Morita Equivalence for Unary Varieties

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## Contents

Preface v

1 Introduction 1
   1.1 Universal Algebra and Varieties 2
   1.2 Lawvere Theories 4
   1.3 Varietal Generators 5
   1.4 Matrix Power and u-Modification 7

2 The Matrix Power 11
   2.1 The Variety $\mathcal{V}^{(n)}$ 12
   2.1.1 $\mathcal{V}^{(n)}$ is the Matrix Power of $\mathcal{V}$ 13
   2.2 An Alternative Description of the Matrix Power: $\mathcal{V}^{\{n\}}$ 15
   2.2.1 $\mathcal{V}^{\{n\}}$ is the Matrix Power of $\mathcal{V}$ 16

3 The Matrix Powers of Set 19
   3.1 $\mathbf{n-Set}_1$ 20
   3.1.1 The Equivalence of $\mathbf{n-Set}_1$ to $\mathbf{Set}$ 22
   3.2 $\mathbf{n-Set}_2$ 24
   3.2.1 The Equivalence of $\mathbf{n-Set}_2$ to $\mathbf{Set}$ 25
   3.3 $\mathbf{n-Set}_3$ 27
   3.3.1 The Equivalence of $\mathbf{n-Set}_3$ to $\mathbf{Set}$ 28
   3.4 $\mathbf{n-Set}_4$ 30
   3.4.1 The Equivalence of $\mathbf{n-Set}_4$ to $\mathbf{Set}$ 31
CONTENTS

4  u-Modification for Unary Varieties 33
   4.1 Varietal Generators and Invertible Unary Operations . . . . . . . . 34
   4.2 Clifford Monoids and $\mathcal{V}(M,u)$ . . . . . . . . . . . . . . . 35
      4.2.1 $\mathcal{V}(M,u)$ is the $u$-Modification of $M\text{Act}^{[u]}$ . . . 38
   4.3 Equivalence for $M$-Acts . . . . . . . . . . . . . . . . . . . . . . . 44

5  Morita Equivalence for Boolean Algebras 47
   5.1 The Matrix Powers of $\text{BOOL}$ . . . . . . . . . . . . . . . . . . . 47
   5.2 The $u$-Modification . . . . . . . . . . . . . . . . . . . . . . . . . . 48
   5.3 The Equivalence of Post Algebras and Boolean Algebras . . . . . . 49
   5.4 Hu’s Primal Algebra Theorem . . . . . . . . . . . . . . . . . . . . . 54

References 57

Index of Symbols 61

Index 63
Preface

The problem of determining all varieties \( W \) (categorically) equivalent to a given variety \( V \) has been addressed in numerous ways. Isbell first posed the problem in the early 1970’s in [15]. There are several solutions for special cases. The most prominent example is classical Morita theory which describes for a given ring \( R \) all rings \( S \) such that the varieties of \( R \)-modules and \( S \)-modules are equivalent (see e.g. in [16]). At first sight, this seems to be a more restricted problem. But since any variety equivalent to a category of modules is itself a variety of \( S \)-modules for a suitable ring \( S \) (cf. e.g. [12]), classical Morita theory characterizes all varieties equivalent to the variety of \( R \)-modules for a given ring \( R \).

Classical Morita theory is based on certain generators in the category of \( R \)-modules. This has been generalized. General Morita theory, as described in [27], is based on the characterization of varietal generators. This characterization gives rise to two constructions: the \( n \)-th matrix power \( V^{(n)} \) of a variety \( V \) for natural numbers \( n \geq 2 \) and the \( u \)-modification \( V(u) \) of \( V \) for an idempotent and invertible term \( u \) in \( V \). The varieties equivalent to a given variety \( V \) are, up to concrete isomorphism, precisely the varieties \( V^{(n)}(u) \) for some \( n \in \mathbb{N} \), \( n \geq 1 \), and some idempotent and invertible term \( u \) for \( V^{(n)} \). These constructions are already described in R. N. McKenzie’s paper [25], which is based on results from J. J. Dukarm [9]. The construction of the matrix power was already known to F. E. J. Linton during the 1960’s according to R. N. McKenzie (cf. [25]).

The role played by the varietal generators and how the concepts of the matrix power and \( u \)-modification arise from them only becomes fully clear in H.-E. Porst’s paper [27]. Using basic results from categorical algebra, as developed mainly by Lawvere, Isbell and Linton, he characterizes all Lawvere theories \( S \) Morita equivalent to a given Lawvere theory \( T \). Lawvere theories \( T \) and \( S \) are called Morita equivalent provided that the categories of their models \( \text{Mod}_T \) and \( \text{Mod}_S \) are equivalent categories. Thus the Lawvere theories correspond to the rings in classical Morita theory.

In this thesis we give new descriptions for the \( n \)-th matrix power \( V^{(n)} \) of a (finitary) variety \( V \) for natural numbers \( n \geq 2 \) and the \( u \)-modification \( V(u) \) of \( V \) for an idempotent and invertible term \( u \) in \( V \). The aim is to simplify the syntax. Especially in the case of the matrix power we have succeeded in giving a very simple characterization by adding just one binary operation to the original
operations of a given variety $V$ as well as adding equations to the original ones. The $u$-modification is more elusive and we have to restrict ourselves to the unary case. Again we gain a characterization by adding operations and equations to the originally given ones. Furthermore we treat some special cases like the variety of Boolean algebras as illustrating examples.

The outline of this thesis is the following: The first chapter provides a brief summary of the fundamental concepts needed in the following chapters. After sketching both Birkhoff’s as well as Lawvere’s approach to varieties, we give a characterization of varietal generators. Finally, the notions of $n$-th matrix power $V^n$ of a variety $V$ for natural numbers $n \geq 2$ and the $u$-modification $V(u)$ of $V$ for an idempotent and invertible term $u$ in $V$ are introduced.

After the first chapter we leave it to the reader to decide in which order to read Chapter 2 and Chapter 3. Both chapters can be read on their own. Whereas Chapter 2 describes the matrix power in the general case, Chapter 3 contains the matrix powers of the category $\textbf{Set}$ of sets and maps.

In the second chapter we characterize the $n$-th matrix power $V^n$ of a variety $V$ for all natural numbers $n \geq 2$. We give two different descriptions, both times by adding operations and equations to the existing syntax. After explaining how these operations work for a given algebra, we show that they indeed completely characterize the matrix powers. The first description $V^{(n)}$ is more elegant whereas the second characterization $V^{\{n\}}$ is easier to work with in the case of unary varieties in Chapter 4. The results in the second chapter are a generalization of the results in Chapter 3.

Chapter 3 contains four different descriptions of the matrix powers $\textbf{Set}^n$ of $\textbf{Set}$ for all natural numbers $n \geq 2$. As in the second chapter we add operations and equations to the existing ones. The characterizations are generalizations of special instances of the matrix powers of $\textbf{Set}$. The first two, $n\text{-Set}_1$ and $n\text{-Set}_2$, are direct generalizations of R. Börger’s example in [5], the third, $n\text{-Set}_3$, uses R. N. MacKenzie’s operations taken from Example 2 in [25]. The fourth and most refined solution $n\text{-Set}_4$, is based on a variety constructed by Saade [31]. Börger’s and MacKenzie’s examples just give the $n$-th matrix power $\textbf{Set}^n$ of $\textbf{Set}$ for $n = 2$.

Chapter 4 is dedicated to finding all varieties equivalent to a given unary variety. For this it is sufficient to deal with $M$-acts, since all unary varieties can be described as varieties of $M$-acts. First we determine the varietal generators in $M\textbf{Act}$ and thereby find the idempotent invertible unary operations $u$ needed for the $u$-modification. We do not fully achieve our goal of describing Morita equivalence for all unary varieties since we have to restrict ourselves to the varieties $M\textbf{Act}$ of $M$-acts where $M$ is a Clifford monoid. But for those cases we construct the $u$-modification of the matrix powers $M\textbf{Act}^{(n)}$ of $M\textbf{Act}$ for all suitable idempotent and invertible terms $u$ in $M\textbf{Act}^{(n)}$. We define a variety $V(M, u)$ via explicitly given operations and equations and show that it is indeed the $u$-modification of the $n$-th matrix powers of $M\textbf{Act}$ if $M$ is a Clifford Monoid. We finish the fourth chapter by answering a similar question
to the one Morita treated: Given a monoid $M$, for which monoids $N$ is the variety $N\text{Act}$ equivalent to $M\text{Act}$? Knauer [19] and Banachschewsky [4] both independently found an answer to this question in 1972. We give a short proof of their result using categorical algebra.

In the fifth and last chapter we prove that the varieties equivalent to the variety $\text{BOOL}$ of Boolean algebras are (up to concrete isomorphism) precisely the varieties $\mathcal{P}_n$ of Post algebras of order $n$ for $n \in \mathbb{N}$, $n \geq 2$. This is done by writing down the constructions of the matrix power and the $u$-modification explicitly, thus giving a complete characterization through operations and equations. It turns out that we get exactly the description of Post algebras by axioms which were given by T. Traczyk in [32]. Of course, we also show the reverse direction, that every Post algebra of order $n+1$ ($n \in \mathbb{N}$) is isomorphic to the $u$-modification of an $n$-th matrix power of a Boolean algebra. The result is already contained in [27] but we make the construction of the matrix power and $u$-modification of the variety $\text{BOOL}$ of Boolean algebras and some other parts of the proof more explicit than they are given by Porst. We finish the chapter by shortly pointing out the connection to Hu’s primal algebra theorem.

### Notation

Some symbols describe different operations in different sections, but only when the operations have the same underlying principle. We refrained from using indices in these cases since we had to use more than enough already.

The categorical algebra notations are mostly from [27, 29, 28]. Other categorical notations have been taken from [1].

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Chapter 1

Introduction

In this first chapter we introduce the fundamental concepts which are used throughout this thesis. We shortly sketch traditional universal algebra and finitary varieties in Birkhoff’s sense. It is important to have at least an intuitive understanding of these concepts because they will be needed constantly. We continue by describing the more modern approach to varieties by Lawvere’s theories [20] and the concept of varietal generators. These concepts are important for the description of equivalence between varieties. Knowing the varietal generators of a given variety $V$ provides at least theoretical knowledge of all varieties equivalent to $V$. These results mostly go back to Lawvere [20], Isbell [15] and Linton [21].

Most important of all we introduce the $n$-th matrix power $V^{[n]}$ of a variety $V$ for $n \in \mathbb{N}$ and the $u$-modification $V(u)$ of $V$ for an idempotent and invertible term $u$ in $V$. They are the fundamental concepts of this thesis. The varieties equivalent to a given variety $V$ are, up to concrete isomorphism, precisely the varieties $V^{[n]}(u)$ for some $n \in \mathbb{N}, n \geq 1$, and some idempotent and invertible term $u$ for $V^{[n]}$.

We have taken most definitions and theorems in this chapter from [27, 28, 29]. Porst’s paper [27], in which he makes the $n$-th matrix power construction of a variety and McKenzie’s $u$-modification of a variety [25] more transparent, is the basis for this thesis. The proofs omitted here can be found in [27] (unless mentioned otherwise).

We assume that the reader has an understanding of basic categorical notions. Especially equivalence is of course a fundamental concept for this thesis. Be aware that we use the term in the categorical sense, i.e. what is called categorical equivalence in the terminology of universal algebra. Equivalence in the sense of algebraical texts like [25, 24] is concrete equivalence in category theory. For more information on concrete equivalence we refer to [26]. In general we use categorical language as found in [27]. For more information on category theory we refer to [1, 12] or [22].
1.1 Universal Algebra and Varieties

The concept of varieties goes back to Birkhoff who wrote the first papers on universal algebra in the 1930s. Here we just recall the basic notions. Readers who are not familiar with universal algebra may refer to [24]. A very concise introduction can be found in [17].

Let \(A\) be a set and \(n\) a natural number. An \(n\)-ary operation on \(A\) is a map from \(A^n\) to \(A\). A signature is a pair \((F, E)\) where \(F\) is a set of operations and \(E\) is a set of equations. An \(F\)-algebra is a pair \((A, \bar{F})\) where \(A\) is a set and \(\bar{F}\) is the set which contains the \(A\)-interpretations \(f^A : A^n \to A\) of the operations in \(F\). It is called an \((F, E)\)-algebra if all equations from \(E\) are satisfied. An \(F\)-homomorphism \(\varphi : A \to B\) between two \(F\)-algebras \(A\) and \(B\) is a map such that for each \(n \in \mathbb{N}\) and each \(f \in F\) the following diagram commutes:

\[
\begin{array}{ccc}
A^n & \xrightarrow{\bar{f}^A} & B^n \\
\downarrow{\varphi^A} & & \downarrow{\bar{f}^B} \\
A & \xrightarrow{\varphi} & B
\end{array}
\]

Now we can define a variety:

**Definition 1.1** Let \((F, E)\) be a signature. By \(\text{Alg}(F, E)\) we denote the category of all \((F, E)\)-algebras and all \(F\)-homomorphisms. A category of the form \(\text{Alg}(F, E)\) is called a variety.

Please note that a variety \(\mathcal{V}\) is always a concrete category via the canonical functor \(|-| : \mathcal{V} \to \text{Set}\) which sends each algebra to its underlying set and each homomorphism to its underlying map.

As an example we define \(M\)-acts and the variety \(\text{Set}\), which we will need in Chapter 4.

**Definition 1.2** Let \(M\) be a monoid. An \(M\)-act is an \((F, E)\)-algebra where the set of operations \(F\) contains one unary operation \(m\) for every \(m \in M\) and the set of equations \(E\) consists of the following equations:

- \(ex = x\) where \(e\) is the unary operation which corresponds to the neutral element of \(M\).
- \(m(nx) = (mn)x\) for all unary operations \(m, n\) and \(mn\) where \(m, n\) and \(mn\) are the unary operations which correspond to the elements \(m, n\) and \(mn\) \(\in M\).

By \(M\text{-Act}\) we denote the variety of all \(M\)-acts \(\text{Alg}(F, E)\).

\(M\)-acts can be defined without making the set \(F\) of operations explicit. In fact the following is the more common definition which is equivalent to the first:
**Definition 1.3** Let $M$ be a monoid. An $M$-act is a set $X$ with a map 

\[ \phi : M \times X \to X, \]

such that 

\[ \phi(e, x) = x \quad \text{and} \quad \phi(m, \phi(n, x)) = \phi(mn, x) \]

for $e, n, m \in M$, $e$ the neutral element, $x \in X$. We write $ax$ instead of $\phi(a, x)$. By $M\text{Act}$ we denote the variety of all $M$-acts.

$M$-acts describe all unary varieties:

**Lemma 1.4** Each unary variety is concretely isomorphic to $M\text{Act}$ for a suitable monoid $M$.

**Proof:** Let $V$ be a unary variety and let $F$ be the set of operations and $E$ the set of equations defining $V$. Let $M$ be the free monoid generated by the elements of $F$ subject to the equations of $E$, where the binary monoid-operation is the composition of the unary operations. Then each term in $V$ corresponds to an element of $M$ and vice versa. Now let $A$ be any $V$-algebra. We can fully determine how $M$ acts on a set by defining how the generating elements act on the set. If we define that the generating elements act on the underlying set of $A$ like the unary operations which they are, it is obvious that the such defined $M$-act and $A$ have the same underlying set and the same clone. Also each $M$-act is a unary algebra defined by the operations in $F$ and satisfying the equations of $E$ and is thus a $V$-algebra. 

Unary varieties can also be characterized by a categorical property:

**Lemma 1.5** A variety is unary if and only if its underlying functor preserves coproducts.

**Proof:** Coproducts in $\text{Set}$ are disjoint unions. Thus the underlying functor of a variety preserves coproducts if and only if the coproduct of a family $(A_i)_{i \in I}$ in the variety has the disjoint union of the family $(A_i)_{i \in I}$ as underlying set.

It is easy to see that this is the case for unary varieties: Let $V$ be a unary variety and let $(A_i)_{i \in I}$ be a family of $V$-algebras. The disjoint union $\biguplus (A_i)_{i \in I}$ is a $V$-algebra if we define the unary operations to act as in the $A_i$, i.e. for an element $a \in (\biguplus (A_i)_{i \in I})$ there exists a $k \in I$ such that $a \in A_k$ and each operation $f$ acts as $f^{A_k}$ on $a$. This is indeed the coproduct of the family $(A_i)_{i \in I}$. Let $C$ be any $V$-algebra and $(h_i : A_i \to C)_{i \in I}$ a family of homomorphisms. Then there exists exactly one map $h$ such that the following diagram commutes:

\[
\begin{array}{ccc}
A_i & \xrightarrow{\mu_j} & \biguplus (A_i)_{i \in I} \\
\downarrow{h_i} & & \downarrow{h} \\
C & & \\
\end{array}
\]
For each element \( a \in \bigsqcup (A_i)_{i \in I} \) there exists a \( k \in I \) such that \( a \in A_k \), thus we must define \( h(a) = h_k(a) \). The map \( h \) is obviously a homomorphism, since for any operation \( f \) the image \( f(a) \) lies again in \( A_k \).

Now let \( \mathcal{V} \) be a variety which has an \( r \)-ary operation \( g \) for an \( r \geq 2 \) which cannot be reduced to a unary one by equations. Let \( (A_i)_{i \in I} \) be a family of \( \mathcal{V} \)-algebras. Assume that the coproduct \( \bigsqcup (A_i)_{i \in I} \) has the disjoint union \( \bigsqcup (A_i)_{i \in I} \) as underlying set. Now let \( C \) be the term algebra over \( \bigsqcup (A_i)_{i \in I} \) modulo the congruence generated by all equations in \( \mathcal{V} \), i.e. the free algebra generated by \( \bigsqcup (A_i)_{i \in I} \). Let \( (h_i : A_i \rightarrow C)_{i \in I} \) be the family of homomorphisms which inserts the generators into \( C \). As above there exists exactly one map \( h \) such that the following diagram commutes:

There exist elements \( a_{k_1}, \ldots, a_{k_r} \) of the coproduct \( \bigsqcup (A_i)_{i \in I} \), where \( a_{k_j} \in A_{k_j} \), such that \( g^C(a_{k_1}, \ldots, a_{k_r}) \) is a term in \( C \) and not one of the generating elements from \( \bigsqcup (A_i)_{i \in I} \). Otherwise there would be equations reducing \( g \) to a unary operation contrary to the assumption. But \( g^\bigsqcup (A_i)_{i \in I}(a_{k_1}, \ldots, a_{k_r}) \) must be an element of \( \bigsqcup (A_i)_{i \in I} \). Thus

\[
h(g^\bigsqcup (A_i)_{i \in I}(a_{k_1}, \ldots, a_{k_r})) \neq g^C(h(a_{k_1}), \ldots, h(a_{k_r})).
\]

Hence \( h \) is not a homomorphism and therefore the coproduct \( \bigsqcup (A_i)_{i \in I} \) cannot have the disjoint union \( \bigsqcup (A_i)_{i \in I} \) as underlying set.

\[\square\]

While Birkhoff’s definition of a variety is certainly the most intuitive one there is a different more modern approach via categories and functors.

### 1.2 Lawvere Theories

Lawvere’s approach to varieties was first described in his Ph.D. thesis [20]. It is fundamental for Porst’s description of Morita equivalence. The concept of a Lawvere theory arises from extending the clone of a variety into a category such that clone composition becomes composition in this category. For further information about theories see e.g. [6]. An introduction can also be found in [27].

**Definitions 1.6** A (Lawvere) theory \( \mathcal{T} \) is a category of countably many objects \( T_0, T_1, T_2, \ldots \) together with a distinguished family of morphisms

\[
(\pi^n_i : T_n \rightarrow T_1)_{1 \leq i \leq n}
\]
1.3 Varietal Generators

for each $n \in \mathbb{N}$ such that $T_n$ is an $n$-fold power of $T_1$.

A theory morphism is a functor $\Phi : \mathbb{S} \to \mathbb{T}$ between theories which preserves the distinguished product families $(\pi^n_i : T_n \to T_1)_{1 \leq i \leq n}$.

A $\mathbb{T}$-model (or $\mathbb{T}$-algebra) is a functor from $\mathbb{T}$ into the category $\textbf{Set}$ of sets and mappings which preserves all finite products.

For a given Lawvere theory $\mathbb{T}$, the $\mathbb{T}$-models and natural transformations between them form the category $\textbf{Mod}\mathbb{T}$. Hence the category of all $\mathbb{T}$-models is a full subcategory of the category of all functors from $\mathbb{T}$ to $\textbf{Set}$.

The category $\textbf{Mod}\mathbb{T}$ is a concrete category. It has a canonical underlying functor $U_\mathbb{T} : \textbf{Mod}\mathbb{T} \to \textbf{Set}$ which is evaluation at $T_1$:

$$U_\mathbb{T}(H \xrightarrow{\mu} K) = H(T_1) \xrightarrow{\mu_{T_1}} K(T_1).$$

**Definition 1.7** Lawvere theories $\mathbb{T}$ and $\mathbb{S}$ are called Morita equivalent provided that $\textbf{Mod}\mathbb{T}$ and $\textbf{Mod}\mathbb{S}$ are equivalent categories.

A paradigmatic way to construct a theory is the following: Let $G$ be an object in a category $\mathbb{V}$ which admits all its finite copowers. If we take the dual of the full subcategory of $\mathbb{V}$ spanned by the chosen $n$-fold copowers $nG$ of $G$ for each $n \in \mathbb{N}$ we have a Lawvere theory. We call this theory the Lawvere theory generated by $G$ and we denote it by $\text{Th}_\mathbb{V}(G)$.

Let $\mathbb{V}$ be a variety and $G = F1$ the freely generated algebra on one generator. Then the theory $\text{Th}_\mathbb{V}(F1)$ is called the theory of $\mathbb{V}$ and is denoted by $\text{Th}_\mathbb{V}$.

The connection between Birkhoff’s and Lawvere’s approaches to varieties is given by the fact that every variety $\mathbb{V}$ is concretely equivalent to $\textbf{Mod}\text{Th}_\mathbb{V}$ and conversely.

1.3 Varietal Generators

Varietal generators are crucial for equivalence between varieties. Knowing the varietal generators of a variety $\mathbb{V}$ means at least theoretical knowledge of all varieties equivalent to $\mathbb{V}$ as Theorem 1.10 shows. However it seems that there is no one-to-one correspondence between varietal generators in $\mathbb{V}$ and varieties $\mathbb{W}$ equivalent to $\mathbb{V}$. The results in this section mostly go back to Lawvere [20], Isbell [15] and Linton [21] except for Theorem 1.12 which is due to Porst [27].

**Definition 1.8** An object $G$ in a cocomplete category is called a varietal generator if $G$ is

(i) a regular generator,
(ii) regularly projective,
(iii) finitely presentable.

Remark 1.9 In this thesis we just need varietal generators in varieties. An algebra $G$ in a variety is a varietal generator provided

(i) $G$ is a generator, i.e. each object is a homomorphic image of some copower of $G$,
(ii) $G$ is projective (with respect to surjective homomorphisms),
(iii) $G$ is presentable by finitely many generators and equations.

In each variety the finitely generated free algebras $F_n$ for $n \in \mathbb{N}$ are varietal generators. They may serve as the paradigmatic example here.

Combining the concepts of theories and varietal generators allows the following characterization of equivalent varieties.

Theorem 1.10 For varieties $\mathcal{V}$ and $\mathcal{W}$ the following are equivalent:

(i) $\mathcal{W}$ is equivalent to $\mathcal{V}$.
(ii) $\text{Th}_\mathcal{W}(F_1) \cong \text{Th}_\mathcal{V}(G)$ for a varietal generator $G$ in $\mathcal{V}$.

For a proof see [6].

With a suitable underlying functor the equivalence even becomes concrete. This will be of importance for finding the right underlying set when trying to construct equivalent varieties.

Theorem 1.11 Let $\Phi : \mathcal{W} \rightarrow \mathcal{V}$ be an equivalence between varieties. Then the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{W} & \xrightarrow{\Phi} & \mathcal{V} \\
\downarrow{|-|} & & \downarrow{\text{hom}(G,-)} \\
\text{Set} & & \\
\end{array}
$$

where $G$ is the varietal generator from Theorem 1.10, i.e. $\Phi$ is a concrete equivalence between $(\mathcal{W}, |-|)$ and $(\mathcal{V}, \text{hom}(G,-))$.

For a proof see [6].

We have seen that the knowledge of all varietal generators is fundamental. The following theorem (cf. Porst [27]) characterizes the varietal generators as retracts of the finitely generated free algebras $F_n$. The morphism $u$ in this theorem delivers the operation $u$ needed for the $u$-modification in the next section. We use the same notation for morphisms and operations.
Theorem 1.12 For an object \( G \) in \( \mathcal{V} \) the following are equivalent:

(i) \( G \) is a varietal generator.

(ii) There are retractions

\[
G \xrightarrow{s} F_n \xrightarrow{r} G = \text{id}_G \quad \text{and} \quad F_1 \xrightarrow{p'} mG \xrightarrow{d'} F_1 = \text{id}_{F_1}
\]

with \( n, m \) natural numbers.

(iii) For some natural number \( n \) there exists a morphism \( r : F_n \to G \) such that \((r, G)\) is a coequalizer of \((u, \text{id}_{F_n})\), where \( u \) is an idempotent endomorphism of \( F_n \) for which there exists for some natural number \( m \) a retraction

\[
F_1 \xrightarrow{p} F(nm) \xrightarrow{d} F_1 = \text{id}_{F_1}
\]

such that \( d \circ (mu) \circ p = \text{id}_{F_1} \).

1.4 Matrix Power and \( u \)-Modification

In this section we present two constructions: the \( n \)-th matrix power \( A^{[n]} \) of an algebra \( A \) for natural numbers \( n \geq 2 \) and the \( u \)-modification \( A(u) \) of an algebra \( A \) for an idempotent and invertible term \( u \). They are the fundamental concepts of this thesis. In this section we just give the definitions as they can be found in [27] and the main theorem from [25] which characterizes all varieties equivalent to a given one.

Definition 1.13 Let \( A \) be an algebra in a given variety \( \mathcal{V} \) and \( n \in \mathbb{N}, n \geq 2 \).

The \( n \)-th matrix power \( A^{[n]} \) of \( A \) is the following algebra:

1. The underlying set of \( A^{[n]} \) is \( |A|^n \).

2. The \( r \)-ary operations \(|A^{[n]}|^r = |A|^{nr} \to |A|^n = |A^{[n]}|\) are those maps \( f \) whose composition with the projections yield \( nr \)-ary \( \mathcal{V} \)-operations \( f_i \)

\[
\begin{array}{c}
A^{nr} \\
\xrightarrow{f_i} A^n \\
\xrightarrow{\pi_i} A
\end{array}
\]

By \( \mathcal{V}^{[n]} \) we denote the category of all algebras isomorphic to some \( A^{[n]} \) for \( A \in \mathcal{V} \) and all homomorphisms between them.
\( V[n] \) is in fact a variety (cf. [27]).

In Chapter 2 we will present two new ways to describe the \( n \)-th matrix power \( V[n] \) of a variety \( V \) for \( n \in \mathbb{N} \).

**Definition 1.14** An idempotent morphism \( u : A \to A \) in any category \( V \) admitting finite products of \( A \) is called invertible if there exists some \( m \in \mathbb{N} \) and morphisms \( p : A^m \to A \) and \( d : A \to A^m \) such that

\[
A \xrightarrow{d} A^m \xrightarrow{um} A^m \xrightarrow{p} A = 1_A
\]

The idempotent endomorphisms \( u : Fn \to Fn \) obtained in Theorem 1.12 (iii) correspond exactly to the idempotent and invertible terms \( u \) needed for the \( u \)-modification defined in the following definition. For more details and proofs please refer to [27].

**Definition 1.15** Let \( A \) be an algebra in a given variety \( V \) and \( u \) be an idempotent and invertible term in \( V \). Then the \( u \)-modification \( A(u) \) of \( A \) is the following algebra:

1. The underlying set of \( A(u) \) is \( u^A[\vert A \vert] \).
2. The \( r \)-ary operations \( (u^A[\vert A \vert])^r \to u^A[\vert A \vert] \) are those maps \( f^A_u \) which arise as restrictions to \( (u^A[\vert A \vert])^r \) of maps \( \vert A \vert^r \xrightarrow{f^A} \vert A \vert \xrightarrow{u^A} u^A[\vert A \vert] \) where \( f^A \) is the \( A \)-interpretation of a operation \( f \) in \( V \).

By \( V(u) \) we denote the category of all algebras isomorphic to some \( A(u) \) for \( A \in V \) and all homomorphisms between them.

\( V(u) \) is in fact a variety (cf. [27]).

In Chapter 4 we examine the \( u \)-modification of unary varieties. We show a new way to construct the \( u \)-modification for a special class of unary varieties.

**Remark 1.16** As we have seen in Theorem 1.12 the varietal generators \( G \) in a variety \( V \) correspond to retractions

\[
G \xrightarrow{s} Fn \xrightarrow{r} G = \text{id}_G.
\]

In the case of \( r = \text{id}_{Fn} \) the equivalent variety determined by \( G \) is \( V[n] \). The nontrivial retractions

\[
G \xrightarrow{s} F1 \xrightarrow{r} G = \text{id}_G
\]

correspond to \( V(u) \) where \( u = sr(1) \in F1 \). These correspondences become transparent in [27].
The following theorem is one of the main results of McKenzie’s paper [25]. It characterizes all varieties equivalent to a given one via the construction of the matrix power and the $u$-modification. We present it as Porst in [27] where you can also find a proof which makes use of categorical algebra and clarifies the necessary constructions much better than McKenzie’s original proof.

**Theorem 1.17** The varieties equivalent to a given variety $\mathcal{V}$ are, up to concrete isomorphism, precisely the varieties $\mathcal{V}^{[n]}(u)$ for some $n \in \mathbb{N}, n \geq 1$, and some idempotent and invertible term $u$ for $\mathcal{V}^{[n]}$. 
Chapter 1. Introduction
As we have seen in the introduction we are already able to construct the matrix power for arbitrary varieties. But this construction is very abstract and difficult to apply in concrete cases since we need the whole clone for its description.

Here we provide a different approach, a characterization of the matrix power via “a few” operations and equations. We give two similar descriptions. The first one, $\mathcal{V}^{(n)}$, is more elegant, since it consists of adding only one binary operation and “three” basically different equations to the already existing operations and equations. The “real” number of equations added depends on the number of operations the variety we start with has: one of the equations is in fact a family of equations, one equation for every original operation. The second, $\mathcal{V}^{\{n\}}$, needs $2n - 1$ operations for the $n$-th matrix power ($n \in \mathbb{N}$), which are basically just two families of operations, and we have more equations as well. However it turns out to be a lot easier to work with when we construct the $u$-modification for unary varieties in Chapter 4, since the binary operations are less complex.

The matrix power is sufficient to describe all Morita equivalent varieties for those varieties in which each finitely generated projective algebra is free, i.e. where the finitely generated free algebras are all varietal generators. This is for example the case in the category of all (Abelian) groups or in the category of all sets (cf. [27]).

The results in this chapter are a generalization of the description of varieties equivalent to the variety Set of sets and maps which we present in the next chapter. Thus the next chapter can be read first, as already mentioned in the preface, depending on whether one prefers the theory or the example first. Both chapters can be read on their own. In the next chapter we develop several different descriptions of the matrix powers of Set. They are generalizations of examples of $n$-th matrix powers which were published earlier for special values of $n$ (Example 1.2 in [5] and Example 2 in [25]). Each description is a refinement of the previous and might help to understand how the operations work. The last description in the next chapter, $\text{n-Set}_4$, is due to results by Marshall Saade [31]. He did not work on the same problem as far as I can discern from his paper.
but his operations work very well for our problem. All the motivation given in his paper is the following sentence: “We describe in this paper, for each integer \( n > 2 \), a curious variety \( V \) of groupoids originally introduced in [10] by Evans for the purpose of describing the spectrum.”. The binary operation we use for our variety \( \mathcal{V}^{(n)} \) is from his “curious variety”. The variety \( \mathcal{V}^{(n)} \) which is our second description of the \( n \)-th matrix power of a variety \( \mathcal{V} \) is a generalization of the first construction \( n \text{-Set}_1 \) in the next chapter.

Some ideas for these generalizations are due to Fajtlowicz. In his paper on \( n \)-dimensional dice [11] he finds necessary and sufficient conditions for an algebra to have an \( n \)-th Cartesian power as underlying set. Especially the idea for equation (i), which is needed in both descriptions, comes from his paper.

### 2.1 The Variety \( \mathcal{V}^{(n)} \)

**Definition 2.1** Let \( \mathcal{V} \) be a variety which is defined by a set \( F \) of operations and a set \( E \) of equations. For each natural number \( n \geq 2 \) let \( \mathcal{V}^{(n)} \) be the variety defined by adding one binary operation, which we do not denote by a symbol, to \( F \) and adding the following equations to \( E \):

\[
\begin{align*}
(i) & \quad f(x_1, \ldots, x_k) f(y_1, \ldots, y_k) = f(x_1y_1, \ldots, x_ky_k) \text{ for all } f \in F \\
(ii) & \quad x_1 \ldots x_n y = x_1 \ldots x_n z \\
(iii) & \quad (x_1 \ldots x_{n-1} x)(x_n \ldots x_{2n-3} x) \ldots (x_{p-2}x_{p-1} x)(x_p x)(xy) = x \\
& \text{where } p = \frac{n(n-1)}{2}
\end{align*}
\]

Here \( x_1x_2\ldots x_m \) denotes the product \( (x_1(x_2(\ldots(x_{m-2}(x_{m-1}x_m))\ldots))) \); we leave out the brackets wherever possible. By \( x^k \) we denote the product

\[
x x \ldots x, \quad \text{k-times}
\]

**Remark 2.2** We can turn any \( \mathcal{V} \)-algebra \( A \) into a \( \mathcal{V}^{(n)} \)-algebra \( A^{(n)} \) by taking its Cartesian power \( A^n \) with operations acting coordinatewise and defining the binary operation by

\[
(x_1, \ldots, x_n)(y_1, \ldots, y_n) \mapsto (x_n, y_1, y_2, \ldots, y_{n-1}).
\]

It is straightforward to check that this is indeed a \( \mathcal{V}^{(n)} \)-algebra.

**Definition 2.3** Through repeated application of the binary operation on a single variable, we define new unary operations \( D_i \) by \( D_i(x) := (x^{n+1-i})^{n+1}, i = 1, \ldots, n. \)

**Remark 2.4** In \( A^{(n)} \) it is straightforward to see that the \( D_i \) operate as

\[
(x_1, \ldots, x_n) \mapsto (x_i, \ldots, x_i).
\]
2.1 The Variety \( \mathcal{V}^{(n)} \)

\( x^{n+1-i} \) has the \( i \)-th coordinate of \( x \) as its last coordinate. Applying the binary operation on \( x^{n+1-i} \) \( n+1 \) times, copies the \( i \)-th coordinate of \( x \) into every coordinate.

**Lemma 2.5** The following equations can be derived from equations (ii) and (iii) in \( \mathcal{V}^{(n)} \):

\[
\begin{align*}
(iv) \quad (x_1 \ldots x_n y) z &= x_n z \\
(v) \quad (x_1 \ldots x_{n-1} y x_n)^{n+1} &= y^{n+1} \\
(vi) \quad x^{n+1} &= ((x y)^n)^{n+1} \\
(vii) \quad (y^{n-k})^{n+1} &= ((x y^{n-k-1})^{n+1} \text{ for } k = 0, 1, \ldots, n-2 \\
(viii) \quad (x^n)^{n+1} (x^{n-1})^{n+1} \ldots (x^2)^{n+1} x^{n+1} &= x \\
(ix) \quad (x^{n+1})^2 &= x^{n+1} \\
(x) \quad ((x_{k+1})^{n-k})^{n+1} &= ((x_1^n) (x_2^{n-1}) \ldots (x_{t+1}^{n-t}) \ldots (x_{n-1}^2) (x_n^2)^{n-k})^{n+1} \text{ for } k = 0, 1, \ldots, n-1
\end{align*}
\]

**Proof:** The proof consists of straightforward calculations and can be found in [31].

2.1.1 \( \mathcal{V}^{(n)} \) is the Matrix Power of \( \mathcal{V} \)

**Definition 2.6** For \( A \in \mathcal{V}^{(n)} \) we define the set

\[
D_A := \{ x \in A | xx = x \} \subset A.
\]

A simple calculation shows that \( D_A \) is closed under all operations \( f \in F \) due to equation (i). Thus, it is even a \( \mathcal{V} \)-algebra.

It follows from equation (ix) that the \( D_i \) give maps from \( A \) to \( D_A \).

**Lemma 2.7** Let \( A \) be a \( \mathcal{V}^{(n)} \)-algebra. Then \( A \) is isomorphic to \( (D_A)^{(n)} \).

**Proof:** Define maps \( \varphi : A \to (D_A)^n \) by:

\[
x \mapsto (D_1 x, \ldots, D_n x) = ((x^n)^{n+1}, (x^{n-1})^{n+1}, \ldots, x^{n+1})
\]

and \( \psi : (D_A)^n \to A \) by:

\[
(x_1, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n
\]

Then \( \varphi \circ \psi = \text{id}_{(D_A)^n} : \)

\[
\begin{align*}
\varphi \circ \psi(x_1, \ldots, x_n) & = ((x_1 x_2 \ldots x_n)^{n+1}, ((x_1 x_2 \ldots x_n)^{n-1})^{n+1}, \ldots, ((x_1 x_2 \ldots x_n)^{1})^{n+1}) \\
& = ((x_1^{n+1}, \ldots, x_n^{n+1}) \quad \text{(since all } x_i \in D_A \text{ we have } x_{t+1} = (x_{t+1})^{n-t}) \\
& = (x_1, \ldots, x_n) \quad \text{since all } x_i \in D_A
\end{align*}
\]


as well as \( \psi \circ \varphi = \text{id}_A \):
\[
\psi \circ \varphi(x) = (x^n)^{n+1}(x^{n-1})^{n+1} \ldots (x^2)^{n+1}x^{n+1} \quad \text{(viii)}
\]

Hence \( A \cong (D_A)^n \) in \textbf{Set}.

If we turn \((D_A)^n\) into a \(\mathcal{V}^{(n)}\)-Algebra \(D_A^{(n)}\) according to Remark 2.2 then \( \varphi \) is a homomorphism:
\[
\varphi(xy) = (((xy)^{n+1}, ((xy)^{n-1})^{n+1}, \ldots, (xy)^{n+1})
\]
\[
= (((x)^{n+1}, ((y)^{n+1}, \ldots, ((y)^2)^{n+1})
\]
\[
= \varphi(x)\varphi(y)
\]

Let \( f \in F \)
\[
\varphi(f(x_1, \ldots, x_k)) = (D_1f(x_1, \ldots, x_k), \ldots, D_nf(x_1, \ldots, x_k))
\]
\[
= f^n((D_1x_1, \ldots, D_1x_k), \ldots, (D_nx_1, \ldots, D_nx_k))
\]
\[
= f^n(\varphi(x_1), \ldots, \varphi(x_n))
\]

Thus \( \varphi \) is a \(\mathcal{V}^{(n)}\)-isomorphism. \( \square \)

**Theorem 2.8** For a given variety \( \mathcal{V} \) and any natural number \( n \geq 2 \) the variety \( \mathcal{V}^{(n)} \) and the matrix power \( \mathcal{V}^{[n]} \) are concretely isomorphic.

**Proof:** The lemma above shows that every algebra in \( \mathcal{V}^{(n)} \) is isomorphic to an algebra of the form \( A^{(n)} \) for an algebra \( A \) in \( \mathcal{V} \). Every algebra in \( \mathcal{V}^{[n]} \) is isomorphic to an algebra of the form \( A^{[n]} \) for an algebra \( A \) in \( \mathcal{V} \) by definition. Since \( A^{(n)} \) and \( A^{[n]} \) have the same underlying set it remains to show that they have the same clone. The operations of the Cartesian power \( A^n \) which are the operations from \( A \) just acting coordinatwise are obviously contained in the clone of \( A^{[n]} \) and the binary operation as defined in Remark 2.2 can be easily derived from the appropriate projections and thus also lies in the clone of \( A^{[n]} \).

Now let \( f \) be an \( r \)-ary \( A^{[n]} \) operation. By definition of the matrix power, composing \( f \) with the projections gives \( nr \)-ary \( A \) operations \( f_i = \pi_i \circ f \) for \( 1 \leq i \leq n \).
2.2 An Alternative Description of the Matrix Power: $V^{(n)}$

The operations $f_i^{(n)} : A^{nnr} \to A^n$ which operate like the $f_i$ on the single coordinates are contained in the clone of $A^{(n)}$. For a given $nr$-tuple $x = (x_1, \ldots, x_r)$ with $x_i \in A^n$ we introduce the following notation:

$$\bar{x} = (x_{11}, \ldots, x_{12}, \ldots, x_{rn}, \ldots, x_{rn}) \in A^{nnr}$$

Obviously

$$f_i^{(n)}(\bar{x}) = (f_i(x), \ldots, f_i(x)),$$

thus

$$f(x) = f_1^{(n)}(\bar{x})f_2^{(n)}(\bar{x}) \ldots f_n^{(n)}(\bar{x}).$$

Therefore $f$ is an element of the clone of $A^{(n)}$.

Now we introduce a second way to construct the matrix power. This approach needs more operations and equations but is better suited when we construct the $u$-modification of $M_{\text{Act}}^{[n]}$ in Chapter 4.

**Definition 2.9** Let $\mathcal{V}$ be a variety which is defined by a set $F$ of operations and a set $E$ of equations. For each natural number $n \geq 2$ let $\mathcal{V}^{(n)}$ be the variety defined by adding unary operations $D_i$ for $i = 1, \ldots, n$ and binary operations $\circ_j$ for $j = 1, \ldots, n-1$ to $F$ and adding the following equations to $E$:

1. $D_i(f(x_1, \ldots, x_k)) = f(D_ix_1, \ldots, D_ix_k)$ $\forall f \in F, \forall i \in \{1, \ldots, n\}$
2. $D_jD_ix = D_ix$ $\forall i, j \in \{1, \ldots, n\}$
3. $D_1x_1 \circ_1 D_2x_2 \circ_2 D_3x_3 \cdots \circ_{n-1} D_nx = x$
4. $D_i(x_1 \circ_1 \cdots \circ_{n-1} x_n) = D_ix_i$ $\forall i \in \{1, \ldots, n\}$
5. $D_i(x \circ_j y) = \begin{cases} D_ix & \text{for } i = j \\ D_iy & \text{for } i \neq j \end{cases} \forall i \in \{1, \ldots, n\}, \forall j \in \{1, \ldots, n-1\}$

$x_1 \circ x_2 \cdots \circ x_m$ denotes the product $(x_1 \circ (x_2 \circ (\cdots \circ (x_{m-2} \circ (x_{m-1} \circ x_m)) \cdots)))$, again we leave out the brackets wherever possible.

**Remark 2.10** The $D_i$ in this section are the same operations as the derived operations $D_i$ in the previous section. Thus we use the same notation. The binary operations $\circ_i$ are very similar to the binary operation in $V^{(n)}$. They basically just lack the shifting of coordinates which makes them easier to handle.
Remark 2.11 We can turn any $\mathcal{V}$-algebra $A$ into a $\mathcal{V}^{\{n\}}$-algebra $A^{\{n\}}$ by taking its Cartesian power $A^n$ with operations acting coordinatewise and defining the unary operations $D_i$ by
\[ D_i(x_1, \ldots, x_n) = (x_i, \ldots, x_i) \]
and the binary operations $\circ_i$ by
\[ (x_1, \ldots, x_n) \circ_i (y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n). \]
It is straightforward to check that this is indeed a $\mathcal{V}^{\{n\}}$-algebra.

2.2.1 $\mathcal{V}^{\{n\}}$ is the Matrix Power of $\mathcal{V}$

Definition 2.12 For $A \in \mathcal{V}^{\{n\}}$ we define the sets
\[ D_i^A := \{ x \in A | D_i x = x \} \subset A \quad \text{for } i = \{1, \ldots, n\}. \]
From equation (ii) we get $D_i^A := D_1^A = D_2^A = \cdots = D_n^A$. $D^A$ is closed under all operations $f \in F$ due to equation (i), and thus it is even a $\mathcal{V}$-algebra.

The $D_i$ give maps from $A$ to $D^A$.

Lemma 2.13 Let $A$ be a $\mathcal{V}^{\{n\}}$-algebra. Then $A$ is isomorphic to $(D^A)^{\{n\}}$.

Proof: Define maps
\[ \varphi : A \rightarrow (D^A)^n \quad \text{by} \quad x \mapsto (D_1 x, \ldots, D_n x) \]
and
\[ \psi : (D^A)^n \rightarrow A \quad \text{by} \quad (x_1, \ldots, x_n) \mapsto x_1 \circ_1 x_2 \circ_2 \cdots \circ_{n-1} x_n. \]
Then $\psi \circ \varphi = \text{id}_A$:
\[ \psi \circ \varphi(x) = D_1 x \circ_1 D_2 x \circ_2 D_3 x \circ_3 \cdots \circ_{n-1} D_n x \overset{(iii)}{=} x \]
as well as $\varphi \circ \psi = \text{id}_{(D^A)^n}$:
\[ \varphi \circ \psi(x_1, \ldots, x_n) = (D_1(x_1 \circ_1 \cdots \circ_{n-1} x_n), \ldots, D_n(x_1 \circ_1 \cdots \circ_{n-1} x_n)) \overset{(iv)}{=} (D_1 x_1, \ldots, D_n x_n) x_i \in D^A \overset{(vi)}{=} (x_1, \ldots, x_n). \]
Hence $A \cong D^A$ in Set.
2.2 An Alternative Description of the Matrix Power: \( V^{(n)} \)

If we turn \((D^A)^n\) into a \( V^{(n)} \)-algebra \( D^{(n)}_A \) according to Remark 2.11 then \( \varphi \) is even a homomorphism:

\[
\varphi(D_i x) = (D_1 D_i x, \ldots, D_n D_i x) = (D_1 x, \ldots, D_n x) = D_i \varphi(x)
\]

\[
\varphi(x \circ_i y) = (D_1 (x \circ_i y), \ldots, D_n (x \circ_i y)) = (D_1 y, \ldots, D_{i-1} y, D_i x, D_{i+1} y, \ldots, D_n y) = (D_1 x, \ldots, D_n x) \circ_i (D_1 y, \ldots, D_n y) = \varphi(x) \circ_i \varphi(y)
\]

Let \( f \in F \)

\[
\varphi(f(x_1, \ldots, x_k)) = (D_1 f(x_1, \ldots, x_k), \ldots, D_n f(x_1, \ldots, x_k))
\]

\[
= (f(D_1 x_1, \ldots, D_1 x_k), \ldots, f(D_n x_1, \ldots, D_n x_k))
\]

\[
= f^n((D_1 x_1, \ldots, D_n x_1), \ldots, (D_1 x_k, \ldots, D_n x_k))
\]

Thus \( \varphi \) is a \( V^{(n)} \)-isomorphism.

\[ \square \]

**Theorem 2.14** For a given variety \( V \) and any natural number \( n \geq 2 \) the variety \( V^{(n)} \) and the matrix power \( V^{[n]} \) are concretely isomorphic.

**Proof:** The lemma above shows that every algebra in \( V^{(n)} \) is isomorphic to an algebra of the form \( A^{(n)} \) for an algebra \( A \) in \( V \). Every algebra in \( V^{[n]} \) is isomorphic to an algebra of the form \( A^{[n]} \) for an algebra \( A \) in \( V \) by definition. Since \( A^{(n)} \) and \( A^{[n]} \) have the same underlying set it remains to show that they have the same clone. This can be done in the same way as in the case of \( V^{(n)} \). The operations of the Cartesian power \( A^n \) which are the operations from \( A \) just acting coordinatewise are obviously contained in the clone of \( A^{[n]} \) and the unary operation \( D_i \) as well as the binary operations \( \circ_i \) as defined in Remark 2.11 can be easily derived from the appropriate projections and thus also lie in the clone of \( A^{[n]} \).

Now let \( f \) be an \( r \)-ary \( A^{[n]} \) operation. By definition of the matrix power composing \( f \) with the projections gives \( nr \)-ary \( A \) operations \( f_i = \pi_i \circ f \) for \( 1 \leq i \leq n \).
Chapter 2. The Matrix Power

The operations $f_i^{(n)} : A^{nnr} \to A^n$ which operate like the $f_i$ on the single coordinates are contained in the clone of $A^{(n)}$. For a given $nr$-tuple $x = (x_1, \ldots, x_r)$ with $x_i \in A^n$ we introduce the following notation

$$\bar{x} = (x_{11}, \ldots, x_{11}, x_{12}, \ldots, x_{12}, \ldots, x_{1n}, \ldots, x_{1n}) \in A^{nnr}$$

Obviously $f_i^{(n)}(\bar{x}) = (f_i(x), \ldots, f_i(x))$ thus

$$f(x) = f_1^{(n)}(\bar{x}) \circ f_2^{(n)}(\bar{x}) \circ \cdots \circ f_n^{(n)}(\bar{x}).$$

Therefore $f$ is an element of the clone of $A^{(n)}$. 

□
Chapter 3

The Matrix Powers of Set

This chapter provides four different descriptions of the matrix powers of $\text{Set}$, the variety of sets and maps. Each one is described independently, but their order shows how I initially got to understand and refine the operations needed to describe the matrix powers. This chapter can be read on its own to get a better understanding of the operations before reading Chapter 2 or it can be treated as an example for the theory in Chapter 2.

The varieties equivalent to $\text{Set}$ are known: the nonempty finite sets are the varietal generators of $\text{Set}$, i.e. up to isomorphism the natural numbers without zero are all varietal generators. The corresponding Lawvere theories are therefore easy to describe. Thus every variety $\mathcal{V}$ equivalent to $\text{Set}$ is concretely equivalent to $(\text{Set}, \text{hom}(n, -))$ for some natural number $n \neq 0$ (see Theorem 1.11). But this only gives us a very abstract description. The aim of this chapter is to find a small number of operations and equations which define the varieties in question. When first starting to work on this problem our interest was sparked by two existing solutions for the special case of the second matrix power. One from Börger (Example 1.2 in [5]) and one from McKenzie (Example 2 in [25]). For the general case we present four solutions. The first two, $\mathbf{n-Set}_1$ and $\mathbf{n-Set}_2$, are direct generalizations of Börger’s example whereas the third, $\mathbf{n-Set}_3$, uses McKenzie’s operations. The fourth description, $\mathbf{n-Set}_4$, is based on a variety constructed by Saade [31]. It seems that Saade did not realize the significance of his construction in this context because he only talks about describing a “curious variety”. The description $\mathbf{n-Set}_4$ is an example of the first characterization $\mathcal{V}^{(n)}$ in the previous chapter.

Many of the proofs are analogous in the different sections of this chapter. Nevertheless, they are all given explicitly, so that it is not necessary to read every section. In this chapter, we use the symbols of some operations repeatedly. They can represent different operations in different sections. But they are always based on the same principle, especially the operations $D_i$. We refrain from distinguishing between them via indices since the differences should be clear and we have to use more than enough indices already.
Remark 3.1 Due to their concrete equivalence to \((\text{Set}, \text{hom}(n, -))\) for some natural number \(n \neq 0\), the varieties equivalent to \(\text{Set}\) contain empty sets and thus they cannot contain nullary operations (constants).

It is impossible to describe those varieties only by unary operations since otherwise unions of subalgebras would always be subalgebras. But that is not true in general because of the cardinality restriction induced by \(\text{hom}(n, -)\) as forgetful functor. Only sets that are an \(n\)-th Cartesian power for \(n \geq 2\) appear as underlying sets.

3.1 \(n\)-Set

Definition 3.2 For each natural number \(n \geq 2\) let \(n\text{-Set}_1\) be the variety defined by the following operations and equations:

Unary operations: \(D_i\) for \(i = 1, \ldots, n\)

Binary operations: \(\circ_j\) for \(j = 1, \ldots, n - 1\)

Equations:

\[
\alpha_{ij} : D_j D_i = D_i \quad \forall i, j \in \{1, \ldots, n\} \text{ with } i \neq j
\]

\[
\beta_{ij} : D_i (x \circ_j y) = \begin{cases} D_i x & \text{for } i \leq j \\ D_i y & \text{for } i > j \end{cases} \quad \forall i \in \{1, \ldots, n\}, \forall j \in \{1, \ldots, n - 1\}
\]

\[
\gamma : D_1 x \circ_1 D_2 x \circ_2 D_3 x \circ_3 \cdots \circ_{n-1} D_n x = x
\]

Remark 3.3 We use the following conventions for greater clearness of representation: the \(D_i\) have higher priority than the \(\circ_j\) and

\[x_1 \circ_1 x_2 \circ_2 \cdots \circ_{n-1} x_n := ((x_1 \circ_1 x_2) \circ_2 x_3) \circ_3 \cdots \circ_{n-1} x_n.
\]

Note that this implies that in this section the terms have to be read from the left to the right opposite to Chapter 2 where equations had to be read from the right to the left.

Remark 3.4 We can turn a set of the form \(X^n\) into an \(n\text{-Set}_1\)-algebra by defining

\[D_i^{X^n} : X^n \to X^n \quad \text{by} \quad (x_1, \ldots, x_n) \mapsto (x_i, \ldots, x_i)
\]

for all \(i = 1, \ldots, n\) and

\[\circ_j^{X^n} : (X^n)^2 \to X^n \quad \text{by} \quad (x, y) \mapsto (x_1, \ldots, x_j, y_{j+1}, \ldots, y_n)
\]

for all \(j = 1, \ldots, n - 1\). The such defined operations obviously satisfy the required equations.

Lemma 3.5 If \(D_i x = D_i y\) for all \(i = 1, \ldots, n\) then \(x = y\).
3.1 n-Set

**Proof:** Let \( D_i x = D_i y \) for all \( i = 1, \ldots, n \). Then
\[
x \gamma = D_1 x \circ_1 \cdots \circ_{n-1} D_n x = D_1 y \circ_1 \cdots \circ_{n-1} D_n y = y.
\]

\[\square\]

**Lemma 3.6** In n-Set\(_1\) the following equations are satisfied:

\( \alpha_{ii} : \ D_i D_i = D_i \quad \forall i \in \{1, \ldots, n\} \)
\( \beta_i : \ D_i(x_1 \circ_1 x_2 \circ_2 \cdots \circ_{n-1} x_n) = D_i x_i \quad \forall i \in \{1, \ldots, n\} \)
\( \delta_j : \ (x_1 \circ_1 \cdots \circ_{n-1} x_n) \circ_j (y_1 \circ_1 \cdots \circ_{n-1} y_n) \quad \forall j \in \{1, \ldots, n-1\} \)
\( \varepsilon_j : \ x \circ_j x = x \quad \forall j \in \{1, \ldots, n-1\} \)

**Proof:** Repeated application of \( \beta_{ij} \) gives \( \hat{\beta}_i \). For all \( i = 1, \ldots, n \):
\[
D_i D_i x = D_i(D_1 x \circ_1 D_2 x \circ_2 D_3 x \circ_3 \cdots \circ_{n-1} D_n x) = D_i x
\]
which proves \( \alpha_{ii} \).

Ad \( \delta_j \): Let \( j = 1, \ldots, n - 1 \)
Case 1: For all \( i \leq j \)
\[
D_i((x_1 \circ_1 \cdots \circ_{n-1} x_n) \circ_j (y_1 \circ_1 \cdots \circ_{n-1} y_n)) \quad \hat{\beta}_i = D_i(x_1 \circ_1 \cdots \circ_{n-1} x_n)
\]
Case 2: For all \( i > j \)
\[
D_i((x_1 \circ_1 \cdots \circ_{n-1} x_n) \circ_j (y_1 \circ_1 \cdots \circ_{n-1} y_n)) \quad \hat{\beta}_i = D_i y_i
\]

Cases 1 and 2 give \( \delta_j \) by Lemma 3.5.

For all \( i = 1, \ldots, n \) we have \( D_i(x \circ_j x) = D_i x \) according to \( \beta_{ij} \). Hence Lemma 3.5 gives \( \varepsilon_j \).

\[\square\]

**Remark 3.7** For \( n = 2 \) we get Example 1.2 from Börger in [5] where \( p = D_1, q = D_2 \) and \( * = \circ_1 \). The equations are exactly the same.
Remark 3.8 For \( n = 2 \) we can derive an associative law for the binary operation by applying \( \delta \):

\[
(x \circ y) \circ z = ((x \circ y) \circ (z \circ z)) \overset{\delta}{=} x \circ (y \circ z) = ((x \circ x) \circ (y \circ z)) \overset{\delta}{=} x \circ (y \circ z).
\]

Hence for \( n = 2 \) we get semigroups with two additional unary operations (cf. Example 2 in [25]).

3.1.1 The Equivalence of \( n \text{-Set}_1 \) to \( \text{Set} \)

Definition 3.9 For \( A \in n \text{-Set}_1 \) we define the set

\[ D_A := \{ x \in A | D_1x = x \} \subset A. \]

Obviously \( D_i x = x \) for all \( x \in D_A \) and all \( i = 1, \ldots, n \) due to equation \( \alpha_{ij} \).

Lemma 3.10 Let \( A \) be an \( n \text{-Set}_1 \)-algebra. Then \( A \) is isomorphic to \( (D_A)^n \).

Proof: Define maps

\[ \varphi : A \to (D_A)^n \text{ by } x \mapsto (D_1x, \ldots, D_nx) \]

and

\[ \psi : (D_A)^n \to A \text{ by } (x_1, \ldots, x_n) \mapsto x_1 \circ_1 \cdots \circ_{n-1} x_n. \]

Then \( \varphi \circ \psi = \text{id}_{(D_A)^n} \):

\[
\varphi \circ \psi(x_1, \ldots, x_n) = (D_1(x_1 \circ_1 \cdots \circ_{n-1} x_n), \ldots, D_n(x_1 \circ_1 \cdots \circ_{n-1} x_n))
\]

\[
= (D_1x_1, \ldots, D_nx_n)
\]

\[
= (x_1, \ldots, x_n) \quad \text{since } x_i \in D_A
\]

as well as \( \psi \circ \varphi = \text{id}_A \):

\[
\psi \circ \varphi(x) = D_1x \circ_1 D_2x \circ_2 D_3x \circ_3 \cdots \circ_{n-1} D_nx \overset{\gamma}{=} x.
\]

Hence \( A \cong (D_A)^n \) in \( \text{Set} \).

If we turn \( (D_A)^n \) into an \( n \text{-Set}_1 \)-algebra according to Remark 3.4 then \( \varphi \) is a homomorphism since for all \( i = 1, \ldots, n \)

\[
\varphi(D_i x) = (D_1 D_i x, \ldots, D_n D_i x) \overset{\alpha_{ij}}{=} (D_i x, \ldots, D_i x) = D_i (D_1 x, \ldots, D_n x) = D_i \varphi(x)
\]
and for all $j = 1, \ldots, n - 1$

$$\varphi(x \circ y) = (D_1(x \circ y), \ldots, D_n(x \circ y)) \overset{\beta_j}{=} (D_1x, \ldots, D_jx, D_{j+1}y, \ldots, D_ny) \overset{\delta_j}{=} (D_1x, \ldots, D_ny) \circ_j (D_1y, \ldots, D_ny) = \varphi(x) \circ_j \varphi(y).$$

Thus $\varphi$ is an $n$-$\text{Set}_1$-isomorphism.

\[\square\]

**Proposition 3.11** ($n$-$\text{Set}_1, | - |$) is concretely equivalent to ($\text{Set}, \text{hom}(n, -)$).

**Proof:** We define a functor $F : \text{Set} \to n$-$\text{Set}_1$ which maps each map $f : X \to Y$ to the $n$-$\text{Set}_1$-homomorphism $f^n : X^n \to Y^n$, where $X^n$ and $Y^n$ are $n$-$\text{Set}_1$-algebras as defined in Remark 3.4. It is clear that $F$ is faithful.

Let $g : X^n \to Y^n$ be an $n$-$\text{Set}_1$-homomorphism. Then for $(x, \ldots, x) \in D_{X^n}$ all coordinates of $g(x, \ldots, x)$ are the same since

$$g(x, \ldots, x) = gD_i(x, \ldots, x) = D_ig(x, \ldots, x)$$

for all $i = 1, \ldots, n$. Thus we can define a map $f : X \to Y$ which maps each $x \in X$ to the first coordinate of $g(x, \ldots, x)$. Obviously $f^n$ equals $g$ on $D_{X^n}$. But this implies $f^n = g$ on the whole of $X^n$ by applying equation $\gamma$. Therefore $F$ is full.

$F$ is isomorphism dense by Lemma 3.10. Thus $F$ is an equivalence. Concreteness is obvious.

\[\square\]

**Remark 3.12** Here is an alternative proof of the equivalence between $n$-$\text{Set}_1$ and $\text{Set}$. We construct a functor $G$ which is an equivalence inverse to the functor $F$ in the previous proof.

**Proof:** We construct a functor $G : n$-$\text{Set}_1 \to \text{Set}$ which assigns to each $n$-$\text{Set}_1$-morphism $f : A \to B$ the map $f|_{D_A} : D_A \to D_B$ ($\text{Im} f|_{D_A} \subset D_B$, since for each $x \in D_A$ we have $f(x) = f(D_1x) = D_1f(x) \in D_B$).

Each map $g : D_A \to D_B$ yields a unique homomorphism $\bar{g} : A \to B$ in $n$-$\text{Set}_1$ defined by $x \mapsto g(D_1x) \circ_1 \cdots \circ_{n-1} g(D_nx)$.

(i) $\bar{g} : A \to B$ is a homomorphism:

(a) For all $i = 1, \ldots, n$ and all $x \in A$ we get $\bar{g}(D_i x) = D_i \bar{g}(x)$, since

$$\bar{g}(D_i x) = g(D_1D_i x) \circ_1 \cdots \circ_{n-1} g(D_nD_i x) \overset{\alpha_{ij}}{=} g(D_i x) \circ_1 \cdots \circ_{n-1} g(D_i x) \overset{\epsilon_i}{=} g(D_i x) \overset{\delta_i}{=} D_i g(D_i x) \text{ since } g(D_i x) \in D_B \overset{\beta_i}{=} D_i (g(D_1 x) \circ_1 \cdots \circ_{n-1} g(D_n x)) = D_i \bar{g}(x)$$
Chapter 3. The Matrix Powers of Set

(b) For \( j = 1, \ldots, n - 1 \) we have \( \bar{g}(x \circ y) = \bar{g}(x) \circ_j \bar{g}(y) \). Define \( (x/y)_{ij} \) by \( (x/y)_{ij} = x \) for \( i \leq j \) and \( (x/y)_{ij} = y \) for \( i > j \). Then

\[
D_i \bar{g}(x \circ y) = \bar{g}(D_i(x \circ y)) = \bar{g}(D_i((x/y)_{ij})) = D_i \bar{g}(y/x)_{ij} = D_i(\bar{g}(x) \circ_j \bar{g}(y))
\]

for all \( i = 1, \ldots, n \). Hence \( \bar{g}(x \circ y) = \bar{g}(x) \circ_j \bar{g}(y) \) by Lemma 3.5.

(ii) \( \bar{g}|_{D_B} = g \) since for each \( x \in D_A \) we have

\[
\bar{g}(x) = \bar{g}(D_1x) \overset{(i)[a]}{=} g(D_1x) = g(x).
\]

(iii) Let \( f : A \to B \) be an \( \mathbf{n-Set}_1 \)-homomorphism and let \( g := G(f) = f|_{D_A} \). Then \( \bar{g} = f \) since

\[
f(x) = f(D_1x \circ_1 \cdots \circ_{n-1} D_nx) = f(D_1x) \circ_1 \cdots \circ_{n-1} f(D_nx) = g(D_1x) \circ_1 \cdots \circ_{n-1} g(D_nx) = \bar{g}(x)
\]

Thus \( G \) is fully faithful.

\( G \) is isomorphism dense. Let \( X \) be a set and let \( A = X^n \) an \( \mathbf{n-Set}_1 \)-algebra with operations defined according to Remark 3.4. Then

\[
D_A = \{(x, \ldots, x) \in A|x \in X\} \cong X.
\]

Therefore \( G \) is an equivalence.

3.2 \( \mathbf{n-Set}_2 \)

Instead of \( n - 1 \) binary equations we can alternatively use just one \( n \)-ary operation. It is left to the reader to decide whether it is preferable to have a small (and fixed) arity or a smaller number of operations. This case is analogous to the previous \( \mathbf{n-Set}_1 \) case except for minor details.

**Definition 3.13** For each natural number \( n \geq 2 \) let \( \mathbf{n-Set}_2 \) be the variety defined by the following operations and equations:

Unary operations: \( D_i \) for \( i = 1, \ldots, n \)

\( n \)-ary operation: \( * \)

Equations:

\[
\begin{align*}
\alpha_{ij} : & \quad D_jD_i = D_i & \forall i, j \in \{1, \ldots, n\} \text{ with } i \neq j \\
\beta_i : & \quad D_i * (x_1, \ldots, x_n) = D_ix_i \\
\gamma : & \quad *(D_1x, \ldots, D_nx) = x
\end{align*}
\]
We can turn a set of the form $X^n$ into an $\text{n-Set}_2$-algebra by defining

$$D_i^{X^n} : X^n \to X^n \quad \text{by} \quad (x_1, \ldots, x_n) \mapsto (x_i, \ldots, x_i)$$

for all $i = 1, \ldots, n$ and

$$*^{X^n} : (X^n)^n \to X^n \quad \text{by} \quad (\bar{x}_1, \ldots, \bar{x}_n) \mapsto (x_{11}, \ldots, x_{nn}).$$

The such defined operations obviously satisfy the required equations.

**Lemma 3.15** If $D_ix = D_iy$ for all $i = 1, \ldots, n$ then $x = y$.

**Proof:** Let $D_ix = D_iy$ for all $i = 1, \ldots, n$. Then

$$x \triangleq *^{D_1x, \ldots, D_nx} = *^{D_1y, \ldots, D_ny} \triangleq y.$$  \hfill $\square$

**Lemma 3.16** In $\text{n-Set}_2$ the following equations are satisfied for $i = 1, \ldots, n$:

$$\alpha_{ii} : \quad D_iD_i = D_i$$

**Proof:** For all $i = 1, \ldots, n$: $D_iD_ix \triangleq D_i*^{D_1x, \ldots, D_nx} \triangleq D_ix.$ \hfill $\square$

**Remark 3.17** For $n = 2$ we get Example 1.2 from Börger in [5] again, where $p = D_1$, $q = D_2$ and $* = *$. The equations are the same.

### 3.2.1 The Equivalence of $\text{n-Set}_2$ to Set

**Definition 3.18** For $A \in \text{n-Set}_2$ we define the set

$$D_A := \{x \in A | D_ix = x\} \subset A.$$ 

Obviously we have $D_ix = x$ for all $x \in D_A$ and all $i = 1, \ldots, n$.

**Lemma 3.19** Let $A$ be an $\text{n-Set}_2$-algebra. Then $A$ is isomorphic to $(D_A)^n$.

**Proof:** Define maps

$$\varphi : A \to (D_A)^n \quad \text{by} \quad x \mapsto (D_1x, \ldots, D_nx)$$

and

$$\psi : (D_A)^n \to A \quad \text{by} \quad (x_1, \ldots, x_n) \mapsto *^{x_1, \ldots, x_n}.$$
Then $\varphi \circ \psi = \text{id}_{(D_A)^n}$:

$$\varphi \circ \psi(x_1, \ldots, x_n) = (D_1 \ast (x_1, \ldots, x_n), \ldots, D_n \ast (x_1, \ldots, x_n))$$

$$\beta_i = (D_1 x_1, \ldots, D_n x_n) = (x_1, \ldots, x_n) \quad \text{since } x_i \in D_A$$

as well as $\psi \circ \varphi = \text{id}_A$:

$$\psi \circ \varphi(x) = \ast(D_1 x, \ldots, D_n x) = x$$

Hence $A \cong (D_A)^n$ in $\text{Set}$. If we turn $(D_A)^n$ into an $\text{n-Set}_2$-algebra according to Remark 3.14 then $\varphi$ is a homomorphism. Because for all $i = 1, \ldots, n$

$$\varphi(D_i x) = (D_1 D_i x, \ldots, D_n D_i x) \stackrel{\alpha_{ij}}{=} (D_i x, \ldots, D_i x) = D_i(D_1 x, \ldots, D_n x) = D_i \varphi(x)$$

and for all $j = 1, \ldots, n - 1$

$$\varphi(\ast(x_1, \ldots, x_n)) = (D_1 \ast (x_1, \ldots, x_n), \ldots, D_n \ast (x_1, \ldots, x_n))$$

$$\beta_i = (D_1 x_1, \ldots, D_n x_n) = \ast((D_1 x_1, \ldots, D_n x_1), \ldots, (D_1 x_n, \ldots, D_n x_n))$$

Thus $\varphi$ is even an $\text{n-Set}_2$-isomorphism. 

**Proposition 3.20** ($\text{n-Set}_2, | - |$) is concretely equivalent to ($\text{Set}, \text{hom}(n, -)$).

**Proof:** We define a functor $F : \text{Set} \to \text{n-Set}_2$ which maps each map $f : X \to Y$ to the $\text{n-Set}_2$-homomorphism $f^n : X^n \to Y^n$, where $X^n$ and $Y^n$ are defined as in Remark 3.14. It is clear that $F$ is faithful.

Let $g : X^n \to Y^n$ be an $\text{n-Set}_2$-homomorphism. Then for $(x, \ldots, x) \in D_X^n$ all coordinates of $g(x, \ldots, x)$ are the same since we have

$$g(x, \ldots, x) = gD_i(x, \ldots, x) = D_i g(x, \ldots, x)$$

for all $i = 1, \ldots, n$. Thus we can define a map $f : X \to Y$ which maps each $x \in X$ to the first coordinate of $g(x, \ldots, x)$. Obviously $f^n$ equals $g$ on $D_X^n$. But this implies $f^n = g$ on the whole of $X^n$ by applying equation $\gamma$. Therefore $F$ is full.

$F$ is isomorphism dense by Lemma 3.19. Thus $F$ is an equivalence. Concreteness is obvious. \qed
3.3 \textit{n-Set}_3

We can diminish the number of operations further, without having to fall back on large arities. In this approach we only need one unary and one binary operation. At the moment it seems that the equations can only be described in a readable manner with the help of derived operations which is not really satisfactory.

**Definition 3.21** For each natural number \( n \geq 2 \) let \( n\text{-Set}_3 \) be the variety defined by the following operations and equations:

**Unary operation:** \( \tau \)

**Binary operation:** \( \star \)

Let \( D_i x := \tau \left( \ldots \tau \left( \tau \left( D_i x \star D_2 x \star D_3 x \star \ldots \right) \star D_2 x \right) \star D_3 x \right) \star \ldots \) for \( i = 1, \ldots, n \).

**Equations:**

\[
\begin{align*}
\alpha_i & : \quad \tau D_i = D_i \quad \forall i \in \{1, \ldots, n\} \\
\beta_i & : \quad D_i(x \star y) = \begin{cases} 
D_i x & \text{for } i < n \\
D_i y & \text{for } i = n 
\end{cases} \quad \forall i \in \{1, \ldots, n\} \\
\gamma & : \quad \tau \left( \ldots \tau \left( \tau \left( D_i x \star D_2 x \star D_3 x \star \ldots \right) \star D_2 x \right) \star D_3 x \right) \star \ldots \star D_n x = x \\
\delta & : \quad x \star x = x \\
\varepsilon & : \quad \tau^n x = x
\end{align*}
\]

**Remark 3.22** We can turn a set of the form \( X^n \) into an \( n\text{-Set}_3 \)-algebra by defining

\[
\tau^{X^n} : X^n \to X^n \quad \text{by} \quad (x_1, \ldots, x_n) \mapsto (x_2, x_3, \ldots, x_n, x_1)
\]

and

\[
\star^{X^n} : (X^n)^2 \to X^n \quad \text{by} \quad (x, y) \mapsto (x_1, \ldots, x_{n-1}, y_n).
\]

Now the \( D_i \) are the same operations as in the two cases above, i.e.

\[
D_i^{X^n} : X^n \to X^n \quad \text{is given by} \quad (x_1, \ldots, x_n) \mapsto (x_i, \ldots, x_i)
\]

for all \( i = 1, \ldots, n \). That is easy to see if you realize that by the above definition \( \tau^i \) rotates the \( i \)-th coordinate to the last position. Equally \( \tau^{i+1} \) rotates the \( i \)-th coordinate to the penultimate position. \( \star \tau^i x \) thus inserts the \( i \)-th coordinate of \( x \) into the last position. Then everything is shifted one position forward by \( \tau \), \( \star \tau^i x \) inserts the \( i \)-th coordinate of \( x \) into the last position, etc. until all coordinates read \( x_i \).

Thus it is now easy to see that the equations hold for \( X^n \) with the operations defined as above.
Lemma 3.23 If $D_ix = D_iy$ for all $i = 1,\ldots,n$ then $x = y$.

Proof: Let $D_ix = D_iy$ for all $i = 1,\ldots,n$. Then

$$x \overset{\gamma}{=} \tau(\ldots\tau(D_1x \star D_2x) \star D_3x) \star \ldots) \star D_nx$$

$$= \tau(\ldots\tau(D_1y \star D_2y) \star D_3y) \star \ldots) \star D_ny \overset{\gamma}{=} y.$$

Lemma 3.24 In $n\text{-Set}_3$ the following equations are satisfied:

$\hat{\alpha}_{ij} : \quad D_jD_i = D_i \quad$ for $i,j = 1,\ldots,n$

$\hat{\beta}_i : \quad D_i(\tau(\ldots\tau(x_1 \star x_2) \star x_3) \star \ldots) \star x_n) = \begin{cases} D_{n-1}x_1 & \text{for } i = 1 \\ D_nx_i & \text{for } i = 2,\ldots,n \end{cases}$

for all $i = 1,\ldots,n$

$\zeta : \quad D_i\tau x = \begin{cases} D_{i+1}x & \text{for } i = 1,\ldots,n-1 \\ D_1x & \text{for } i = n \end{cases}$

Proof: Ad $\hat{\alpha}_{ij}$: For all $i,j = 1,\ldots,n$:

$$D_jD_ix = \tau(\ldots\tau(\tau^{j+1}(D_ix) \star \tau^{j}(D_ix)) \star \ldots) \star \tau^{j}(D_ix)$$

$$\overset{\alpha_i}{=} \tau(\ldots\tau(D_ix \star D_ix) \star \ldots) \star \tau^{j}(D_ix)$$

$$\overset{\delta,\alpha_i}{=} D_ix$$

Ad $\zeta$: Follows trivially from the definition of the $D_i$ (and $\varepsilon$ in the case $i = n$).

Ad $\hat{\beta}_i$: Follows by repeated application of $\beta_i$ and $\zeta$.

3.3.1 The Equivalence of $n\text{-Set}_3$ to Set

Definition 3.25 For $A \in n\text{-Set}_3$ we define the set

$$D_A := \{ x \in A | \tau x = x \} \subset A.$$  

We have $D_ix = x$ for all $x \in D_A$ and all $i = 1,\ldots,n$ because of equation $\zeta$ and Lemma 3.23. Also it is obvious from $\alpha_i$ that the $D_i$ give maps from $A$ to $D_A$.

Lemma 3.26 Let $A$ be an $n\text{-Set}_3$-algebra. Then $A$ is isomorphic to $(D_A)^n$. 
Proof: Define maps
\[ \varphi : A \to (D_A)^n \quad \text{by} \quad x \mapsto (D_1 x, \ldots, D_n x) \]
and
\[ \psi : (D_A)^n \to A \quad \text{by} \quad (x_1, \ldots, x_n) \mapsto \tau(\ldots \tau(x_1 \ast x_2) \ast x_3) \ast \ldots \ast x_n. \]

Then \( \varphi \circ \psi = \text{id}_{(D_A)^n} \):
\[
\varphi \circ \psi(x_1, \ldots, x_n) \\
= (D_1(\tau(\ldots \tau(x_1 \ast x_2) \ast x_3) \ast \ldots \ast x_n)) \\
= (D_1 x_1, D_2 x_2, \ldots, D_n x_n) \\
\overset{\beta_i}{=} (D_{n-1} x_1, D_n x_2, \ldots, D_n x_n) \\
= (x_1, \ldots, x_n) \quad \text{since} \ x_i \in D_A
\]
as well as \( \psi \circ \varphi = \text{id}_A \):
\[
\psi \circ \varphi(x) = \tau(\ldots \tau(D_1 x \ast D_2 x) \ast D_3 x) \ast \ldots \ast D_n x \overset{\gamma_i}{=} x.
\]

Hence \( A \cong (D_A)^n \) in \( \text{Set} \).

If we turn \( (D_A)^n \) into an \( \text{n-Set}_3 \)-algebra according to Lemma 3.22 then \( \varphi \) is a homomorphism. Because for all \( i = 1, \ldots, n \)
\[
\varphi(\tau x) = (D_1 \tau x, \ldots, D_n \tau x) \overset{\xi}{=} (D_2 x, \ldots, D_n x, D_1 x) \\
= \tau(D_1 x, \ldots, D_n x) \\
= \tau \varphi(x)
\]
and
\[
\varphi(x \ast y) = (D_1(x \ast y), \ldots, D_n(x \ast y)) \overset{\beta_i}{=} (D_1 x, \ldots, D_{n-1} x, D_n y) \\
= (D_1 x, \ldots, D_n x) \ast (D_1 y, \ldots, D_n y) \\
= \varphi(x) \ast \varphi(y).
\]

Thus \( \varphi \) is an \( \text{n-Set}_3 \)-isomorphism. \( \square \)

Proposition 3.27 \( (\text{n-Set}_3, | - |) \) is concretely equivalent to \( (\text{Set}, \text{hom}(n, -)) \).

Proof: We define a functor \( F : \text{Set} \to \text{n-Set}_3 \) which maps each map \( f : X \to Y \) to the \( \text{n-Set}_3 \)-homomorphism \( f^n : X^n \to Y^n \) where \( X^n \) and \( Y^n \) are defined according to Lemma 3.22. It is clear that \( F \) is faithful.
Let $g : X^n \to Y^n$ be an $\textbf{n-Set}_3$-homomorphism. Then for $(x, \ldots, x) \in D_{X^n}$ all coordinates of $g(x, \ldots, x)$ are the same since we have

$$g(x, \ldots, x) = gD_i(x, \ldots, x) = D_ig(x, \ldots, x)$$

for all $i = 1, \ldots, n$. Thus we can define a map $f : X \to Y$ which maps each $x \in X$ to the first coordinate of $g(x, \ldots, x)$. Obviously $f^n = g$ on $D_{X^n}$. But that implies that $f^n = g$ on the whole of $X^n$ by applying equation $\gamma$. Therefore $F$ is full.

$F$ is isomorphism dense by Lemma 3.26. Thus $F$ is an equivalence. Concreteness is obvious.

\[\square\]

**Remark 3.28** For $n = 2$ we get the same operations as McKenzie in Example 2 in [25] but we have different equations. McKenzie needs less equations which is only possible because he makes use of the special symmetry in the case $n = 2$.

### 3.4 $\textbf{n-Set}_4$

In this section we introduce a solution already found by Marshall Saade [31] in 1969. It needs only one binary operation, which we do not denote, and two equations. This seems to be the best possible solution.

**Definition 3.29** For each natural number $n \geq 2$ let $\textbf{n-Set}_4$ be the variety defined by one binary operation and the following equations:

\[(i)\quad x_1 \ldots x_n y = x_1 \ldots x_n z\]

\[(ii)\quad (x_1 \ldots x_{n-1}x)(x_n \ldots x_{2n-3}x) \ldots (x_{p-2}x_{p-1}x)(x_{p}x)(xy) = x\]

where $p = \frac{n(n-1)}{2}$.

$x_1x_2\ldots x_m$ denotes the product $(x_1(x_2(\ldots(x_{m-2}(x_{m-1}x_m))\ldots)))$, we leave out the brackets wherever possible. By $x^k$ we denote the product

$$\underbrace{x \ldots x}_k.$$

**Remark 3.30** $\textbf{n-Set}_4$ is an example for the variety $\mathcal{V}^{(n)}$, our first description of the $n$-th matrix power of a variety $\mathcal{V}$. Of course we do not need the first equation from $\mathcal{V}^{(n)}$, since we do not have operations in $\textbf{Set}$.

**Remark 3.31** We can turn a set of the form $X^n$ into an $\textbf{n-Set}_4$-algebra by defining the binary operation by

$$(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_n, y_1, y_2, \ldots, y_{n-1}).$$

Checking this is straightforward.
3.4 n-Set

The $D_i$ from the previous descriptions can now be defined by

$$D_i = (x^{n+1-i})^{n+1}$$

as an easy calculation shows.

**Lemma 3.32** The following equations are satisfied in n-Set$_4$:

(iii) $(x_1 \ldots x_n y) z = x_n z$

(iv) $(x_1 \ldots x_{n-1} y x_n)^{n+1} = y^{n+1}$

(v) $x^{n+1} = ((xy)^{n+1})^{n+1}$

(vi) $(y^{n-k})^{n+1} = ((xy^{n-k-1})^{n+1}$ for $k = 0, 1, \ldots, n - 2$

(vii) $(x^n)^{n+1}(x^{n-1})^{n+1} \ldots (x^2)^{n+1}x^{n+1} = x$

(viii) $(x^{n+1})^2 = x^{n+1}$

(ix) $((x_{k+1})^{n-k})^{n+1} = ((x_1^n)(x_2^{n-1}) \ldots (x_{t+1}^{n-t}) \ldots (x_{n-1}^2)(x_n^2))^{n-k})^{n+1}$ for $k = 0, 1, \ldots, n - 1$

**Proof:** The proof consists of straightforward calculations and can be found in [31].

3.4.1 The Equivalence of n-Set$_4$ to Set

**Definition 3.33** For $A \in$ n-Set$_4$ we define the set

$$D_A := \{x \in A | xx = x\} \subset A.$$

It follows from equation (vii) that the $D_i$ (as defined in Remark 3.31) give maps from $A$ to $D_A$.

**Lemma 3.34** Let $A$ be an n-Set$_4$-algebra. Then $A$ is isomorphic to $(D_A)^n$.

**Proof:** Define maps

$$\varphi : A \to (D_A)^n \text{ by } x \mapsto (D_1 x, \ldots, D_n x) = ((x^n)^{n+1}, (x^{n-1})^{n+1}, \ldots, x^{n+1})$$

and

$$\psi : (D_A)^n \to A \text{ by } (x_1, \ldots, x_n) \mapsto x_1 x_2 \ldots x_n$$

Then $\varphi \circ \psi = \text{id}_{(D_A)^n}$:

$$\varphi \circ \psi(x_1, \ldots, x_n)$$

$$= (((x_1 x_2 \ldots x_n)^n)^{n+1}, ((x_1 x_2 \ldots x_n)^{n-1})^{n+1}, \ldots, ((x_1 x_2 \ldots x_n)^1)^{n+1})$$

$$= ((x_1)^{n+1}, \ldots, (x_n)^{n+1}) \quad \text{(since all } x_i \in D_A \text{ we have } x_{t+1} = (x_{t+1})^{n-t})$$

$$= (x_1, \ldots, x_n) \quad \text{since all } x_i \in D_A$$
as well as $\psi \circ \varphi = \text{id}_A$:

$$\psi \circ \varphi(x) = (x^n)^{n+1}(x^{n-1})^{n+1} \cdots (x^2)^{n+1}x^{n+1} \overset{(vii)}{=} x$$

Hence $A \cong (D_A)^n$ in $\text{Set}$.

If we turn $(D_A)^n$ into an $n\text{-Set}_4$-algebra according to Lemma 3.31 then $\varphi$ is a homomorphism.

$$\varphi(xy) = (((xy)^n)^{n+1}, ((xy)^{n-1})^{n+1}, \ldots, (xy)^{n+1})$$

$$\overset{(vii)}{=} (((x)^{n+1}, ((y)^{n+1}, \ldots, ((y)^{2n+1}))$$

$$= \varphi(x)\varphi(y)$$

Thus $\varphi$ is an $n\text{-Set}_4$-isomorphism. \qed

**Proposition 3.35** $(n\text{-Set}_4, |-|)$ is concretely equivalent to $(\text{Set}, \text{hom}(n, -))$.

**Proof:** We define a functor $F : \text{Set} \to n\text{-Set}_4$ which maps each map $f : X \to Y$ to the $n\text{-Set}_4$-homomorphism $f^n : X^n \to Y^n$ where $X^n$ and $Y^n$ are defined according to Lemma 3.31. It is clear that $F$ is faithful.

Let $g : X^n \to Y^n$ be an $n\text{-Set}_4$-homomorphism. Then for $(x, \ldots, x) \in D_{X^n}$ all coordinates of $g(x, \ldots, x)$ are the same since we have

$$g(x, \ldots, x) = g((x, \ldots, x)(x, \ldots, x)) = g(x, \ldots, x)g(x, \ldots, x)$$

which implies that $g(x, \ldots, x) \in D_{X^n}$. Thus we can define a map $f : X \to Y$ which maps each $x \in X$ to the first coordinate of $g(x, \ldots, x)$. Obviously $f^n$ equals $g$ on $D_{X^n}$. But that implies that $f^n = g$ on the whole of $X^n$ by applying equation $(vii)$. Therefore $F$ is full.

$F$ is isomorphism dense by Lemma 3.34. Thus $F$ is an equivalence. Concreteness is obvious. \qed
Chapter 4

\textit{u}-Modification for Unary Varieties

In order to find all varieties equivalent to a given variety we first need to construct its matrix powers and then the \textit{u}-modification of each matrix power for each idempotent invertible unary operation \( u \) in the matrix power. Simplifying the construction of the matrix power was relatively easy to accomplish for all finitary varieties. The \textit{u}-modification is much more complex to describe. Here we just treat the case of unary varieties. But even in this case, which looks easy at first sight, we encounter problems and can only solve the problem of describing the \textit{u}-modification of unary varieties by adding further restrictions: we have to presume that all idempotents are central.

It is sufficient to consider \( M \)-acts instead of all unary varieties as we have shown in Lemma 1.4. From here on we will only consider \( M \)-acts. The properties of \( M \)-acts are well explored. In this chapter we use the results on \( M \)-acts from [18].

First we determine the varietal generators in \( M\text{Act} \) and thereby find the idempotent invertible unary operations \( u \) needed for the \textit{u}-modification. For a Clifford monoid \( M \) we define a variety \( \mathcal{V}(M, u) \) via explicitly given operations and equations. The main result of this chapter is Theorem 4.10 where we show that the variety \( \mathcal{V}(M, u) \) is the \textit{u}-modification of the \( n \)-th matrix powers of \( M\text{Act} \) if \( M \) is a Clifford monoid.

In the last section of this chapter we treat a more restricted question: Given a monoid \( M \), for which monoids \( N \) is the variety \( N\text{Act} \) equivalent to \( M\text{Act} \)? Knauer [19] and Banachschewsky [4] both independently found an answer to this question in 1972. We give a short proof of their result via categorical algebra.
4.1 Varietal Generators and Invertible Unary Operations

To determine the idempotent invertible unary operations \( u \) that are of interest for the \( u \)-modification we need to find all varietal generators in \( M\text-\text{Act} \).

Lemma 4.1 \( G \) is a varietal generator in \( M\text-\text{Act} \) if and only if

\[
G \cong \prod_{i=1}^{n} Me_i
\]

where each \( e_i \in M \) is idempotent and one \( e_k \) has the property \( Me_k M = M \).

Proof: For the proof we fall back upon known results on generators in \( M\text-\text{Act} \). All generators in \( M\text-\text{Act} \) are regular since all epimorphism are surjective in \( M\text-\text{Act} \). We need to find all projective generators that are presentable by finitely many generators and equations to find the varietal generators of a variety (cf. Remark 1.9). An \( M \)-act \( G \) is a projective generator if and only if

\[
G \cong \prod_{i=1}^{n} G_i
\]

for some \( n \in \mathbb{N} \) where every \( G_i \) is a cyclic projective act and at least one of them is a generator (cf. Proposition III.18.5 in [18]). \( G_i \) is a cyclic projective act if and only if \( G_i \cong Me_i \) for some idempotent \( e_i \in M \) (cf. Corollary III.17.9 in [18]; note that all cyclic \( M \)-acts are factor acts of \( M \) cf. Proposition I.5.17 in [18]). \( G_k \) is a cyclic projective generator in \( M\text-\text{Act} \) if and only if \( G_k \cong Me_k \) for some idempotent \( e_k \in M \) with \( Me_k M = M \) (cf. Proposition III.18.6 in [18]). Obviously \( \prod_{i=1}^{n} Me_i \) is generated by finitely many generators, i.e. the \( e_i \), and finitely many equations, i.e. for each \( i \in \{1, \ldots, n\} \) \( e_i e_i = e_i \) and \( Me_k M = M \) for one \( k \in \{1, \ldots, n\} \) (which can be expressed as one equation: there exist \( m, n \in M \) such that \( me_k n = 1 \)).

Lemma 4.2 The idempotent invertible unary operations \( u \) in the \( n \)-th matrix power of an \( M \)-act determined by the varietal generator \( G \cong \prod_{i=1}^{n} Me_i \) are of the form

\[
u = (e_1, \ldots, e_n),\]

i.e. each \( e_i \) acts as an idempotent operation on the \( i \)-th coordinate of the underlying set.

Proof: We know that the coequalizers of

\[
\begin{array}{ccc}
Fn & \xrightarrow{u} & Fn \\
\downarrow & \nearrow & \\
1 & \rightarrow & 1
\end{array}
\]
with $u$ an invertible idempotent endomorphism of $F n$ are the varietal generators (cf. Theorem 1.12). Thus according to Lemma 4.1

$$F n \xrightarrow{u} F n \xrightarrow{r} \bigsqcup_{i=1}^{n} M e_i$$

is a coequalizer in $M \text{Act}$. As already mentioned in Remark 1.16 $u = sr$ with $s$ the section such that $rs = 1$. This can also be seen in the following diagram:

$$
\begin{array}{ccc}
F n & \xrightarrow{u} & F n \\
\downarrow{r} & & \downarrow{1} \\
\bigsqcup_{i=1}^{n} M e_i & \xrightarrow{s} & F n \\
\downarrow{r} & & \downarrow{1} \\
\bigsqcup_{i=1}^{n} M e_i & \xrightarrow{r} & F n \\
\end{array}
$$

We receive a unique morphism $s$ such that $u = sr$. Since $r$ is the coequalizer of $(u, 1)$ we have $ru = r1 = r$. Thus we get $r = ru = rsr$. But we also have $r = 1r$. Hence all triangles in the diagram commute. Thus $rs = 1$ due to the uniqueness condition.

The $F n$ are of the form $\bigsqcup_{i=1}^{n} M x_i$ in $M \text{Act}$. We know how to construct coequalizers in $M \text{Act}$ (cf. Proposition II.2.21 in [18]):

$$F n \xrightarrow{u} F n \xrightarrow{r} F n / \nu$$

is a coequalizer in $M \text{Act}$ where $r$ is the canonical epimorphism and $\nu$ is the congruence relation generated by all pairs $(u(x), 1(x))$ for $x \in F n$. We also have

$$(F n) / \nu = \left( \bigsqcup_{i=1}^{n} M x_i \right) / \nu = \bigsqcup_{i=1}^{n} M e_i = \left( \bigsqcup_{i=1}^{n} M x_i \right) / \mu$$

with $\mu$ the congruence relation generated by the pairs $(e_i, x_i)$ for $i \in \{1, \ldots, n\}$. Thus we have $u(x_i) = x_i$ as well as $e_i = x_i$ hence $u(x_i) = e_i$ for all $i \in \{1, \ldots, n\}$.

\[\square\]

### 4.2 Clifford Monoids and $\mathcal{V}(M, u)$

**Definition 4.3** A monoid is called a Clifford monoid if all its idempotents are central, i.e. if each idempotent element commutes with all other elements.

**Lemma 4.4** Let $e$ be an idempotent element of a Clifford monoid $M$. Then $M e M = M$ implies that $e = 1$. 

Proof: Since $MeM = M$, there exist $m, n \in M$ such that $men = 1$. Thus $1 = men = meen = eme = e1 = e$. □

Remark 4.5 If we restrict ourselves to $M$-acts where $M$ is a Clifford monoid, the previous lemma guarantees one of the coordinates of an idempotent invertible operation $u$ as described in Lemma 4.2 to be 1. Without loss of generality we assume from here on that it is the first coordinate $e_1$ which equals 1.

Definition 4.6 For a Clifford monoid $M$ and an idempotent invertible operation $u = (e_1, \ldots, e_n)$ we define the variety $V(M, u)$ by the the following operations and equations:

- a unary operation $D_i$ for each $i \in \{1, \ldots, n\}$,
- each element of the monoid $\prod_{i=1}^n e_i Me_i$ acts as an unary operation,
- a binary operation $\circ_j$ for each $j \in \{1, \ldots, n\}$.

Equations:

(i) : all equations for $\prod_{i=1}^n e_i Me_i$-acts

(ii) : $D_1x \circ_1 D_3x \circ_2 D_3x \circ_3 \cdots \circ_{n-1} D_n x = x$

(iii) : $D_i(x_1 \circ_1 \cdots \circ_{n-1} x_n) = D_i x_i \quad \forall i \in \{1, \ldots, n\}$

(iv) : $D_i(\bar{m}x) = D_i(\bar{m}(D_i x)) \quad \forall i \in \{1, \ldots, n\}, \bar{m} \in \prod e_j Me_j$

(v) : $D_i(x \circ_j y) = \begin{cases} D_ix & \text{for } i = j \\ D_iy & \text{for } i \neq j \end{cases} \quad \forall i \in \{1, \ldots, n\}, \forall j \in \{1, \ldots, n - 1\}$

(vi) : $D_i D_i x = D_i x \quad \forall i \in \{1, \ldots, n\}$

(vii) : $D_i((l_1, \ldots, l_{i-1}, a, l_{i+1}, \ldots, l_n) x) = D_i((m_1, \ldots, m_{i-1}, a, m_{i+1}, \ldots, m_n) x) \quad \forall i \in \{1, \ldots, n\}, l_j, m_j \in e_j Me_j$

(viii) : $D_i D_j x = D_j(e_1, \ldots, e_{j-1}, e_j e_j, e_{j+1}, \ldots, e_n) x \quad \forall i, j \in \{1, \ldots, n\}$

(ix) : $D_j(m, e_2, \ldots, e_n) D_j x = D_j(e_1, \ldots, e_j e_j, e_{j+1}, \ldots, e_n) D_j x \quad \forall j \in \{1, \ldots, n\}, \forall m \in M$

(x) : $D_k(e_i, e_2, \ldots, e_n) D_i x = D_k D_i x \quad \forall i, j, k \in \{1, \ldots, n\}$

Definition 4.7 Let $A$ be an $M$-act and let $u = (e_1, \ldots, e_n)$ be an invertible idempotent operation as in Lemma 4.2. By $A(M, u)$ we denote the algebra which has the Cartesian product $\prod_{i=1}^n e_i A$ as underlying set and the following operations:

- Each element of the monoid $\prod_{i=1}^n e_i Me_i$ acts as an unary operation coordinatewise on $\prod_{i=1}^n e_i A$ such that the elements of the submonoids $e_i Me_i$
of $M$ act as unary operations on the subsets $e_iA$ of $A$ in the same way as in the $M$-act $A$, i.e. $(e,mie_i) \in \prod_{i=1}^n e_iMe_i$ acts on $(e_i a_i) \in \prod_{i=1}^n e_iA$ by 

$$ekmkekeka_k = ekmekeka_k = ekmka_k$$

in each coordinate $k$ (since the $e_k$ are central and idempotent).

- **Unary operations $D_i$** which are defined by

  $$D_i(x_1, \ldots, x_n) = (e_1x_i, e_2x_i, \ldots, e_nx_i)$$

  for all $i = 1, \ldots, n$.

- **Binary operations $\circ_i$** which are defined by

  $$(x_1, \ldots, x_n) \circ_i (y_1, \ldots, y_n) = (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n)$$

  for all $i = 1, \ldots, n - 1$.

**Lemma 4.8** $A(M, u)$ is a $V(M, u)$-algebra.

**Proof:** We just need to check the equations. (i) to (vii) are obvious.

$$(\text{viii}) \quad D_iD_jx = D_i(e_1x_j, \ldots, e_nx_j) = (e_1e_ix_j, \ldots, e_ne_ix_j) = (e_1e_je_ix_j, \ldots, e_ne_je_ix_j) (e_j \text{ is central and } e_jx_j = x_j) = D_j(e_1, \ldots, e_{j-1}, e_je_i, e_{j+1}, \ldots, e_n)x$$

$$(\text{ix}) \quad D_1(m, e_2, \ldots, e_n)D_jx = D_1(m, e_2, \ldots, e_n)(e_1x_j, \ldots, e_nx_j) = (e_1me_1x_j, \ldots, e_ne_1x_j) = (e_1e_je_1x_j, \ldots, e_ne_je_1x_j) (\text{since } e_1 = 1) = D_j(e_1, \ldots, e_{j-1}, e_je_j, e_{j+1}, \ldots, e_n)D_jx$$

$$(\text{x}) \quad D_1(e_k, e_2, e_3, \ldots, e_n)D_ix = D_1(e_k, e_2, e_3, \ldots, e_n)(e_1x_i, \ldots, e_nx_i) = (e_1e_ke_1x_i, \ldots, e_ne_ke_1x_i) = (e_1e_ke_1x_i, \ldots, e_ne_ke_1x_i) (\text{since } e_1 = 1) = D_k((e_1x_i, \ldots, e_nx_i) = D_kD_ix$$

$\square$
4.2.1 \( \mathcal{V}(M, u) \) is the \( u \)-Modification of \( M\text{Act}^n \)

**Lemma 4.9** For each \( M\)-act \( A \) the \( u \)-modification of its \( n \)-th matrix power \( A^n(u) \) belongs to \( \mathcal{V}(M, u) \).

**Proof:** Let \( A \) be an \( M \)-act. We show that \( A^{(n)}(u) = A(M, u) \). In Chapter 2 in Remark 2.11 we describe how to construct the matrix power \( A^n \). We get the Cartesian power \( A^2 \) in Remark 2.11 we describe how to construct the matrix power \( V_{4.2.1} \). Composing the unary operations acting coordinatewise on \( \prod \) acting coordinatewise and for each \( i \in \{1, \ldots, n\} \) we get an unary operation \( D_i \) defined by

\[
D_i(x_1, \ldots, x_n) = (x_i, \ldots, x_i)
\]

and a binary operation \( \circ_i \) defined by

\[
(x_1, \ldots, x_n) \circ_i (y_1, \ldots, y_n) = (y_1, \ldots, y_i-1, x_i, y_i+1, \ldots, y_n).
\]

The \( u \)-modification of this matrix power has the Cartesian product \( \prod_{i=1}^n e_iA \) as underlying set. The unary operations from the \( M \)-act structure have to be composed with \( u \) from the right, i.e. each coordinate by the corresponding \( e_i \). Thus in each coordinate \( i \) we apply a unary \( M \)-act operation \( m_i \) on some \( e_i a_i \) from the underlying set and then apply \( e_i \). But since the \( e_i \) are idempotent we have \( e_i m_i e_i a_i = e_i m_i e_i e_i a_i \). For each \( i \in \{1, \ldots, n\} \) \( e_i M e_i \) is a monoid. Thus the new unary operations correspond to the elements of the monoid \( \prod_{i=1}^n e_i M e_i \) acting coordinatewise on \( \prod_{i=1}^n e_i A \).

Composing the unary operations \( D_i \) with \( u = (e_1, \ldots, e_n) \) we get

\[
u D_i(x_1, \ldots, x_n) = u(x_i, \ldots, x_i) = (e_1 x_i, e_2 x_i, \ldots, e_n x_i).
\]

For the binary operations \( \circ_i \) we obtain

\[
u((x_1, \ldots, x_n) \circ_i (y_1, \ldots, y_n)) = u(y_1, \ldots, y_i-1, x_i, y_i+1, \ldots, y_n) = (e_1 y_1, \ldots, e_i y_i-1, e_i x_i, e_i+1 y_i+1, \ldots, e_n y_n) = (y_1, \ldots, y_i-1, x_i, y_i+1, \ldots, y_n)
\]

since \( (x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^n e_i A \).

Thus we have shown that \( A^{(n)}(u) \) equals \( A(M, u) \) as defined in Definition 4.7. Hence \( A^{(n)}(u) \) is a \( \mathcal{V}(M, u) \)-algebra as shown in Lemma 4.8.

\[\square\]

**Theorem 4.10** Let \( M \) be a Clifford monoid, then \( \mathcal{V}(M, u) \) is the \( u \)-modification of the \( n \)-th matrix power of \( M\text{Act} \).

**Proof:** As we have already shown in Lemma 4.9, the \( u \)-modification of the \( n \)-th matrix power \( A^n(u) \) of an \( M \)-act \( A \) is a \( \mathcal{V}(M, u) \)-algebra. It remains to show that each \( \mathcal{V}(M, u) \)-algebra is the \( u \)-modification of the \( n \)-th matrix power of an \( M \)-act. The proof consists of two parts:
4.2 Clifford Monoids and $\mathcal{V}(M, u)$

- In part a) we show that each algebra in $\mathcal{V}(M, u)$ is isomorphic to a product of $M$-acts.
- In part b) we show that this product is the $u$-modification of the $n$-th matrix power of an $M$-act.

a) In this part we examine the structure of $\mathcal{V}(M, u)$-algebras. Let $X$ be a $\mathcal{V}(M, u)$-algebra. For each $i \in \{1, \ldots, n\}$ we define the set

$$ D_i := \{ x \in X | D_i x = x \}. $$

We can turn the $D_i$ into $M$-acts by defining for each $m \in M$ the unary operation

$$ m^i x := D_i(e_1, \ldots, e_{i-1}, e_i me_i, e_{i+1}, \ldots, e_n)x \quad \text{for } x \in D_i. $$

For the definition of the $m^i$ it is essential that the $e_i$ are central, otherwise the $m^i$ are not $M$-act operations. It is straightforward to check that the operations $m^i$ satisfy the equations for $M$-acts.

Now we turn the product $D := \prod_{i=1}^n D_i$ into a $\mathcal{V}(M, u)$-algebra by defining the operations as follows:

- For each $\bar{m} = (m_1, \ldots, m_n) \in \prod_{i=1}^n e_i Me_i$ a unary operation

$$ \bar{m}^D(x_1, \ldots, x_n) := (D_1 \bar{m} x_1, \ldots, D_n \bar{m} x_n) $$

(note that we have $(D_1 \bar{m} x_1, \ldots, D_n \bar{m} x_n) = (m_1 x_1, \ldots, m_n x_n)$ due to equation (vii), hence this definition is compatible with the $M$-act structure of the $D_i$).

- A unary operation $D_i^D$ for each $i \in \{1, \ldots, n\}$ defined by

$$ D_i^D(x_1, \ldots, x_n) := (D_1 x_i, D_2 x_i, \ldots, D_n x_i). $$

- For each $i \in \{1, \ldots, n\}$ a binary operation $\circ_i^D$ defined by

$$ (x_1, \ldots, x_n) \circ_i^D (y_1, \ldots, y_n) := (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n). $$

We check that these operations fulfil the equations for $\mathcal{V}(M, u)$-algebras:

(i) the equations for $\prod_{i=1}^n e_i Me_i$-acts:

- $(e_1, \ldots, e_n)^D$ is obviously the neutral element,
- $\bar{m}^D n^D = m^D \bar{n}$ follows from equation (iv).

(ii) is trivial since $D_i x_i = x_i$

(iii) for $i \in \{1, \ldots, n\}$
\[ D_i^D (\bar{x}_1 \circ D_1 \circ \cdots \circ D_{n-1} \bar{x}_n) = D_i^D (x_1, x_2, \ldots, x_{nn}) = (D_1 x_{ii}, \ldots, D_n x_{ii}) = D_i \bar{x}_i \]

(iv) for \( i \in \{1, \ldots, n\} \)

\[ D_i^D ((m)^D (x_1, \ldots, x_n)) = (D_1 D_i m x_i, \ldots, D_n D_i m x_i) = D_i^D (D_1^X m x_i, \ldots, D_n^X m x_i) = D_i^D ((m)^D (D_i^D \bar{x})) \]

(v) clear

(vi) clear

(vii) clear

(viii) Let \( m = (e_1, \ldots, e_j, e_j, \ldots, e_n) \) then for all \( i, j \in \{1, \ldots, n\} \)

\[ D_i^D D_j^D \bar{x} = (D_1 D_i D_j x_j, \ldots, D_n D_i D_j x_j) = (D_1 D_j m x_j, \ldots, D_n D_j m x_j) = D_j^D (D_1 m x_1, \ldots, D_n m x_n) = D_j^D m^D \bar{x} \]

(ix) Let \( m_1 = (m, e_2, \ldots, e_n) \) and \( m_2 = (e_1, \ldots, e_j m e_j, \ldots, e_n) \)

for some \( m \in M \) then for all \( j \in \{1, \ldots, n\} \)

\[ D_i^D m_1^D D_j^D \bar{x} = D_i^D (D_1 m_1 D_1 x_j, \ldots, D_n m_1 D_n x_j) = (D_1 D_1 m_1 D_1 x_j, \ldots, D_n D_1 m_1 D_1 x_j) = (D_1 D_1 m_1 x_j, \ldots, D_n D_1 m_1 x_j) = D_j^D (D_1 m_2 D_1 x_j, \ldots, D_n m_2 D_1 x_j) = D_j^D m_2^D D_j^D \bar{x} \]

(x) Let \( m = (e_k, e_2, e_3, \ldots, e_n) \) then for all \( i \in \{1, \ldots, n\} \)
4.2 Clifford Monoids and $\mathcal{V}(M,u)$

Next we show that the $\mathcal{V}(M,u)$-algebra $X$ is isomorphic to $D := \prod_{i=1}^{n} D_i$. We define morphisms

$$\varphi : X \rightarrow D \quad \text{by} \quad x \mapsto (D_1x, \ldots, D_nx)$$

and

$$\psi : D \rightarrow X \quad \text{by} \quad (x_1, \ldots, x_n) \mapsto x_1 \circ_{o_1 \cdot \cdot \cdot o_{n-1}} x_n.$$ 

$\varphi$ and $\psi$ are inverse to each other:

$$\psi \circ \varphi(x) = D_1x \circ_{o_1 \cdot \cdot \cdot o_{n-1}} D_nx \overset{(ii)}{=} x$$

as well as

$$\varphi \circ \psi(x_1, \ldots, x_n) \overset{(iii)}{=} (D_1(x_1 \circ_{o_1 \cdot \cdot \cdot o_{n-1}} x_n), \ldots, D_n(x_1 \circ_{o_1 \cdot \cdot \cdot o_{n-1}} x_n))$$

$$\overset{=}{} = (D_1x_1, \ldots, D_nx_n)$$

$$\overset{}{=} = (x_1, \ldots, x_n) \quad \text{since} \quad x_i \in D_i.$$

Thus $\varphi$ is a bijection. We show that $\varphi$ is a homomorphism. $\varphi$ is compatible with the monoid actions:

$$\varphi(\bar{m}x) \overset{(iv)}{=} (D_1(\bar{m}x), \ldots, D_n(\bar{m}x))$$

$$\overset{}{=} = (D_1mD_1x, \ldots, D_nmD_nx)$$

$$\overset{}{=} = \bar{m} D(D_1x, \ldots, D_nx)$$

$$\overset{}{=} = \bar{m} D \varphi(x)$$

$\varphi$ is compatible with each $o_i$:

$$\varphi(x \circ_i y) \overset{}{=} = (D_1(x \circ_i y), \ldots, D_n(x \circ_i y))$$

$$\overset{}{=} = (D_1y, \ldots, D_{i-1}y, D_ix, D_{i+1}y, \ldots, D_ny)$$

$$\overset{}{=} = (D_1x, \ldots, D_nx) o_i D(D_1y, \ldots, D_ny)$$

$$\overset{}{=} = \varphi(x) o_i D \varphi(y)$$
\( \varphi \) is compatible with each \( D_i \):
\[
\varphi(D_i x) = (D_1 D_i x, \ldots, D_n D_i x) = D_i^D \varphi(x).
\]
Thus \( \varphi \) is a \( \mathcal{V}(M, u) \)-isomorphism between \( X \) and \( D \).

b) It remains to show that the \( \mathcal{V}(M, u) \)-algebra \( X \) is the \( u \)-modification of a matrix power of an \( M \)-act. For this purpose we show that the \( u \)-modification of the \( n \)-th matrix power of the \( M \)-act \( D_1 \) is the \( \mathcal{V}(M, u) \)-algebra \( D \) as defined in part a). We use \( D_1 \) since its unary operations are simpler than the unary operations of the other \( D_i \). Because \( e_1 = 1 \) the unary operations \( m^1 \) of \( D_1 \) corresponding to the elements \( m \) of \( M \) are defined by
\[
m^1 x = D_1(m, e_2, e_3, \ldots, e_n) x.
\]
We define an algebra \( MP \) with underlying set \( (D_1)^n \) and the following operations:

- For each \((m_1, \ldots, m_n) \in M^n\) we define an unary operation \((m_1, \ldots, m_n)^{MP}\) by
  \[
  (m_1, \ldots, m_n)^{MP}(x_1, \ldots, x_n) := (m_1^1 x_1, \ldots, m_n^1 x_n).
  \]
- For each \( m \in M \) we define an unary operation \( m^{MP} \) by
  \[
  m^{MP}(x_1, \ldots, x_n) := (m, \ldots, m)^{MP}(x_1, \ldots, x_n) = (m^1 x_1, \ldots, m^1 x_n).
  \]
- For each \( i \in \{1, \ldots, n\} \) we define an unary operation \( D_i^{MP} \) by
  \[
  D_i^{MP}(x_1, \ldots, x_n) := (x_i, \ldots, x_i).
  \]
- For each \( i \in \{1, \ldots, n\} \) we define a binary operation \( \circ_i^{MP} \) by
  \[
  (x_1, \ldots, x_n) \circ_i^{MP} (y_1, \ldots, y_n) := (y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n).
  \]
Thus \( MP \) is \((D_1)^n\) with the \((m_1, \ldots, m_n)^{MP}\) as extra operations, i.e. \( MP \) is the \( n \)-th matrix power of \( D_1 \) (cf. Remark 2.11).

Now we construct the \( u \)-modification of \( MP \). First examine the underlying set
\[
u((D_1)^n) = \prod_{i=1}^n e_i^1 D_1.
\]
We claim that \( e_i^1 D_1 = D_i \) as a set for each \( i \in \{1, \ldots, n\} \). First we show \( e_i^1 D_1 \subset D_i \). Using equation (viii) with \( j = 1 \) we get for \( x \in e_i^1 D_1 \):
\[
x = e_i^1 x = D_1(e_i, e_2, \ldots, e_n) x = D_i D_1 x = D_i x.
\]
Thus \( x \in D_i \) and therefore \( e_i^1 D_1 \subset D_i \).
4.2 Clifford Monoids and $\mathcal{V}(M, u)$

To show that $D_i \subset e_1^i D_1$, let $x \in D_i$. Then we have $D_i x = x$ and $e_1^i x \in e_1^i D_i$ which gives us

$$e_1^i x = D_i(e_1, \ldots, e_i e_1 e_i, \ldots, e_n) x \overset{(vi)}{=} D_1 D_i x = D_1 x = x. \quad (*)$$

Thus $x = e_1^i x = D_1 x \in D_1$.

$$e_1^i D_1 x = D_1(e_i, e_2, \ldots, e_n) D_1 x 
\overset{(x)}{=} D_i D_1 x = D_i D_i(\ldots, e_i e_1 e_i, \ldots) x \overset{\text{see above}(*)}{=} D_i(\ldots) x = e_1^i x = D_1 x = x$$

Thus $x = D_1 x \in e_1^i D_1$. Therefore $D_i \subset e_1^i D_1$ which together with the above gives $e_1^i D_1 = D_i$ as claimed.

Hence the underlying set of the $u$-modification of $MP$ is the same as the underlying set of $D$

$$u((D_1)^n) = \prod_{i=1}^{n} e_1^i D_1 = \prod_{i=1}^{n} D_i.$$

The only thing left to show is that the composition of the $MP$ operations with $u$ gives the $D$ operations. First we consider

$$e_1^i m_1^i x_j = (e_j m_j)^i x_j = D_1(e_j m_j, \ldots) x_j$$

$x_j \in D_j \overset{\text{def}}{=} D_1(e_j m_j, \ldots) D_j x_j$

$$\overset{(ix)}{=} D_j(\ldots, e_j m_j e_j, \ldots) D_j x_j = m_j^i x_j.$$

Using this we treat the unary operations $\bar{m}^{MP}$ for $\bar{m} = (m_1, \ldots, m_n) \in M^n$

$$u^{MP} \bar{m}^{MP}(x_1, \ldots, x_n) = (e_1^1, \ldots, e_n^1)(m_1^1, \ldots, m_n^1)^{MP}(x_1, \ldots, x_n)$$

$$= (e_1^1 m_1^1 x_1, \ldots, e_n^1 m_n^1 x_n)$$

$$\overset{(vi)}{=} (m_1^1 x_1, \ldots, m_n^1 x_n)$$

Next we treat the $D_i$: first we consider

$$e_1^i x_i = D_1(e_k, e_2, \ldots, e_n) x_i = D_1(e_k, \ldots) D_i x_i \overset{(x)}{=} D_k D_i x_i = D_k x_i.$$
And last the $c_i$:
\[
\begin{align*}
\mu^{MP}(\bar{x} \circ_i^{MP} \bar{y}) &= \mu^{MP}(y_1, \ldots, y_{i-1}, x_i, y_{i+1}, \ldots, y_n) \\
&= \left( e_1^1y_1, \ldots, e_{i-1}^1y_{i-1}, e_i^1x_i, e_{i+1}^1y_{i+1}, \ldots, y_n \right) \\
&= (D_1y_1, \ldots, D_{i-1}y_{i-1}, D_ix_i, D_{i+1}y_{i+1}, \ldots, D_ny_n) \\
&= (\bar{x} \circ_i^D \bar{y})
\end{align*}
\]

Thus we have proven that up to isomorphism each algebra in $\mathcal{V}(M, \mu)$ is the $\mu$-modification of an $n$-th matrix power. 

\[\square\]

**Remark 4.11** In the proof it becomes obvious why we need to restrict ourselves to $M$-acts where $M$ is a Clifford monoid:

- In part a) the $D_i$ are only $M$-acts if the $e_i$ are central. We have encountered constructions similar to these $D_i$ in Chapter 2 and Chapter 3 and will encounter it again in Chapter 5. They have always been essential in showing the underlying structure of our algebras. We have found no way to do without them.

- In part b) we use $e_1 = 1$. Otherwise too much information might be lost when applying the unary operation $\mu$. But in this case all information can be pertained by copying into the first coordinate (via the appropriate $D_i$) before applying $\mu$.

### 4.3 Equivalence for $M$-Acts

In the previous sections of this chapter we have searched for all varieties equivalent to a given variety of $M$-acts. Here we treat a more restricted question: Given a monoid $M$, for which monoids $N$ is the variety $N\mathbf{Act}$ equivalent to $M\mathbf{Act}$? The answer to this question is a lot easier and there are several ways to find it. Knauer [19] and Banachschewsky [4] both independently found the following answer in 1972:

**Theorem 4.12** For monoids $M$ and $N$ the following are equivalent:

(i) The varieties $M\mathbf{Act}$ and $N\mathbf{Act}$ are equivalent.

(ii) $N$ is isomorphic to $uMu$ for some idempotent $u \in M$ for which exist $d, p \in M$ such that $pd = e_M$ and $ud = d$.

The same result can be easily achieved by using categorical algebra: The basis for our previous results was finding the varietal generators. We had to find all varietal generators in $M\mathbf{Act}$ in order to find all varieties equivalent to $M\mathbf{Act}$. But not all varieties which are equivalent to $M\mathbf{Act}$ are of the form $N\mathbf{Act}$ for a
monoid $N$. Can we find special varietal generators which only give varieties of the desired form? The answer is positive.

Varieties of the form $N\text{Act}$ can be characterized by the fact that their underlying functors preserve coproducts: All unary varieties are up to concrete isomorphism $M$-acts (cf. Lemma 1.4) and as we have shown in Lemma 1.5 unary varieties are characterized by the fact that their underlying functors preserve coproducts. Thus according to Theorem 1.11 a varietal generator $G$ gives rise to a variety of the form $N\text{Act}$ (via $\text{Th}_{M\text{Act}}(G)$) if and only if $\text{hom}_{M\text{Act}}(G, -)$ preserves coproducts.

If $G$ is a retract of $F_1$ then $\text{hom}_{M\text{Act}}(G, -)$ preserves coproducts since $\text{hom}_{M\text{Act}}(F_1, -)$ preserves coproducts. If $G$ is not a retract of $F_1$ but only of $F_n, n \geq 2$ then $\text{hom}_{M\text{Act}}(G, -)$ does not preserve coproducts. For example let $G$ be an retract of $F_2$ then $G$ can be decomposed into $G = G_1 + G_2$ for non-empty $M$-acts $G_1$ and $G_2$.

If $\text{hom}_{M\text{Act}}(G, -)$ preserves coproducts then the decomposition $G = G_1 + G_2$ implies that

$$
\begin{array}{ccc}
\text{hom}(G, G_1) & \xrightarrow{\text{hom}(G, \mu_1)} & \text{hom}(G, G) \\
\text{hom}(G, G_2) & \xrightarrow{\text{hom}(G, \mu_2)} & \text{hom}(G, G)
\end{array}
$$

is a coproduct sink in $\textbf{Set}$. But that is a contradiction since it implies that the identity of $G$ factors over one of the $G_i$.

$$
F_1 \xrightarrow{s} mG \xrightarrow{r} F_1 = \text{id}_{F_1}
$$

The image of $s$ lies in one of the copies of $G$ thus the restriction of $r$ to $G$ also gives a retraction. Therefore $n = m = 1$ in Theorem 1.12 if we are looking for varieties of the form $N\text{Act}$.

We know the right generators which makes it easy to determine the monoids in question. We have $\text{Th}_{N\text{Act}}(F_1) \cong \text{Th}_{M\text{Act}}(G)$ and

$$
\text{hom}_{N\text{Act}}(F_1)(F_1, F_1) = \text{End}_{N\text{Act}}(F_1)^{op} \cong F_1 = N
$$

Thus $N \cong \text{End}_{M\text{Act}}(G)$. We have a retraction

$$
F_1 \xrightarrow{s} G \xrightarrow{r} F_1 = \text{id}_{F_1} \text{ and } u := s \circ r
$$

which gives rise to a injective map $\text{End}(G) \to \text{End}(F_1) \cong M$ via $f \mapsto sfr$ with image $u\text{End}(F_1)u$. Hence

$$
N \cong \text{End}_{M\text{Act}}(G) \cong u\text{End}_{M\text{Act}}(F_1)u \cong uMu.
$$

This result can also be found in [29].
Chapter 4. \textit{u}-Modification for Unary Varieties
Chapter 5

Morita Equivalence for Boolean Algebras

In this chapter we make the construction of the matrix power and u-modification of the variety BOOL of Boolean algebras more explicit than they are given by Porst in [27]. Porst uses the equivalence of BOOL to the variety of Post algebras as his leading example to illustrate his methods of constructing the matrix power and u-modification. Here we concentrate on writing down the constructions explicitly by giving a complete characterization through operations and equations.

As main result we get that the varieties equivalent to the variety BOOL of Boolean algebras are (up to concrete isomorphism) precisely the varieties $\text{POST}_n$, the varieties of Post algebras of order $n$, for natural numbers $n \geq 2$.

At the end of the chapter we shortly show the connection to Hu’s primal algebra theorem [13, 14]. Hu’s results follow and are even refined as a corollary of the results presented in this chapter.

This whole chapter serves as an example illustrating the principles of the constructions used throughout this thesis. It is especially well suited since it is easy to see how the operations act on the underlying sets. The u-modification is a lot easier to construct in this case than in the more general case of unary varieties in Chapter 4.

A Boolean algebra is a complemented distributive lattice with 0 and 1. We refer to [3] and [30] as introductionary texts on Boolean algebras or Post algebras since we use results from both books.

5.1 The Matrix Powers of BOOL

We construct the $n$-th matrix power $\text{BOOL}^{(n)}$ for natural numbers $n \geq 2$ as defined in Chapter 2. For each Boolean algebra $(B, \land, \lor, \neg, 0, 1)$ and each natural number, we get a new algebra in $\text{BOOL}^{(n)}$ which has the $n$-th Cartesian
power $B^n$ as underlying set. The “old” operations $\land, \lor, \neg, 0$ and 1 operate coordinatewise. We obtain a new binary operation which operates as 

$$(x_1, \ldots, x_n)(y_1, \ldots, y_n) = (x_n, y_1, \ldots, y_{n-1}).$$

and we get a new unary operation $D_i$ for each $i \in \{1, \ldots, n\}$ which acts as 

$$D_i(x_1, \ldots, x_n) = (x_i, \ldots, x_i)$$

(see Definition 2.9 and Remark 2.11). It is easy to see that composition of the two nullary operations 1 and 0 with the new binary operation gives $n+1$ nullary operations in $\text{BOOL}^{(n)}$

$$c_i = (1, \ldots, 1, 0, \ldots, 0)$$

for $i \in \{0, \ldots, n\}$ where $c_0 = 0$ and $c_n = 1$.

### 5.2 The $u$-Modification

The unary invertible idempotent operations $u$ in the $n$-th matrix power of $\text{BOOL}$ needed for its $u$-modification are known (see Porst [27], leading example (d)). They are the unary operations $u$ which operate as 

$$u(x_1, \ldots, x_n) = (x_1, x_1 \land x_2, \ldots, x_1 \land \cdots \land x_n).$$

It is relatively easy to see what happens if we take the $u$-modification of $\text{BOOL}^{(n)}$. Applying $u$ to the underlying set leaves us with a subset of the original set where each element is an $n$-tuple $x = (x_1, \ldots, x_n)$ satisfying $x_n \leq x_{n-1} \leq \cdots \leq x_1$. The operations are the operations we already have in $\text{BOOL}^{(n)}$, restricted to this new underlying set, composed with $u$. The operations $\land$ and $\lor$ which operate coordinatewise in $\text{BOOL}^{(n)}$ are not changed by composition with $u$:

$$u(x \land y) = u(x_1 \land y_1, \ldots, x_n \land y_n) = (x_1 \land y_1, (x_1 \land y_1) \land (x_2 \land y_2), \ldots, (x_1 \land y_1) \land \cdots \land (x_n \land y_n)) = (x_1 \land y_1, (x_1 \land x_2) \land (y_1 \land y_2), \ldots, (x_1 \land \cdots \land x_n) \land (y_1 \land \cdots \land y_n)) = (x_1 \land y_1, \ldots, x_n \land y_n) = x \land y$$

Thus $u(x \land y) = x \land y$ and we also get $u(x \lor y) = x \lor y$:

$$u(x \lor y) = u(x_1 \lor y_1, \ldots, x_n \lor y_n) = (x_1 \lor y_1, (x_1 \lor y_1) \land (x_2 \lor y_2), \ldots, (x_1 \lor y_1) \land \cdots \land (x_n \lor y_n)).$$
We restrict ourselves to showing what happens in the second coordinate. The other coordinates follow by induction.

\[
(x_1 \lor y_1) \land (x_2 \lor y_2) = ((x_1 \lor y_1) \land x_2) \lor ((x_1 \lor y_1) \land y_2)
\]

\[
= (x_1 \land x_2) \lor (y_1 \land x_2) \lor (x_1 \land y_2) \lor (y_1 \land y_2)
\]

\[
= x_2 \lor y_2 \lor (y_1 \land x_2) \lor (x_1 \land y_2)
\]

\[
= x_2 \lor y_2 \lor ((y_1 \land x_2) \lor x_1) \land ((y_1 \land x_2) \lor y_2)
\]

\[
= x_2 \lor y_2 \lor (x_1 \land y_2)
\]

\[
= x_2 \lor y_2
\]

due to absorption.

The operations \(D_i\) for \(i \in \{1, \ldots, n\}\) are obviously not changed by composition with \(u\). Neither are the operations \(c_i\) for \(i \in \{0, \ldots, n\}\) changed by composition with \(u\).

The complementation \(\neg\) in \(\text{BOOL}^{(n)}\) gives rise to a new operation \(C\) when composed with \(u\):

\[
C = u \circ \neg
\]

which acts as

\[
C(x_1, \ldots, x_n) = (\neg x_1, \neg x_1 \land \neg x_2, \ldots, \neg x_1 \land \cdots \land \neg x_n)
\]

\[
= (\neg x_1, \ldots, \neg x_1).
\]

We show that the underlying set as defined above together with the operations

\(\lor, \land, C, D_1, \ldots, D_n, c_0, \ldots, c_n\)

is a Post algebra. This is easy to show when we take the following definition of Post algebras by operations and equations.

### 5.3 The Equivalence of Post Algebras and Boolean Algebras

The definition of Post algebras as in [3] or [30] is equivalent to the following description by axioms which were given by Traczyk in [32] (cf. [3]).

**Definition 5.1** The \((2n + 5)\)-tuple

\[
\mathcal{P}_{n+1} = (P, \lor, \land, C, D_1, \ldots, D_n, c_0, \ldots, c_n)
\]

where \(P\) is a set, \(\land\) and \(\lor\) are binary operations, \(C, D_1, \ldots, D_n\) unary operations and \(c_0, \ldots, c_n\) nullary operations, is a Post algebra of order \(n + 1\) if it satisfies
the following equations:

1. \((P, \lor, \land)\) is a distributive lattice*
2. \(c_0 \lor x = x\)
   \[c_i \land c_j = c_i \quad \text{for} \quad 1 \leq i \leq j \leq n\]
   \[x \land c_n = x\]
3. \(C x \land D_1 x = c_0, \quad C x \lor D_1 x = c_n\)
4. \(D_i x \land D_j x = D_j x \quad \text{for} \quad 1 \leq i \leq j \leq n\)
5. \(D_i (x \lor y) = D_i x \lor D_i y, \quad D_i (x \land y) = D_i x \land D_i y \quad \text{for} \quad 1 \leq i \leq n\)
6. \(D_i D_j x = D_j x \quad \text{for} \quad 1 \leq i, j \leq n\)
7. \(D_i c_j = \begin{cases} c_n & \text{if} \ i \leq j \\ c_0 & \text{if} \ i > j \end{cases} \quad i, j \in \{1, \ldots, n\}\)
8. \(x = (D_1 x \land c_1) \lor \cdots \lor (D_n x \land c_n)\)

Lemma 5.2

\[\mathcal{P}_{n+1} = (P, \lor, \land, C, D_1, \ldots, D_n, c_0, \ldots, c_n)\]

with the underlying set and the operations as given in the u-modification above satisfies the equations (1) to (8) given in Definition 5.1. Thus it is a Post algebra of order \(n + 1\).

Proof: 1) \((P, \lor, \land)\) is a distributive lattice with unit element \(1 = c_n\) and zero element \(0 = c_0\) since the operations \(\lor\) and \(\land\) are not changed by composition with \(u\) and thus still satisfy the same equations as before.

2) It is easy to see that
   \[c_0 \lor x = (0 \land x_1, \ldots, 0 \land x_n) = (x_1, \ldots, x_n) = x\]
   and for \(1 \leq i \leq j \leq n\) we see that
   \[c_i \land c_j = \left(1 \land \underbrace{1, \ldots, 1}_{\text{i times}}, 0 \land 0, \ldots, 0 \land 0\right)\]
   \[= \left(1, \ldots, 1, 0 \ldots, 0\right) = c_i\]
   as well as
   \[x \land c_n = (x_1 \land 1, \ldots, x_n \land 1) = x.\]

3) The first part of equation (3):
   \[C x \land D_1 x = (\neg x_1, \ldots, \neg x_1) \land (x_1, \ldots, x_1)\]
   \[= (0, \ldots, 0) = c_0.\]

*with unit element \(1 = c_n\) and zero element \(0 = c_0\) (as follows from equation (2)).
5.3 The Equivalence of Post Algebras and Boolean Algebras

The second part:

$$C x \lor D_1 x = (\neg x_1 \lor x_1, \ldots, \neg x_1 \lor x_1) = c_n.$$  

4) For $1 \leq i \leq j \leq n$

$$D_i x \land D_i y = (x_i, \ldots, x_i) \land (x_j, \ldots, x_j) = (x_j, \ldots, x_j),$$

since $x_j = (x_1 \land \cdots \land x_i \land \cdots \land x_j)$ due to the structure of the underlying set.

5) For $1 \leq i \leq n$

$$D_i(x \lor y) = (x \lor y_i, \ldots, x \lor y_i) = (x_i, \ldots, x_i) \lor (y_i, \ldots, y_i) = D_i y \lor D_i y.$$ Analogously for $\land$.

6) Checking equation (6) is straightforward and we have already shown it in Chapter 2 and Chapter 3 where we use $D_i$ which are defined in the same way.

7) Obvious.

8) For each $i \in \{1, \ldots, n\}$

$$D_i x \land c_i = \left(x_i \land 1, \ldots, x_i \land 1, x_i \land 0, \ldots, x_i \land 0\right) = \left(x_i, \ldots, x_i, 0, \ldots, 0\right).$$

Thus we get

$$(D_1 x \land c_1) \lor \cdots \lor (D_n x \land c_n)$$

$$= (x_1, 0, \ldots, 0) \lor \cdots \lor (x_i, \ldots, x_i, 0, \ldots, 0) \lor \cdots \lor (x_n, \ldots, x_n)$$

$$= (x_1 \lor \cdots \lor x_n, 0 \lor x_2 \lor \cdots \lor x_n, \ldots, 0 \lor \cdots \lor 0 \lor x_n)$$

$$= (x_1, \ldots, x_n)$$

$$= x$$

The lemma shows that the $u$-modification of the $n$-th matrix power of a Boolean algebra is a Post algebra of order $n + 1$. The reverse holds as well:

**Lemma 5.3** Each Post algebra of order $n + 1$ ($n \in \mathbb{N}$) is isomorphic to the $u$-modification of an $n$-th matrix power of a Boolean algebra.

**Proof:** Let

$$\mathcal{P}_{n+1} = (P, \lor, \land, C, D_1, \ldots, D_n, c_0, \ldots, c_n)$$

be a Post algebra of order $n + 1$ as defined in 5.1. We define a subset $D_P$ of $P$ by

$$D_P = \{x \in P | D_1 x = x\}.$$
Using equation (6) one readily checks that for each $i \in \{1, \ldots, n\}$ the unary operation $D_i$ provides a map from $P$ to $D_P$. We claim that

$$B = (D_P, \lor, \land, \neg, 0, 1)$$

is a Boolean algebra if we define $\neg := C|_{D_P}$, $0 := c_0$ and $1 := c_n$. $D_P$ is closed under the operations $\land$ and $\lor$ due to equation (5). $c_0$ is in $D_P$ since due to equation (7) $D_1c_0 = c_0$. Equation (7) also provides $D_1c_n = c_n$ thus $c_n$ is in $D_P$. Equation (2) guarantees that $c_0$ acts as 0 and that $c_n$ acts as 1. According to equations (3) and (5) we have $D_1Cx \land D_1D_1x = D_1c_0$. Equation (7) also provides $D_1Cx$ is the complement of each $x$ in $D_P$. Because of equation (3) $C$ is also the complement of each $x$ in $D_P$. But complements are unique in distributive lattices, hence we have $D_1Cx = Cx$ for each $x$ in $D_P$ and thus $D_P$ is closed under $C$. Since $(D_P, \lor, \land)$ is a distributive lattice by equation (1), $B$ is indeed a Boolean algebra.

If we construct the $u$-modification of the $n$-th matrix power $B(n)(u)$ of $B$ as we have done above, we get the subset $\hat{D}_nP$ of $DnP$ where each element $x = (x_1, \ldots, x_n)$ satisfies $x_n \leq x_{n-1} \leq \cdots \leq x_1$ as underlying set and the following operations:

- the binary operations $\land$ and $\lor$ which operate coordinatewise:
  $$(x_1, \ldots, x_n) \land (y_1, \ldots, y_n) = (x_1 \land y_1, \ldots, x_n \land y_n) \text{ and}$$
  $$(x_1, \ldots, x_n) \lor (y_1, \ldots, y_n) = (x_1 \lor y_1, \ldots, x_n \lor y_n)$$

- for each $i \in \{1, \ldots, n\}$ a unary operation $D_i$ which operates like this
  $$D_i(x_1, \ldots, x_n) = (x_i, \ldots, x_i)$$

- the unary operation $C$ which acts as
  $$C(x_1, \ldots, x_n) = (\neg x_1, \neg x_1 \land \neg x_2, \ldots, \neg x_1 \land \cdots \land \neg x_n)$$
  $$= (\neg x_1, \ldots, \neg x_1)$$

- for each $i \in \{0, \ldots, n\}$ a nullary operation $c_i$ given by
  $$c_i = (1, \ldots, 1, 0, \ldots, 0)$$

as we have seen above. We have already shown in Lemma 5.2 that this is a Post algebra of order $n + 1$.

It remains to show that $P_{n+1}$ is isomorphic to $B(n)(u)$. For this purpose we define the maps

$$\varphi : P \to \hat{D}_nP \quad \text{by} \quad x \mapsto (D_1x, \ldots, D_nx)$$
5.3 The Equivalence of Post Algebras and Boolean Algebras

and

\[ \psi : \tilde{D}_p^n \rightarrow P \text{ by } (x_1, \ldots, x_n) \mapsto (x_1 \land c_1) \lor \cdots \lor (x_n \land c_n). \]

Please observe that \( D_n x \leq D_{n-1} x \leq \cdots \leq D_1 x \) due to equation (4). Also be warned that we use the same operation symbols in \( B^{(n)}(u) \) and \( P_{n+1} \) as an attempt to avoid too many indices. It should always be clear from the context which one is meant. We show that \( \varphi \) and \( \psi \) are inverse to each other:

\[
\begin{align*}
\varphi \circ \psi(x_1, \ldots, x_n) &= (D_1((x_1 \land c_1) \lor \cdots \lor (x_n \land c_n)), \ldots, D_n((x_1 \land c_1) \lor \cdots \lor (x_n \land c_n))) \\
&= (D_1(x_1 \land D_1c_1) \lor \cdots \lor (D_1x_n \land D_1c_n), \ldots, (D_nx_1 \land D_nc_1) \lor \cdots \lor (D_nx_n \land D.nc_n)) \\
&\overset{(5)}{=} ((x_1 \land c_n) \lor \cdots \lor (x_n \land c_n), \ldots, (x_1 \land c_0) \lor \cdots \lor (x_{n-1} \land c_0) \lor (x_n \land c_n)) \\
&\overset{(7)}{=} (x_1 \lor x_2 \lor \cdots \lor x_n, 0 \lor x_2 \lor \cdots \lor x_n, \ldots, 0 \lor \cdots \lor 0 \lor x_n) \\
&= (x_1, \ldots, x_n) \quad \text{due to the structure of } P
\end{align*}
\]

and

\[
\psi \circ \varphi(x) = (D_1 x \land c_1) \lor \cdots \lor (D_n x \land c_n) \overset{(8)}{=} x.
\]

Hence \( \varphi \) is a bijective map and we continue to prove that it is an isomorphism between \( P \) and \( \tilde{D}_p^n \) by showing that it is a homomorphism. We show that it commutes with all operations. We start with the binary operation \( \land \):

\[
\varphi(x \land y) = (D_1(x \land y), \ldots, D_n(x \land y)) \\
\overset{(5)}{=} (D_1x, \ldots, D_nx) \land (D_1y, \ldots, D_ny) \\
= \varphi(x) \land \varphi(y).
\]

\( \varphi(x \lor y) = \varphi(x) \lor \varphi(y) \) follows analogously. To check the unary operation \( C \) we need to recall that \( Cx = \neg D_1x \) due to equation (3). Thus

\[
\begin{align*}
C\varphi(x) &= C(D_1x, \ldots, D_nx) \\
&= (\neg D_1x, \ldots, \neg D_1x) \\
&= (C, \ldots, C) \quad \text{since } Cx = \neg D_1x \\
&= (D_1Cx, \ldots, D_1Cx) \quad \text{since } Cx \in \tilde{D}_p \\
&\overset{(6)}{=} (D_1D_1Cx, \ldots, D_nD_1Cx) \\
&= (D_1Cx, \ldots, D_nCx) \\
&= \varphi(Cx)
\end{align*}
\]

For each \( i \in \{1, \ldots, n\} \) we have

\[
\begin{align*}
\varphi(D_ix) &= (D_1D_ix, \ldots, D_nD_ix) \overset{(6)}{=} (D_ix, \ldots, D_ix) \\
&= D_i(D_1x, \ldots, D_nx) = D_i\varphi(x).
\end{align*}
\]
And finally, for each $i \in \{1, \ldots, n\}$ we have
\[
\varphi(c_i) = (D_1 c_i, \ldots, D_n c_i) \overset{(7)}{=} (1, \ldots, 1, 0, \ldots, 0) = c_i.
\]
Thus as claimed $\mathcal{P}_{n+1}$ is isomorphic to $B^{(n)}(u)$.

It is already implicitly mentioned in [30] that each Post algebra is isomorphic to an algebra of the kind we received as the $u$-modification of the matrix power of a Boolean algebra. Porst was the first to see the connection to the $u$–modification in [27]. Combining the two lemmas we have proven the same theorem Porst already obtained in [27].

**Theorem 5.4** The varieties equivalent to the variety $\text{BOOL}$ of Boolean algebras are (up to concrete isomorphism) precisely the varieties $\mathcal{P}_n$ of Post algebras of order $n$ for $n \in \mathbb{N}$, $n \geq 2$.

**Remark 5.5** The proof of this theorem is relatively easy compared to the more general case in Chapter 4. That is due to the unary invertible idempotent operations $u$ in the $n$-th matrix power of $\text{BOOL}$ needed for its $u$-modification. They leave most of the original operations unchanged. Thus many of the equations are preserved. Especially the fact that the unary operations $D_i$ remain unchanged makes it straightforward to show the structure of the underlying set. This is exactly the point where we had to add the extra requirement of the idempotents being central in Chapter 4.

### 5.4 Hu’s Primal Algebra Theorem

The result on the equivalence of Boolean algebras to Post algebras also answers a question which was still undecided until Porst found Theorem 5.4 in [27]. Hu’s primal algebra theorem [13, 14] characterizes the varieties equivalent to the variety of Boolean algebras. They are the varieties which are generated by a primal algebra. It was known that the variety of Post algebras is equivalent to the variety of Boolean algebras since for each natural number $n \geq 2$ the variety of Post algebras of order $n$ is generated by a primal $n$-element algebra (cf. [3]). But it was not known whether these are all varieties equivalent to $\text{BOOL}$.

Theorem 5.4 now completely answers the question, and combining it with with Hu’s theorem we get:

**Theorem 5.6** The following are equivalent for a variety $\mathcal{V}$ which is not isomorphic to $\text{BOOL}$:

(i) $\mathcal{V}$ is equivalent to $\text{BOOL}$, the variety of Boolean algebras.
(ii) $\mathcal{V}$ is isomorphic to $\text{POST}_n$, the variety of Post algebras of order $n$, for a natural number $n \geq 2$.

(iii) $\mathcal{V}$ is generated by a primal algebra.

A quite elegant and concise proof of Hu’s theorem making use category theory can be found in [28]. The proof is based on translating the property of an algebra being primal into categorical language without needing more than the most basic notions of category theory. It is a lot more straightforward than other algebraic proofs in [8, 25] or the original proof which made use of Stone duality.
Bibliography


## Index of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(F, E)$</td>
<td>2</td>
</tr>
<tr>
<td>$F$</td>
<td>2</td>
</tr>
<tr>
<td>$E$</td>
<td>2</td>
</tr>
<tr>
<td>$\text{Alg}(F, E)$</td>
<td>2</td>
</tr>
<tr>
<td>$</td>
<td>-</td>
</tr>
<tr>
<td>$M \text{Act}$</td>
<td>4</td>
</tr>
<tr>
<td>$T$</td>
<td>4</td>
</tr>
<tr>
<td>$T_i$</td>
<td>4</td>
</tr>
<tr>
<td>$U_T$</td>
<td>5</td>
</tr>
<tr>
<td>$S$</td>
<td>5</td>
</tr>
<tr>
<td>$\text{Mod}S$</td>
<td>5</td>
</tr>
<tr>
<td>$G$</td>
<td>5</td>
</tr>
<tr>
<td>$nG$</td>
<td>5</td>
</tr>
<tr>
<td>$\text{Th}_V(G)$</td>
<td>5</td>
</tr>
<tr>
<td>$V$</td>
<td>5</td>
</tr>
<tr>
<td>$F1$</td>
<td>5</td>
</tr>
<tr>
<td>$\text{Th}V$</td>
<td>5</td>
</tr>
<tr>
<td>$\text{ModTh}V$</td>
<td>5</td>
</tr>
<tr>
<td>$Fn$</td>
<td>6</td>
</tr>
<tr>
<td>$(\mathcal{W},</td>
<td>-</td>
</tr>
<tr>
<td>$(V, \text{hom}(G, -))$</td>
<td>6</td>
</tr>
<tr>
<td>$A^{[n]}$</td>
<td>7</td>
</tr>
<tr>
<td>$\mathcal{V}^{[n]}$</td>
<td>7</td>
</tr>
<tr>
<td>$u$</td>
<td>8</td>
</tr>
<tr>
<td>$A(u)$</td>
<td>8</td>
</tr>
<tr>
<td>$\mathcal{V}(u)$</td>
<td>8</td>
</tr>
<tr>
<td>$\mathcal{V}^{[n]}(u)$</td>
<td>9</td>
</tr>
<tr>
<td>$\mathcal{V}^{(n)}$</td>
<td>12</td>
</tr>
<tr>
<td>$A^{(n)}$</td>
<td>12</td>
</tr>
<tr>
<td>$D_i$</td>
<td>12</td>
</tr>
<tr>
<td>$D_A$</td>
<td>12</td>
</tr>
<tr>
<td>$(D_A)^{(n)}$</td>
<td>13</td>
</tr>
<tr>
<td>$f_i$</td>
<td>14</td>
</tr>
<tr>
<td>$\mathcal{V}^{(n)}$</td>
<td>15</td>
</tr>
<tr>
<td>$D_i$</td>
<td>15</td>
</tr>
<tr>
<td>$\circ_j$</td>
<td>15</td>
</tr>
<tr>
<td>$A^{[n]}$</td>
<td>16</td>
</tr>
<tr>
<td>$D_i^A$</td>
<td>16</td>
</tr>
<tr>
<td>$D^A$</td>
<td>16</td>
</tr>
<tr>
<td>$(D^A)^{[n]}$</td>
<td>16</td>
</tr>
<tr>
<td>$n\text{-Set}_1$</td>
<td>20</td>
</tr>
<tr>
<td>$D_i$</td>
<td>20</td>
</tr>
<tr>
<td>$D_i^n$</td>
<td>20</td>
</tr>
<tr>
<td>$\circ_j^n$</td>
<td>20</td>
</tr>
<tr>
<td>$D_A$</td>
<td>22</td>
</tr>
<tr>
<td>$(D_A)^n$</td>
<td>22</td>
</tr>
<tr>
<td>$(n\text{-Set}_1,</td>
<td>-</td>
</tr>
<tr>
<td>$n\text{-Set}_2$</td>
<td>24</td>
</tr>
<tr>
<td>$D_i$</td>
<td>24</td>
</tr>
<tr>
<td>$*$</td>
<td>24</td>
</tr>
<tr>
<td>$D_A$</td>
<td>25</td>
</tr>
<tr>
<td>$(n\text{-Set}_2,</td>
<td>-</td>
</tr>
<tr>
<td>$n\text{-Set}_3$</td>
<td>27</td>
</tr>
<tr>
<td>$\tau$</td>
<td>27</td>
</tr>
<tr>
<td>$n\text{-Set}_4$</td>
<td>27</td>
</tr>
<tr>
<td>$D_i$</td>
<td>27</td>
</tr>
<tr>
<td>$D_A$</td>
<td>28</td>
</tr>
<tr>
<td>$(n\text{-Set}_3,</td>
<td>-</td>
</tr>
<tr>
<td>$\mathcal{V}(M, u)$</td>
<td>30</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>30</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>30</td>
</tr>
<tr>
<td>$D_i$</td>
<td>31</td>
</tr>
<tr>
<td>$D_A$</td>
<td>31</td>
</tr>
<tr>
<td>$(n\text{-Set}_4,</td>
<td>-</td>
</tr>
<tr>
<td>$G$</td>
<td>34</td>
</tr>
<tr>
<td>$\mathcal{V}(M, u)$</td>
<td>36</td>
</tr>
<tr>
<td>$\bar{m}$</td>
<td>36</td>
</tr>
<tr>
<td>$D_i$</td>
<td>37</td>
</tr>
<tr>
<td>$D_i$</td>
<td>37</td>
</tr>
<tr>
<td>$\circ_j$</td>
<td>37</td>
</tr>
<tr>
<td>$m^i$</td>
<td>37</td>
</tr>
<tr>
<td>$f_i$</td>
<td>39</td>
</tr>
<tr>
<td>$D_i$</td>
<td>39</td>
</tr>
<tr>
<td>$m^i$</td>
<td>39</td>
</tr>
<tr>
<td>$D_i$</td>
<td>39</td>
</tr>
<tr>
<td>$m^i$</td>
<td>39</td>
</tr>
<tr>
<td>$D_i$</td>
<td>39</td>
</tr>
<tr>
<td>$m^i$</td>
<td>40</td>
</tr>
</tbody>
</table>

61
### Index of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>MP</td>
<td>42</td>
</tr>
<tr>
<td>( m_{MP} )</td>
<td>42</td>
</tr>
<tr>
<td>( D_i^{MP} )</td>
<td>42</td>
</tr>
<tr>
<td>( o_i^{MP} )</td>
<td>42</td>
</tr>
<tr>
<td>( N \text{Act} )</td>
<td>44</td>
</tr>
<tr>
<td>( M \text{Act} )</td>
<td>44</td>
</tr>
<tr>
<td>( G )</td>
<td>45</td>
</tr>
<tr>
<td>( \text{Th}_{M \text{Act}}(G) )</td>
<td>45</td>
</tr>
<tr>
<td>( \text{hom}_{M \text{Act}}(G, -) )</td>
<td>45</td>
</tr>
<tr>
<td>( \text{End} )</td>
<td>45</td>
</tr>
<tr>
<td>BOOL</td>
<td>47</td>
</tr>
<tr>
<td>POST(_n)</td>
<td>47</td>
</tr>
<tr>
<td>BOOL(_{(n)})</td>
<td>47</td>
</tr>
<tr>
<td>( \land )</td>
<td>47</td>
</tr>
<tr>
<td>( \lor )</td>
<td>47</td>
</tr>
<tr>
<td>( \lnot )</td>
<td>47</td>
</tr>
<tr>
<td>( 0 )</td>
<td>47</td>
</tr>
<tr>
<td>( 1 )</td>
<td>47</td>
</tr>
<tr>
<td>( D_i )</td>
<td>48</td>
</tr>
<tr>
<td>( c_i )</td>
<td>48</td>
</tr>
<tr>
<td>( C )</td>
<td>49</td>
</tr>
<tr>
<td>( \mathcal{P}_{n+1} )</td>
<td>50</td>
</tr>
<tr>
<td>( DP )</td>
<td>52</td>
</tr>
<tr>
<td>( B )</td>
<td>52</td>
</tr>
<tr>
<td>( B^{(n)}(u) )</td>
<td>52</td>
</tr>
</tbody>
</table>
Index

act, 2
\( M \text{Act}, 2 \)
algebra, 2
\((F,E), 2 \)
in \( \text{n-Set}_1, 20 \)
in \( \text{n-Set}_2, 25 \)
in \( \text{n-Set}_3, 27 \)
in \( \text{n-Set}_4, 30 \)
in \( \text{V}^{(n)}, 12 \)
in \( \text{V}^{(n)}, 16 \)
\( \mathbb{T}, 5 \)
\( \text{Alg}(F,E), 2 \)
arity, 2
associative law, 22
Birkhoff, 2
BOOL, 47
Boolean algebra, 47
BOOL\((n), 47 \)
Clifford monoid, 35
clon, 4
concrete equivalence
between varieties, 6
coproduct
preservation of, 3
derived equations
in \( \text{n-Set}_1, 21 \)
in \( \text{n-Set}_2, 25 \)
in \( \text{n-Set}_3, 28 \)
in \( \text{n-Set}_4, 31 \)
in \( \text{V}^{(n)}, 13 \)
distributive lattice, 50
equations, 2
in \( \text{n-Set}_1, 20 \)
in \( \text{n-Set}_2, 24 \)
in \( \text{n-Set}_3, 27 \)
in \( \text{n-Set}_4, 30 \)
in Post algebras, 50
in \( \text{V}^{(n)}, 12 \)
in \( \text{V}^{(n)}, 15 \)
in \( \text{V}(M,u), 36 \)
equivalence
Morita, 5
of \( M \)-acts, 44
equivalent varieties
characterization, 9
generator
varietal, 5
homomorphism, 2
Hu’s theorem, 54
idempotent and invertible
operation, 8
term, 8
unary operation in BOOL\((n), 48 \)
unary operation in \( M \text{Act}^{(n)}, 34 \)
invertible, 8
Lawvere theory, 4
generated by \( G, 5 \)
\( M \)-act, 2
\( M \text{Act}, 2, 44 \)
matrix power, 7
first characterization, 12
of BOOL, 47
second characterization, 15
matrix power of \( \text{Set} \)
first description, 20
second description, 24
third description, 27
fourth description, 30
model, 5
modification, \( u, 8 \)
of BOOL, 48
of unary varieties, 36
Morita equivalent, 5

\( N \, \text{Act} \), 44
\( n \)-ary, 2
\( n \)-th matrix power, 7
\( n \)-Set\(_1\), 20
\( n \)-Set\(_2\), 24
\( n \)-Set\(_3\), 27
\( n \)-Set\(_4\), 30

operation, 2
idempotent and invertible, 8

operations
\( \text{in } \text{BOOL}^{(n)} \), 48
\( \text{in } M \, \text{Act} \), 3
\( \text{in } n \)-Set\(_1\), 20
\( \text{in } n \)-Set\(_2\), 24
\( \text{in } n \)-Set\(_3\), 27
\( \text{in } n \)-Set\(_4\), 30
\( \text{in } \text{Post algebras} \), 49
\( \text{in } \mathcal{V}^{(n)} \), 15
\( \text{in } \mathcal{V}^{(n)} \), 12

Post algebra, 47
equational definition, 49
\( \text{POST}_n \), 47
primal algebra, 54
primal algebra theorem, 54

signature, 2

\( \mathbb{T} \)-algebra, 5
\( \mathbb{T} \)-model, 5
term
idempotent and invertible, 8
theory, 4
construction of, 5
of \( \mathcal{V} \), 5
theory morphism, 5

\( u \)-modification, 8
\( \text{of } \text{BOOL} \), 48
\( \text{of } M \, \text{Act}^{[n]} \), 38

unary variety, 3, 33
categorical characterization, 3
characterization as \( M \, \text{Act} \), 3
\( u \)-modification, 33