition. □

3.4. Ahlfors-David regular measures

In this section, we prove that for any measure ν which is absolutely continuous with respect to an s-regular measure μ, the upper and lower s-dimensional quantization coefficient $\overline{Q}_r^s(\nu), \underline{Q}_r^s(\nu)$ are positive and finite. In particular, $D_r(\nu)$ exists and equals s for all $r \geq 1$. Recall that a measure μ with compact support is called s-regular if there exists constants $C, t_0 > 0$ such that for any $x \in \operatorname{supp}(\mu)$ and any $t \leq t_0$, we have

$$C^{-1}t^s \leq \mu(B(x,t)) \leq Ct^s.$$

We begin with the following proposition.

PROPOSITION 3.4.1. *Let μ be an s-regular measure and $\nu \ll \mu$. Then for all $r \geq 1$ we have $D_r(\nu) = s$. In particular, we have $D_r^{\flat}(\mu) = s$.*

PROOF. Let $F \subset K$ be any Borel set with $\nu(F) = 1$. Since $\nu \ll \mu$, we have $\mu(F) > 0$. It follows from the s-regularity of μ and [**5**, Proposition 2.3] that $\dim_H F = s$. Thus

$$\dim_H^* \nu = \inf\{\dim_H F : \ F \in \mathcal{B}, \ \nu(F) = 1\} = s.$$

On the other hand, we have $\operatorname{supp}(\nu) \subset \operatorname{supp}(\mu)$, this implies that

$$D_\infty(\nu) \leq D_\infty(\mu) = s.$$

The proposition follows since using [**9**, Corollary 12.16, Lemma 10.1(a)], we have

$$s = \dim_H^* \nu \leq D_r(\nu) \leq D_\infty(\nu) \leq s.$$

□

The following theorem shows that the quantization coefficient of these measures are positive and finite.

THEOREM 3.4.2. *Let μ be an s-regular measure and $\nu \ll \mu$. Then*

$$0 < \underline{Q}_r^s(\nu) \leq \overline{Q}_r^s(\nu) < \infty. \tag{3.4.1}$$

PROOF. We denote by h the Radon-Nikodym derivative of ν with respect to μ. Then h is a μ-measurable function. Denote by K the topological support of ν. Then we know that $K \subset \operatorname{supp}(\mu)$. Using [**9**, Lemma 10.6, Proposition 12.17] we have

$$\limsup_{n\to\infty} n^{1/s} e_{n,\infty}(\nu) \leq \limsup_{n\to\infty} n^{1/s} e_{n,\infty}(\mu) < \infty.$$

Since $e_{n,r}(\nu) \leq e_{n,\infty}(\nu)$, the last inequality of (3.4.1) follows. Now note that h does not vanish μ-a.e., the set $H := \{x : \ h(x) > 0\}$ has a positive μ-measure. By Lusin's Theorem, we may choose a compact set $F \subset \operatorname{supp}(\mu)$ such that h is continuous on F and $\mu(F) \geq 1 - \mu(H)/2$. Hence

$$\mu(F \cap H) = \mu(H) - \mu(H \setminus F) \geq \mu(H)/2 > 0.$$

By the Borel regularity of μ we may choose a compact subset D of $F \cap H$ such that $\mu(D) > 0$. Setting

$$M := \max_{x \in D} h(x), \quad m := \min_{x \in D} h(x).$$

Then we have

$$\nu(D) = \int_D h(x)\, d\mu \geq m\mu(D) > 0.$$

We hence consider the conditional probability measure

$$\nu_D := \nu(\cdot|D) = \frac{\nu(\cdot \cap D)}{\nu(D)}.$$

Then by the s-regularity of μ,

$$\nu_D(B(x,t)) \leq C_2 M \nu(D)^{-1} t^s.$$

Thus [**9**, Proposition 12.15] shows that

$$\liminf_{n\to\infty} n^{r/s} V_{n,r}(\nu_D) > 0. \tag{3.4.2}$$

On the other hand, we have $\nu \geq \nu(D)\nu_D$. This together with (3.4.2) yields

$$\liminf_{n\to\infty} n^{r/s} V_{n,r}(\nu) \geq \nu(D) \liminf_{n\to\infty} n^{r/s} V_{n,r}(\nu_D) > 0.$$

This finishes the proof of the theorem. □

REMARK 3.4.3. We have an optional proof for the above theorem by means of a result of PÖTZELBERGER. We first generalize [**22**, Theorem 1(4)] to any $r \geq 1$. And then make use of the equivalence between μ and the restricted Hausdorff measure (cf. [**9**, Proposition 12.11]).

COROLLARY 3.4.4. *Let μ be an s-regular measure. Let ν be a Borel probability measure with* $\mathrm{supp}(\nu) \subset \mathrm{supp}(\mu)$. *Suppose that the continuous part of ν with respect to μ does not vanish. Then (3.4.1) holds.*

PROOF. Let $\nu = \nu_1 + \nu_2, \nu_1 \ll \mu, \nu_2 \perp \mu$ be the Lebesgue decomposition of ν with respect to μ. We consider the conditional probability $\widetilde{\nu}_1 := (\nu_1(\mathrm{supp}(\nu_1)))^{-1} \nu_1$. It is clear that

$$\widetilde{\nu}_1 \ll \mu, \quad \nu \geq \nu_1(\mathrm{supp}(\nu_1))\, \widetilde{\nu}_1.$$

By Theorem 3.4.2, we have $\underline{Q}_r^s(\nu) > 0$. Since $\mathrm{supp}(\nu) \subset \mathrm{supp}(\mu)$, as in the proof of Proposition 3.4.1, we have $\overline{Q}_r^s(\nu) < \infty$. The corollary now follows. □

3.5. Self-similar measures

In this section, we study the quantization for the measures which are absolutely continuous with respect to self-similar measures .

3.5.1. Definitions and known results.

Let X be a non-empty compact set on $\mathbb{R}^d$ and $f_i : \mathbb{R}^d \to \mathbb{R}^d$, $1 \leq i \leq N$ be contracting similarity maps satisfying

$$|f_i(x) - f_i(y)| = c_i |x - y|, \ 1 \leq i \leq N,$$

where $0 < c_i < 1$ for all $1 \leq i \leq N$. We call $\{f_i, 1 \leq i \leq N\}$ an iterated function system (IFS) on X. Let E denote the corresponding self-similar set E, i.e., the unique non-empty compact set satisfying

$$E = \bigcup_{i=1}^{N} f_i(E).$$

We set

$$\Omega_n = \Pi_{i=1}^{n} \{1, 2, \cdots, N\}, \ \ \Omega = \Pi_{i=1}^{\infty} \{1, 2, \cdots, N\}, \ \ \Omega^* = \bigcup_{i=1}^{\infty} \Omega_n.$$

Let $|\sigma|$ denote the number of the components of σ. For $\sigma = (\sigma_1, \sigma_2, \cdots) \in \Omega \cup \Omega^*$ with $|\sigma| \geq n$, we denote by $\sigma|_n$ the finite word $(\sigma_1, \cdots, \sigma_n) \in \Omega_n$ and set

$$f_{\sigma|_n} = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}.$$

Then we have $E = \lim_{n\to\infty} \bigcup_{\sigma\in\Omega_n} f_\sigma(X)$. For each $n \geq 1$ and $\sigma \in \Omega_n$ we set

$$E_\sigma := f_\sigma(E) = f_{\sigma_1} \circ f_{\sigma_2} \circ \cdots \circ f_{\sigma_n}(E), \quad c_\sigma := c_{\sigma_1} \cdots c_{\sigma_n}.$$

We say the IFS $\{f_i, 1 \leq i \leq N\}$ satisfies the open set condition (OSC) if there exists a bounded non-empty open set U in $\mathbb{R}^d$ such that

$$\bigcup_{i=1}^{N} f_i(U) \subset U, \quad f_i(U) \cap f_j(U) = \emptyset, \ i \neq j.$$

We say the IFS $\{f_i, 1 \leq i \leq N\}$ satisfies the strong separation condition (SSC) if $f_i(E), 1 \leq i \leq N$ are mutually disjoint. The fractal properties of self-similar sets have been extensively studied. Under the OSC, its Hausdorff, packing and upper and lower box-counting dimension coincide and equal s, where s is the unique real number given by $\sum_{i=1}^{N} c_i^s = 1$. The s-dimensional Hausdorff measure is positive and finite. The IFS consisting of infinitely many maps have been systematically studied by MAULDIN and URBANSKI, see, for example, [**19, 20, 21**]. Our recent work [**13**] is also in this field. Quantization for measures supported by the limit sets of infinitely IFS seems to be still open.

For an IFS $\{f_i : 1 \leq i \leq N\}$ on X and any probability vector $(p_1, p_2, \cdots, p_N)$, there corresponds a Borel probability measure μ satisfying

$$\mu = \sum_{i=1}^{N} p_i \mu \circ f_i^{-1}.$$

We call this measure the *self-similar measure associated with* $(p_1, p_2, \cdots, p_N)$.

The study of the quantization properties for self-similar measures goes back to ZADOR ([**26, 27**]). GRAF and LUSCHGY determined in ([**10**]) the quantization properties for self-similar measures under the OSC. Let μ be a self-similar measure determined by $\{f_i : 1 \leq i \leq N\}$ and $(p_1, p_2, \cdots, p_N)$. Suppose that $\{f_i : 1 \leq i \leq N\}$ satisfies OSC. Then

$$0 < \liminf_{n\to\infty} n^{r/s_r} V_{n,r}(\mu) \leq \limsup_{n\to\infty} n^{r/s_r} V_{n,r}(\mu) < \infty,$$

where s_r is the unique number given by the following equation

$$\sum_{i=1}^{N} (p_i c_i^r)^{\frac{s_r}{s_r+r}} = 1. \tag{3.5.1}$$

In particular, the quantization dimension of μ of order r exists and equals s_r.

3.5.2. Measures absolutely continuous with respect to self-similar measures. Let μ be a self-similar measure with respect to an IFS satisfying the open set condition. Let $\nu, \mu \in \mathcal{M}$ with $\nu \ll \mu$. We first prove that $\overline{D}_r(\nu) = \overline{D}_r(\mu)$ if the Radon-Nikodym derivative $h := d\nu/d\mu$ is continuous. Then we show that there exists a Borel probability measure $\nu \ll \mu$ such that $\overline{D}_r(\nu) < \overline{D}_r(\mu)$. In the last subsection, we will prove an analogue of [**9**, Theorem 6.2].

THEOREM 3.5.1. *Let μ denote the self-similar measure on $\mathbb{R}^d$ with respect to the IFS $\{f_1, f_2, \cdots, f_N\}$ satisfying strong separation condition. Let $\nu \ll \mu$ be a Borel probability measure with h continuous on E. Then for s_r defined in (3.5.1),*

$$0 < \underline{Q}_r^{s_r}(\nu) \leq \overline{Q}_r^{s_r}(\nu) < \infty.$$

In particular, $D_r(\nu)$ exists and equals s_r.

PROOF. Recall that $B_k := \{x \in E : h(x) > k\}$. Since $\nu \ll \mu$ and $\nu\left(\mathbb{R}^d\right) = 1$, we know that $\mu(B_0) > 0$. Hence for some $m \in \mathbb{N}$, we have $\mu\left(B_{1/m}\right) > 0$. Choose an arbitrary point $x \in B_{1/m}$. By the continuity of h, there exists a neighbourhood U of x such that $h(x) > 1/m$ for all $x \in U$. Note that U contains a basic set I of some order k_0. We consider again the conditional probability $\nu_I := \nu(\cdot|I)$ and $\mu_I := \mu(\cdot|I)$. By self-similarity we easily deduce that $D_r(\mu_I) = D_r(\mu) =: s$ and

$$0 < \underline{Q}_r^s(\mu_I) = C_\sigma^r \underline{Q}_r^s(\mu) < \infty,$$

where σ satisfies that $I = f_\sigma(E)$. Note that $\nu_I \geq \frac{\nu(I)}{m\mu(I)}\mu_I =: C\mu_I$, we have $Q_r^s(\nu_I) \geq CQ_r^s(\mu_I) > 0$. This implies that $\underline{Q}_r^s(\nu) > 0$ since $\nu \geq \nu(I)\nu_I$. On the other hand, since $h(x)$ is continuous, we know that for some $M > 0, h(x) \leq M$ for all $x \in E$. Hence

$$\overline{Q}_r^s(\nu) \leq M\overline{Q}_r^s(\mu) < \infty.$$

This finishes the proof of this theorem. □

REMARK 3.5.2. From the proof of the above proposition, we actually know a little more: if h is bounded from above on E and bounded away from zero on some cylinder set, then we have $D_r(\nu)$ exists and equals $D_r(\mu)$. However, this can be further extended by using Proposition 3.2.4. In fact, the following holds.

COROLLARY 3.5.3. *Suppose that h is bounded away from zero on some cylinder F and $\underline{R}(h) \geq r/D_r(\mu)$. Then $D_r(\nu)$ exists and $D_r(\nu) = D_r(\mu)$.*

PROOF. This follows immediately from Theorem 3.2.4 and Remark 3.5.2. □

COROLLARY 3.5.4. *Let μ be the homogeneous Cantor measure on $([0,1], (2), (c_k))$ with $\sup_{k\geq 1} c_k \leq 1/4$. Let ν be a Borel probability measure with $\nu \ll \mu$ and h continuous. Then $\overline{D}_r(\nu) = \overline{D}_r(\mu)$.*

PROOF. As in the proof of Theorem 3.5.1, there exists a cylinder I of some order k such that $h(x) \geq m > 0$ for all $x \in I$ and for some $m > 0$. It follows that $\overline{D}_r(\nu(\cdot|I)) \geq \overline{D}_r(\mu(\cdot|I))$. On the other hand, we have

$$\overline{D}_r(\mu) = \max_{F\in\mathcal{D}_k} \overline{D}_r(\mu(\cdot|F)) = \overline{D}_r(\mu(\cdot|I)).$$

This implies that $\overline{D}_r(\nu(\cdot|I)) \geq \overline{D}_r(\mu)$. Hence $\overline{D}_r(\nu) \geq \overline{D}_r(\mu)$. Since h is bounded from above, the reverse inequality is clear. □

Now we use the following example to show that by Corollary 3.5.3 we can determine the quantization dimension for many more measures which are absolutely continuous with respect to μ_s and with h unbounded.

EXAMPLE 3.5.5. Let E be a Cantor set $[0,1]$ determined by IFS $\{f_1, f_2\}$ given by

$$f_1(x) = \frac{1}{2}x, \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}.$$

Let μ be the self-similar measure on E associated with the probability vector $(1/3, 2/3)$. Setting

$$c := \left(\frac{2}{3}\sum_{k=1}^{\infty} \log k \cdot 3^{-(k-1)}\right),$$

We define a measurable function by

$$h(x) = \begin{cases} 1 & \text{if } x \in [1] \\ (\log k)/c & \text{if } x \in [0,0,\cdots 0,1], |(0,0,\cdots,0,1)| = k \end{cases},$$

where $|\sigma|$denotes the length of the word σ and $[\sigma_1,\cdots,\sigma_k]$ denotes the cylinder of order k defined by

$$[\sigma_1,\cdots,\sigma_k] := f_{\sigma_1} \circ f_{\sigma_2} \cdots \circ f_{\sigma_k}([0,1]).$$

Define a Borel probability measure by

$$\nu(A) = \int_A h(x)\, d\mu(x), \quad A \in \mathcal{B}.$$

For large enough k, we have

$$\int_{B_k} h(x)\, d\mu \le \sum_{j=[e^{kc}]}^{\infty} j/c \cdot \frac{2}{3} \cdot 3^{-j} \le \frac{2}{3} \sum_{j=[e^{kc}]}^{\infty} 2^{-j} := C_1 2^{-e^{kc}}.$$

It follows that

$$\overline{R}(h) = \infty > \frac{r}{D_r(\mu)}.$$

By Theorem 3.5.3, we have $\overline{D}_r(\nu) \le D_r(\mu)$. Finally, it is clear that h is bounded away from zero, so observing Theorem 3.5.3, we have $\underline{D}_r(\nu) \ge D_r(\mu)$. This implies $D_r(\nu)$ exists and equals $D_r(\mu)$.

By Proposition 3.2.2 (3), in general, the continuity of h doesnot imply the equality of the quantization dimensions. For self-similar measures, one may further ask whether the continuity of h in Proposition 3.5.1 can be dropped. In the following we show that in general this is not the case. Let μ_s denote the self-similar measure determined by the probability vector $(c_1^s,\cdots,c_N^s)$ and $s = \dim_H E$. We have

THEOREM 3.5.6. *Let μ be a self-similar measure on E with $\mu \ne \mu_s$. There exist Borel probability measures $\nu \ll \mu$ such that $\overline{D}_r(\nu) < D_r(\mu)$ for all $r > 1$.*

PROOF. By Lemma 2.4.2 and [**9**, Lemma 14.16], for $\mu \ne \mu_s$ we have

$$\dim_P \mu \le D_1(\mu) < D_r(\mu), \quad r > 1.$$

Thus according to Lemma 3.2.9 there exists a Borel set A such that $\mu(A) > 0$ and $\overline{\dim}_B A < D_r(\mu)$. Let $\nu = \mu_A = \mu(\cdot|A)$. We have

$$\overline{D}_r(\mu_A) \le \overline{\dim}_B^* \mu_A = \overline{\dim}_B A < D_r(\mu).$$

□

3.5.3. Quantization coefficients and the Radon-Nikodym derivative.

The remaining part of this section is aiming at establishing the relationship between the Radon-Nikodym derivative and the upper and lower quantization coefficient of ν. We again denote by μ_s the self-similar measure determined by the probability vector $(c_1^s, \cdots, c_N^s)$, $s = \dim_H E$. Set

$$c_{\max} := \max_{1\leq i\leq m} c_i, \quad c_{\min} := \min_{1\leq i\leq m} c_i.$$

We first prove the following proposition for μ_s, which is analogous to the second step of the proof in [**9**, Theorem 6.2]. This will be used to prove Theorem 3.5.11.

PROPOSITION 3.5.7. *Let $\sigma^i \in \Omega_k$, $1 \leq i \leq m \leq N^k$, $k \geq 1$ and $\nu = \sum_{i=1}^m s_i \mu_s \circ f_{\sigma^i}^{-1}$, where $s_i \geq 0$, $\sum_{i=1}^m s_i = 1$. Let h denote the Radon-Nikodym derivative $d\nu/d\mu_s$. Then we have*

$$\|h\|_{\frac{s}{s+r},\mu_s} \underline{Q}_{n,r}^s(\mu_s) < \underline{Q}_{n,r}^s(\nu) \leq \overline{Q}_{n,r}^s(\nu) \leq \|h\|_{\frac{s}{s+r},\mu_s} \overline{Q}_{n,r}^s(\mu_s).$$

PROOF. It is easy to see that $\mu_s(\cdot|E_{\sigma^i}) = \mu_s \circ f_{\sigma^i}^{-1}$ and

$$h := \frac{d\nu}{d\mu_s} = \sum_{i=1}^m s_i \mu_s^{-1}(E_{\sigma^i}) 1_{E_{\sigma^i}} = \sum_{i=1}^m s_i c_{\sigma^i}^{-s} 1_{E_{\sigma^i}}.$$

We set

$$t_i = \frac{s_i^{\frac{s}{s+r}} c_{\sigma^i}^{\frac{rs}{s+r}}}{\sum_{i=1}^m s_i^{\frac{s}{s+r}} c_{\sigma^i}^{\frac{rs}{s+r}}}, \; n_i = [nt_i].$$

By [**9**, Lemma 4.14 (b), Lemma 3.2], we have

$$\begin{aligned}
\limsup_{n\to\infty} n^{r/s} V_{n,r}(\nu) &\leq \limsup_{n\to\infty} \sum_{i=1}^m s_i n^{r/s} V_{n_i,r}\left(\mu_s \circ f_{\sigma^i}^{-1}\right) \\
&\leq \sum_{i=1}^m s_i \limsup_{n\to\infty} n^{r/s} V_{n_i,r}\left(\mu_s \circ f_{\sigma^i}^{-1}\right) \\
&\leq \sum_{i=1}^m s_i t_i^{-r/s} c_{\sigma^i}^r \limsup_{n\to\infty} (n_i+1)^{r/s} V_{n_i,r}(\mu_s) \\
&\leq \limsup_{n\to\infty} n^{r/s} V_{n,r}(\mu_s) \sum_{i=1}^m s_i t_i^{-r/s} c_{\sigma^i}^r \\
&= \limsup_{n\to\infty} n^{r/s} V_{n,r}(\mu_s) \left(\int h^{\frac{s}{s+r}}(x)\, d\mu_s(x)\right)^{\frac{s+r}{s}} \\
&= \|h\|_{\frac{s}{s+r},\mu_s} \overline{Q}_r^s(\mu_s). \qquad (3.5.2)
\end{aligned}$$

Next we show the other side. By the strong separation condition, there exists a constant $\alpha > 0$ such that

$$\min_{i\neq j} \operatorname{dist}(E_i, E_j) \geq \alpha \operatorname{diam}(E_i), \quad i \neq j.$$

Inductively, for two basic sets $E_{\sigma^i}, E_{\sigma^j}$ of order k which are descendents of the same basic set of order $k-1$, we have

$$\operatorname{dist}(E_{\sigma^i}, E_{\sigma^j}) \geq \alpha c_{\min}^k \operatorname{diam}(E) =: \delta, \quad i \neq j.$$

By estimating the volume we know that there exists a constant $l > 0$ such that E_{σ^i} intersects at most l disjoint balls of radius $\delta/8$. Thus we denote by γ_i the set of the centers of such balls and set $\gamma = \cup_i \gamma_i$. We denote by $\mathcal{F}_{ki}$ the collection of basic sets of order k which are descendents of the same basic sets of order $k-1$ as E_{σ^i}, then for all $x \in E_{\sigma^i}$ we have

$$\min_{a\in\gamma_i} |x-a| \leq \frac{\delta}{4} \leq \frac{1}{4} \min_{F\in\mathcal{F}_{ki}} \operatorname{dist}(E_{\sigma^i}, F).$$

Let A_δ denote the δ-neighbourhood of A. For $\beta \in C_{n,r}(\nu)$. We set

$$\beta_i = \beta \cap (E_{\sigma^i})_{\delta/4}, \; n_i = \operatorname{card}(\beta_i).$$

Note that if $E_{\sigma^i}, E_{\sigma^j}$ are descendents of different basic set of order $k-1$, then

$$\operatorname{dist}(E_{\sigma^j}, E_{\sigma^i}) \geq \alpha c_{\max}^{-1} \operatorname{diam}(E_{\sigma^i}) \geq \delta.$$

This implies that for $x \in E_{\sigma^i}$,

$$\min_{b\in\beta\cup\gamma} |x-b| = \min_{b\in\beta_i\cup\gamma_i} |x-b|.$$

It follows that

$$\begin{aligned} V_{n,r}(\nu) &= \int \min_{b\in\beta} |x-b|^r \, d\nu(x) = \sum_{i=1}^m s_i \int_{E_{\sigma^i}} \min_{b\in\beta} |x-b|^r \, c_{\sigma^i}^{-s} d\mu_s(x) \\ &\geq \sum_{i=1}^m s_i c_{\sigma^i}^{-s} \int_{E_{\sigma^i}} \min_{b\in\beta\cup\gamma} |x-b|^r \, d\mu_s(x) \\ &= \sum_{i=1}^m s_i c_{\sigma^i}^{-s} \int_{E_{\sigma^i}} \min_{b\in\beta_i\cup\gamma_i} |x-b|^r \, d\mu_s(x) \\ &\geq \sum_{i=1}^m s_i c_{\sigma^i}^r V_{n_i+l,r}(\mu_s). \end{aligned} \tag{3.5.3}$$

Take a subsequence (n_k) of (n), we still denote it by (n), such that

$$\underline{Q}_r^s(\nu) = \lim_{n\to\infty} n^{r/s} V_{n,r}(\nu).$$

We then take a further subsequence (n_{kj}) of (n_k), still denoted by (n) such that

$$\lim_{n\to\infty}\frac{n_i}{n}=: v_i \text{ exists, } \ 1\le i\le m.$$

Since $\sum_{i=1}^m n_i \le n$, we have $\sum_{i=1}^m v_i \le 1$. Furthermore, $v_i > 0$, $1 \le i \le m$. Otherwise (3.5.3) yields that

$$\limsup_{n\to\infty} n^{r/s}V_{n,r}(\nu)\ge s_i c_{\sigma^i}^r \limsup_{n\to\infty} n^{r/s}V_{n_i+l,r}(\mu_s)\ge v_i^{-r/s}\underline{Q}_r^s(\mu_s)=\infty.$$

Now by (3.5.3) and [**9**, Lemma 6.8], we have for the subsequence (n_{kj}) which we still denote by (n),

$$\begin{aligned}
\underline{Q}_r^s(\nu) &\ge \limsup_{n\to\infty} n^{r/s}\sum_{i=1}^m s_i c_{\sigma^i}^r V_{n_i+l,r}(\mu_s)\\
&\ge \sum_{i=1}^m s_i c_{\sigma^i}^r \liminf_{n\to\infty} n^{r/s}V_{n_i+l,r}(\mu_s)\\
&\ge \sum_{i=1}^m s_i c_{\sigma^i}^r v_i^{-r/s} \liminf_{n\to\infty} n_i^{r/s}V_{n_i+l,r}(\mu_s)\\
&\ge \sum_{i=1}^m s_i c_{\sigma^i}^r v_i^{-r/s} \liminf_{n\to\infty} n^{r/s}V_{n,r}(\mu_s)\\
&\ge \left(\sum_{i=1}^m s_i^{\frac{s}{s+r}} c_{\sigma^i}^{\frac{rs}{s+r}}\right)^{\frac{r+s}{s}} \liminf_{n\to\infty} n^{r/s}V_{n,r}(\mu_s)\\
&= \|h\|_{\frac{s}{r+s},\mu_s}\underline{Q}_r^s(\mu_s).
\end{aligned}$$

This finishes the proof of the proposition. □

Let μ,ν be two Borel probability measures on $\mathbb{R}^d$. For $x\in E$, let $C_l(x)$ denote the cylinder of order l containing x. We define the upper and lower derivatives of ν with respect to μ at a point $x\in\mathbb{R}^d$ by

$$\overline{\delta}(\nu,\mu,x)=\limsup_{l\to\infty}\frac{\nu(C_l(x))}{\mu(C_l(x))},\ \ \underline{\delta}(\nu,\mu,x)=\liminf_{l\to\infty}\frac{\nu(C_l(x))}{\mu(C_l(x))}.$$

If the upper and lower derivative coincide, then we call the common value the derivative of ν with respect to μ and denote it by $\delta(\nu,\mu,x)$. The following lemma is an analogue of [**18**, Lemma 2.13].

LEMMA 3.5.8. *Let ν,μ_s be as in Proposition 3.5.7. Then for $A\subset\mathbb{R}^d$ we have*

(a) *if $\underline{\delta}(\nu,\mu_s,x)\le t$ for all $x\in A$, then $\nu(A)\le t\mu_s(A)$.*
(b) *if $\overline{\delta}(\nu,\mu_s,x)\ge t$ for all $x\in A$, then $\nu(A)\ge t\mu_s(A)$.*

PROOF. For $A \subset \mathbb{R}^d$ and $\epsilon > 0$, according to the definition of μ, we can always choose a countable class $\{C_i\}$ of mutually disjoint basic sets such that $C_i \cap A \neq \emptyset$ and

$$\sum_i \mu_s(C_i) \le \mu_s(A) + \frac{\epsilon}{2}.$$

By the hypothesis of (a), for each $x \in A$ and for i sufficiently large ,

$$\nu(C_i(x)) \le (t+\epsilon)\,\mu_s(C_i(x)).$$

If for some C_i we already have the above inequality we then leave it as it is, otherwise we consider the basic subsets of C_i of next order which intersects A. Thus we can choose countably many basic sets $\widetilde{C}_i, i \ge 1$ such that $A \subset \bigcup_i \widetilde{C}_i$ and $\nu\left(\widetilde{C}_i\right) \le (t+\epsilon)\,\mu_s\left(\widetilde{C}_i\right)$. It follows that

$$\begin{aligned} \nu(A) &\le \sum_i \nu\left(\widetilde{C}_i\right) \le (t+\epsilon)\sum_i \mu_s\left(\widetilde{C}_i\right) \le (t+\epsilon)\sum_j \mu_s(C_i) \\ &\le (t+\epsilon)\left(\mu_s(A) + \frac{\epsilon}{2}\right). \end{aligned}$$

By the arbitrariness of ϵ we have $\nu(A) \le t\mu_s(A)$. Similarly we may show (b). $\square$

LEMMA 3.5.9. *Let $C_i, 1 \le i \le N^k$ be the basic sets of order k and let $\nu \ll \mu_s$. Define $\nu_k := \sum_{i=1}^{N^k} \nu(C_i)\,\mu_s \circ f_i^{-1}$. Then $\nu_k \ll \mu_s$ and*

$$\lim_{k\to\infty} \frac{d\nu_k}{d\mu_s} = \frac{d\nu}{d\mu_s} \;\; \mu - a.e..$$

PROOF. It follows by Lemma 3.5.8 and an analogous argument as in the proof of [**18**, Theorem 2.12]. $\square$

LEMMA 3.5.10. *For $\epsilon > 0$, let $n_2 = \left[(n\epsilon)^{1/s}\right]^s$. Then there exists a constant $C > 0$ such that for large n we may choose a set $\zeta \subset \mathbb{R}^d$ with $\operatorname{card}(\zeta) \le n_2$ and*

$$\min_{a\in\zeta} |x-a| \le C n_2^{-1/s}, \quad x \in E.$$

PROOF. Let $N_t(E)$ denote the largest number of mutually disjoint balls centered in E and of radii t. Then by the s-regularity of μ_s, there exists $C_1, C_2 > 0$ such that

$$N_t(E)\,C_1 t^s \le 1 \le N_t(E)\,C_2 2^s t^s.$$

This implies that $N_t(E) \le C_1^{-1} t^{-s}$. For $\epsilon > 0$ and large enough n, we choose t such that

$$C_1^{-1} t^{-s} \le n_2 \le 2C_1^{-1} t^{-s}.$$

We denote by γ the centers of the $N_t(E)$ mutually disjoint balls. Then for all $x \in E$, we have

$$\min_{a\in\gamma} \|x-a\| \le 2t \le 2^{1+1/s} C_1^{-1/s} n_2^{-1/s}.$$

The lemma follows by taking $C = 2^{1+1/s} C_1^{-1/s}$. □

The following theorem establishes the relationship between the upper and lower quantization coefficient of μ_s and that of $\nu \ll \mu_s$. Although our purpose here is to establish an analogue of the finite-dimensional Lebesgue case, this theorem extends part of a result of PÖTZELBERGER (cf. [**23**, Theorem 3]). In there, the author states an equality regarding the quantization coefficient of a general self-similar measure μ and that of a probability $\nu \ll \mu$ in case that the vector $(\log p_1 c_1^r, \log p_2 c_2^r, \cdots, \log p_N c_N^r)$ is non-arithmetic. If $\mu = \mu_s$, then the equality is given in terms of the corresponding Radon-Nikodym derivative. We will prove an inequality for $\mu = \mu_s$ in general, allowing us to drop the non-arithmetic condition.

THEOREM 3.5.11. *Let E denote the self-similar set on $\mathbb{R}^d$ with respect to the $\{f_i\}_1^N$ satisfying the strong separation condition. Let ν be a Borel probability measure with $\nu \ll \mu_s$. Let $h := d\nu/d\mu$. Then*

$$\|h\|_{\frac{s}{s+r},\mu_s} \underline{Q}_{n,r}^s(\mu_s) < \underline{Q}_{n,r}^s(\nu) \leq \overline{Q}_{n,r}^s(\nu) \leq \|h\|_{\frac{s}{s+r},\mu_s} \overline{Q}_{n,r}^s(\mu_s). \tag{3.5.4}$$

PROOF. Let $C_{ki}, 1 \leq i \leq N^k$ denote the basic sets of order k. Define

$$\nu_k := \sum_{i=1}^{N^k} \nu(C_{ki})\, \mu_s \circ f_i^{-1}.$$

Then we have $d\nu_k/d\mu_s = h_k$. By Lemma 3.5.9, we have $h_k(x) \to h(x)$ μ_s-a.e. . Using Schéffe's lemma we have $\|h_k - h\|_{1,\mu_s} \to 0$ as $k \to \infty$. Define

$$n_1 = [n(1-\epsilon)],\ n_2 = \left[(n\epsilon)^{1/s}\right]^s.$$

Let $\alpha(n_1) \in C_{n_1,r}(\nu_k)$ and $\delta(n,k,\epsilon) = \alpha(n_1) \cup \zeta$, where ζ is chosen as in Lemma 3.5.10. Then clearly $\operatorname{card}(\delta(n,k,\epsilon)) \leq n$ and

$$\begin{aligned}
\Delta(n,k,\epsilon) &:= n^{r/s} \left| \int \min_{a\in\delta(n,k,\epsilon)} |x-a|^r \, d\nu_k(x) - \int \min_{a\in\delta(n,k,\epsilon)} |x-a|^r \, d\nu(x) \right| \\
&\leq n^{r/s} \int \min_{a\in\delta(n,k,\epsilon)} |x-a|^r \, |h_k(x) - h(x)| \, d\mu_s(x) \\
&\leq c^r \epsilon^{-r/s} \|h_k - h\|_{1,\mu_s}.
\end{aligned}$$

It follows that

$$\begin{aligned}\overline{Q}^{s}_{n,r}(\nu) &\leq \limsup_{n\to\infty} n^{r/s}\int \min_{a\in\delta(n,k,\epsilon)} |x-a|^{r}\, d\nu_{k}(x)+\Delta(n,k,\epsilon)\\ &\leq \limsup_{n\to\infty} n^{r/s}\int \min_{a\in\alpha} |x-a|^{r}\, d\nu_{k}(x)+\Delta(n,k,\epsilon)\\ &= (1-\epsilon)^{-r/s}\limsup_{n\to\infty} n_{1}^{r/s} V_{n_1,r}(\nu_k)+\Delta(n,k,\epsilon)\\ &\leq (1-\epsilon)^{-r/s}\left\|h_k\right\|_{\frac{s}{s+r},\mu_s}\ \overline{Q}^{s}_{r}(\mu_s)+2^{-r}\epsilon^{-r/s}\left\|h_k-h\right\|_{1,\mu_s}.\end{aligned}$$

The last inequality in (3.5.4) follows by letting k tend to infinity and then let ϵ go to zero. For the other side of (3.5.4), it suffices to use Lemma 3.5.7 and follow the proof of [**9**, Theorem 6.2]. □

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