Homogenization of a System of Nonlinear Multi-Species Diffusion-Reaction Equations in an $H^{1,p}$ Setting

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Homogenization of a System of Nonlinear Multi-Species Diffusion-Reaction Equations in an $H^{1,p}$ Setting

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Prof. Dr. habil. Peter Knabner (Universität Erlangen-Nürnberg)
Dedicated in the loving memory of my mother who could not live longer to see me concluding this work but she always believed that I have the courage to face any obstacle.
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Abstract

The processes of chemical transport in porous media are extensively studied in the fields of applied mathematics, material science, chemical engineering etc. A porous medium (e.g. concrete, soil, rocks, reservoir etc.) is a multiscale material/medium where the heterogeneities present in the medium are characterized by the micro scale and the global behaviors of the medium are observed by the macro scale. The upscaling from the micro scale to the macro scale can be done via averaging methods.

The transport process in a porous medium is a complex phenomena. In this thesis, the heterogeneities inside a porous medium are assumed to be periodically distributed and diffusion-reaction of a finite number of chemical species are investigated. Two different models are proposed in this work. In model M1, diffusion-reaction of mobile chemical species are considered. The chemical processes are modeled via mass action kinetics and the modeling leads to a system of multi-species diffusion-reaction equations (nonlinear partial differential equations) at the micro scale. For this system of equations, existence of a unique positive global weak solution is proved by the help of a Lyapunov functional and Schaefer’s fixed point theorem. The upscaled model of this system is obtained using periodic homogenization which is an averaging method.

In model M2, we consider diffusion-advection-reaction of two different types of mobile species (type I and type II). The type II species are supplied via dissolution process due to the presence of immobile species on the surface of the solid parts. The presence of mobile and the immobile species make the model complex and the modeling yields a coupled system of nonlinear partial differential equations. The existence of a unique positive global weak solution of this complex system is shown. Finally, with the help of periodic homogenization, model M2 is upscaled from the micro scale to the macro scale.

Numerical simulations are conducted for both models separately. For the purpose of illustration, we restrict ourselves to relatively simple 2-dimensional situations. For models M1 and M2, simulation results at the micro scale and at the macro scale are compared.
Acknowledgments

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Hari Shankar Mahato
List of Mathematical Notations

\[ \mathbb{R} \quad \text{(resp. } \mathbb{Z}, \mathbb{C}, \mathbb{K}) \] set of real numbers (resp. integers, complex, scalar)

\[ \mathbb{R}^+ \quad \text{(resp. } \mathbb{Z}^+, \mathbb{Z}^-, \mathbb{N}) \] set of positive real numbers (resp. positive integers, negative integers, natural numbers)

\[ \mathbb{R}_0^+ \quad \text{(resp. } \mathbb{Z}_0^+, \mathbb{Z}_0^-, \mathbb{N}_0) \] \[ A - B \] \[ (\ , \ ) \quad \text{(resp. } [\ , \ ], \text{ or } \ , \ ), \text{ or } \ ] open interval (resp. closed, or semi-open intervals)

\[ ||\ , ||_{X} \] norm on the linear space \( X \), see page 18

\[ X^* \] dual space of \( X \)

\[ \mathcal{L}(X,Y) \] set of all continuous linear operators from \( X \) to \( Y \)

\[ (\ , \ , \ )_{H} := (\ , \ , \ ) \] inner product on a Hilbert space \( H \)

\[ (\ , \ , \ )_{X \times X^*} \] duality paring between \( X \) and \( X^* \)

\[ I, I_1, I_2 \] positive integers (number of chemical species)

\[ X^{I} \] \[ |||\ , |||_{X^{I}} \] norm on the vector-valued space \( X^{I} \), see page 19

\[ \langle\ , \ , \ \rangle_{I} \] Euclidean inner product in \( \mathbb{R}^{I} \)

\[ \langle\ , \ , \ \rangle_{X^{I} \times [X^*]^{I}} \] duality paring between \( X^{I} \) and \([X^*]^{I}\)

\[ ||\ , ||_{I} \] Euclidean norm in \( \mathbb{R}^{I} \)

\[ p \] a real number in \((1,\infty)\)

\[ q \] \[ (p)' = \text{dual index of } q, \text{ i.e., } \frac{1}{p} + \frac{1}{q} = 1 \]

\[ \subset \] subset

\[ \rightarrow \] maps to

\[ \rightarrow \] strong convergence

\[ \rightarrow \] weak convergence

\[ \ast \] weak-star convergence

\[ d \subset \] dense subset

\[ \hookrightarrow \] continuous embedding

\[ \hookrightarrow \] compact embedding

\[ \varnothing \] two-scale convergence

\[ \Rightarrow \] implies

\[ \iff \] if and only if

\[ \Rightarrow \] reversible reaction

\[ \delta_{jk} \] Kronecker delta
\(d\sigma_y\) surface measure on \(\Gamma\), see page 15

\(d\sigma_x\) surface measure on \(\Gamma_{\varepsilon}\), see page 15

\(\nabla, \nabla_x, \nabla_y\) gradient operator;

\[
\nabla_x = \left( \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right), \text{ where } x = (x_1, x_2, \ldots, x_n)
\]

\(\text{div}, \text{div}_x, \text{div}_y\) divergence of a vector function

\(\Omega\) bounded domain in \(\mathbb{R}^n\)

\(S\) \([0, T]\), the time interval

\(\frac{\partial u}{\partial t}\) time derivative of \(u\) with respect to \(t\) in the
distributional sense

\(\partial_j u\) \(\frac{\partial u}{\partial x_j}\), where \(x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n\)

\(\alpha\) a multi-index such that \(\alpha_1 + \alpha_2 + \ldots + \alpha_n = |\alpha|\),

where \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_+\)

\(D^\alpha u\) \(\frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}\)

\((\ldots \ldots)_p\) real-interpolation space, where \(0 < \theta < 1\)

\([\ldots \ldots]_p\) complex interpolation space, where \(0 < \theta < 1\)

\(C^k(\bar{\Omega})\) space of all \(k\)-times continuously differentiable function on \(\bar{\Omega}\),

where \(k \in \mathbb{N}\), see page 18

\(C^\gamma(\bar{\Omega})\) Hölder space, where \(0 < \gamma \leq 1\), see page 18

\(L^p(\Omega)\) equivalence class of all measurable functions \(u : \Omega \to \mathbb{R}\)
such that \(|u(\cdot)|^p\) is Lebesgue integrable, see page 18

\(L^\infty(\Omega)\) equivalence class of all measurable functions \(u : \Omega \to \mathbb{R}\)
such that \(\text{ess sup}_{x \in \Omega} |u(x)| < \infty\), see page 18

\(L^p_\Omega\) \(\{ u \in L^p(\Omega) : u \geq 0 \text{ a.e.} \}\), where \(1 \leq p \leq \infty\)

\(H^{1,p}(\Omega)\) space of locally summable functions \(u : \Omega \to \mathbb{R}\) such that for
every multiindex \(\alpha\) with \(|\alpha| \leq 1\), \(D^\alpha u\) exists in the weak
sense and belongs to \(L^p(\Omega)\), see page 18

\(H^{s,p}(\Omega)\) fractional order Sobolev/Sobolev-Slobodecki space,

where \(s \in \mathbb{R}_0^+\), see page 19

\(F\) Bochner space of type \(H^{1,p}((0, T); H^{1,q}(\Omega)^*) \cap L^p((0, T); H^{1,p}(\Omega))\),

see page 19

\(\mathcal{F}_p, \mathcal{G}_p, \mathcal{M}_p, \mathcal{M}_\infty, \mathcal{X}_p, \mathcal{X}_p^\varepsilon, \mathcal{X}_p^{\infty}\) see pages 21 and 21

\(\mathcal{F}_\varepsilon, \mathcal{G}_\varepsilon, \mathcal{H}_\varepsilon, \mathcal{M}_\varepsilon, \mathcal{X}_\varepsilon, \mathcal{X}_\varepsilon^\varepsilon, \mathcal{X}_\varepsilon^{\infty}\) see page 21

\(C, C_i\) generic nonnegative constants but may be different at
different steps of the inequality
List of Modeling Notations

ε scale parameter, see page 14
Y a representative cell in $\mathbb{R}^n$, see page 13
$Y^p$ pore space in $Y$, see page 13
$Y^s$ solid part in $Y$, see page 13
Γ $\partial Y^s$, i.e., boundary of $Y^s$, see page 13
$\vec{n}$ unit outward drawn normal on the boundary
Ω a bounded domain/porous medium in $\mathbb{R}^n$
$\Omega^p$ pore space available for fluid in Ω, see page 8
$\Omega^s$ union of the solid parts in Ω, see page 8
Γ* boundary of $\Omega^s$, see page 8
$\partial \Omega$ outer boundary of Ω
$\partial \Omega^p$ $\partial \Omega \cup \Gamma^*$, see page 8
$\partial \Omega_{in}$ inflow boundary, see page 12
$\partial \Omega_{out}$ outflow boundary, see page 12
$\Omega^p_\varepsilon$ pore space scaled by $\varepsilon$, see page 14
$\Omega^s_\varepsilon$ solid parts scaled by $\varepsilon$, see page 14
Γ_ε union of boundaries of the solid parts scaled by $\varepsilon$, see page 14
$\partial \Omega^p_\varepsilon$ $\partial \Omega \cup \Gamma_\varepsilon$
ϕ porosity constant
$D$ diffusion coefficient
$k_j^f$ forward reaction rate factor in the $j$-th reaction
$k_j^b$ backward reaction rate factor in the $j$-th reaction
$k_d$ dissolution coefficient
$P$ positive definite diffusive tensor, see page 58
$\rho$ density of the fluid
$q$ fluid velocity
$\psi$ positive signum function, see page 11
$\chi_M$ characteristic/indicator function of a set $M$
## List of Abbreviations

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<td>BC (resp. IC)</td>
<td>boundary condition (resp. initial condition)</td>
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<tr>
<td>r.h.s.</td>
<td>right hand side</td>
</tr>
<tr>
<td>l.h.s.</td>
<td>left hand side</td>
</tr>
<tr>
<td>w.r.t.</td>
<td>with respect to</td>
</tr>
<tr>
<td>b.v.p.</td>
<td>boundary value problem</td>
</tr>
<tr>
<td>♦</td>
<td>end of the proof</td>
</tr>
<tr>
<td>‘∵’</td>
<td>because/since</td>
</tr>
<tr>
<td>‘∴’</td>
<td>therefore</td>
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Several physical problems in the fields of physics, chemistry, biology and engineering sciences are governed by \textit{diffusion-reaction equations}. One of the important phenomena that can be explained with the help of these equations is \textit{chemical transport in a porous medium} (e.g. soil, rock, concrete, pellets etc., see figure 1.1.1). The aim of this thesis is to investigate the transport processes of mobile and immobile chemical species present inside a porous medium.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{porous_medium_examples.png}
\caption{Examples of a porous medium.\textsuperscript{1}}
\end{figure}

In general, a porous medium has a complex geometry. It is a heterogeneous medium composed of pore space and union of solid parts (see figure 1.1.2), where the heterogeneities are much smaller compared to the size of the medium. Thus in order to analyze the processes which take place inside the medium, one needs to consider the \textit{microscopic} and the \textit{macroscopic} description of the medium. The size of the microscopic scale can vary from nanometer to micrometer and it is appropriate for describing the heterogeneities of the medium, however, it is not suitable for numerical simulations. On the other hand, the size of the macroscopic scale can vary from meter to kilometer or even larger and the macroscopic description of the medium fits well for numerical computations. Thus, to study the bulk (global) behaviors of a material, one \textit{upscales} a mathematical model (in this thesis it is given by partial differential equations) from the micro scale to the macro scale.

In this thesis, two different models are proposed at the micro scale and the upscaled models (models at the macro scale) are obtained by periodic homogenization. Periodic homogenization refers to an averaging method in which the distribution of the solid pieces comprising the solid part (see sections 2.4 and 2.5.1) in the porous medium is periodic (cf. figure 2.5.4). The periodicity assumption of solid parts in the porous medium is used by many authors for homogenization (cf. [ADH96], [ADH90], [CD99], [Cla98], [ACP08], [HJ91], [Pet06] and references therein). In reality such a distribution of solid parts is very rarely met, however, the assumption of periodicity can be relaxed (cf. [Pet06], [Mei08], [Fat13] etc.)

The transport processes in porous media, for example in soil, have been extensively studied in last decades and it has drawn the attention of geologists, hydrologists, math-

\textsuperscript{1}These images are taken from the website http://purechemicals.co.uk/news/tag/benzo-fury-pellets/ and http://commons.wikimedia.org/wiki/Main_Page.
Chapter 1. Introduction

Mathematicians and others (cf. [BB90], [vDP04], [Kna91], [Kna86], [Krä08], [Log01], [Rub83], [WR87] etc.). Recently Kräutle has shown, on the macroscopic level, the existence and uniqueness of the global solution in \( [H^{1,p}(0,T); L^p(\Omega)] \cap L^p(0,T; H^{2,p}(\Omega)) \) of a system of diffusion-reaction equations for the multi-species reactive transport problem in a porous medium, where \( \Omega \) is the given porous medium, \( I \) is the number of chemical species and \( p > n + 1 \) (cf. [Krä08], [Krä11]). With the help of a Lyapunov functional, he obtained some a-priori estimates (global in time) and showed the existence of a unique solution on the time interval \([0,T] \) for any \( T > 0 \). However, to our knowledge, it seems that this idea has not been excavated to its full strength when the solution \( u(t) \) has derivative only up to the first order, i.e., if only \( u(t) \in H^{1,p}(\Omega) \). In the first part of this work, we also consider diffusion-reaction of a finite number of chemical species\(^2\). Since our porous medium is heterogeneous, we consider the system of diffusion-reaction equations at the micro scale and we prove the existence of a unique positive global weak solution in \( [H^{1,p}(0,T); H^{1,\beta}(\Omega^p)^*) \cap L^p(0,T; H^{1,p}(\Omega^p))] \) for \( p > n + 2 \), (see section 2.5.1 for the definition of \( \Omega^p \)). We upscale the models governed by nonlinear partial differential equations from the micro to the macro scale using two-scale convergence and periodic unfolding (see sections 3.5 and 3.6). In the second part of this thesis, we investigate a rather complex model where we incorporate the previous model with dissolution which takes place on the surface of the solid parts (see page 10).

In this work, we will consider the following type of a porous medium:

![Figure 1.1.2: A typical porous medium with solid parts \( \Omega^s \) and pore space \( \Omega^p \).](image)

The transport processes take place in the pore space. In this work the pore space is assumed to be connected whereas the solid parts are considered as disconnected. It is also possible that the species present on the surface of the solid parts react with the species present in the fluid via dissolution or precipitation. This will not only lead to an extra source term but may also affect the size of the solid parts in the medium (cf. [Pet06], e.g.).

We cite some examples from the literature in which chemical transport in a porous medium has been investigated. The carbonation inside the concrete affects its durability and longevity. The authors in [SS98], [Pet06], [Mun06], [MPM+07], [MPM07], [Mei08], [MB09b], [MB09a] (and references therein) have proposed appropriate mathematical models for the concrete carbonation and investigated the reactions associated with it. Sulfuric acid attack in sewer pipes made of concrete is studied in [BJDR98], [FAZM11], [FM12]. In [NRK08], authors have discussed the dynamics of hematopoietic stem cells (HSCs). The processes of dissolution and precipitation have been examined by many authors in the context of porous media, for example see [KvDH95], [Kna86], [vDP04] etc.

\(^2\)The reaction rates (given by mass action law) are of the form (2.4.7) which is motivated from the work of Kräutle in [Krä08].
1.1 Periodic Homogenization

The goal of homogenization theory is to give a macroscopic description of a material body or of a medium which is microscopically heterogeneous, i.e., the heterogeneous body is replaced by a homogeneous body which is considered as an approximation to the heterogeneous body so that the physical properties associated with the body can be examined. Mathematically speaking, homogenization theory gives the convergence of the solutions of a given b.v.p. which has highly oscillating coefficients to the solution of a limit b.v.p. which is a good approximation of the original b.v.p., i.e., the limit b.v.p. is simpler and does not involve highly oscillating coefficients. For example, let \( \Omega \subset \mathbb{R}^n \) be a bounded domain and assume that the heterogeneities in \( \Omega \) are very small and periodically distributed. Let \( \varepsilon > 0 \) be the scale parameter representing the periodicity. Consider the following b.v.p.

\[
L^\varepsilon u^\varepsilon(x) := -\nabla \cdot (D^\varepsilon(x)\nabla u^\varepsilon(x)) = f(x) \quad \text{in } \Omega \tag{1.1.1}
\]

\[
u^\varepsilon(x) = 0 \quad \text{on } \partial \Omega, \tag{1.1.2}
\]

where \( D^\varepsilon(x) = D(x, \frac{x}{\varepsilon}) \) for a.e. \( x \in \Omega \) and \( D^\varepsilon \) is periodic w.r.t. \( \frac{x}{\varepsilon} \) (cf. [CD99], [Hor97]). Here \( x \) is the macroscopic variable and \( \frac{x}{\varepsilon} \) is the microscopic variable. To illustrate the ideas, consider \( \Omega = (-\pi, \pi) \) and \( D^\varepsilon(x) = 0.8 \cos(x) + \varepsilon \sin(\frac{x}{\varepsilon}) \). For different values of \( \varepsilon \), the graphs of \( D^\varepsilon \) are plotted in figure 1.1.3.

![Figure 1.1.3: Graph of \( D^\varepsilon \) for different \( \varepsilon \).](image)

The variation at the macro scale is given by the part 0.8 \( \cos(x) \) and the oscillation at the micro scale is described by \( \varepsilon \sin(\frac{x}{\varepsilon}) \). By letting \( \varepsilon \to 0 \), we make the oscillations smaller and smaller. The numerical simulation of the model (1.1.1)-(1.1.2) is difficult due to the micro oscillations. Thus we are interested to obtain a homogenized b.v.p. (see chapter 4) which contains an averaged effect of the micro oscillations instead of involving it explicitly in the problem. Let us denote this b.v.p. by

\[
Lu(x) = -\nabla \cdot (\bar{D} \nabla u(x)) = f(x) \quad \text{in } \Omega \tag{1.1.3}
\]

\[
u(x) = 0 \quad \text{on } \partial \Omega, \tag{1.1.4}
\]

where \( \bar{D} \) is the "averaged coefficient" (see equation 4.1.101, e.g.). The homogenized equation (1.1.3)-(1.1.4) is better suitable for numerical simulations and the solution of (1.1.3)-(1.1.4) is an approximation to the solution of (1.1.1)-(1.1.2). However, the convergence of \( u^\varepsilon \) as \( \varepsilon \to 0 \) needs to be established.

To obtain the homogenized b.v.p. (i.e., to understand the convergence as \( \varepsilon \to 0 \)), several methods have been developed:

- The first method is asymptotic expansion (cf. the book of A. Bensoussan, J.L. Lions and G. Papanicolaou [BLP78]). We assume that our unknown function \( u^\varepsilon \) has an asymptotic expansion of the form

\[
u^\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \cdots, \tag{1.1.5}
\]
where the functions $u_i$ for all $i$ depend on $x$ and $\varepsilon_i$, and are periodic w.r.t. the microscopic variable $\varepsilon_i$. Substituting the expansion (1.1.5) in (1.1.1) and comparing different powers of $\varepsilon$, one can obtain the homogenized b.v.p. This method is, however, only formal and does not give any mathematical proof of convergence.

- A mathematical proof of convergence of $u_\varepsilon$ can be given by oscillating test function method developed by Tartar (see chapter 8 in [CD99]). However, for complex problems this method is not suitable.

- The notion of two-scale convergence has been introduced by Nguetseng (cf. [Ngu89]) and later on developed by Allaire (cf. [All92]). This method is suitable for studying the problems of the type above (see section 3.5 for definition and theorems).

- The recently developed periodic unfolding method by Cioranescu, Damlamian and Griso (cf. [CDG02]) has also become a very efficient tool to deal with the problems described above (cf. section 3.6 for definition and theorems). It is suitable for dealing with the nonlinear boundary value problems.

1.2 Outline of the Thesis

This thesis contains six chapters followed by an appendix. After the introductory chapter (chapter 1), we present diffusion-reaction models in chapter 2. Some mathematical tools have been collected in chapter 3. The analysis of models is done in chapter 4 and numerical simulations of models are given in chapter 5. We summarize this work in chapter 6 followed by an appendix.

In chapter 2, we start with a brief discussion on different types of fluxes in section 2.1. A very short illustration of reaction rates is given in section 2.2. We familiarize with the notions of dissolution and precipitation in section 2.3. In section 2.4, two types of diffusion-reaction models (M1 and M2, see sections 2.4.1 and 2.4.2) are introduced. The periodic scaling of the domain $\Omega$ (given porous medium) is shown in section 2.5 and we conclude chapter 2 by deriving diffusion-reaction models at the micro scale in sections 2.5.1 and 2.5.2.

In chapter 3, we collect some mathematical tools to analyze the models M1 and M2 respectively. In section 3.1, several function spaces are introduced followed by some embedding theorems and the weak formulation of models M1 and M2 at the micro scale. The concept of maximal parabolic regularity is given in section 3.3. Some important theorems are derived at the micro scale in section 3.4. The notions of two-scale convergence and periodic unfolding are given in sections 3.5 and 3.6 respectively.

Chapter 4 is the main body of this work. Model M1 is considered in section 4.1. In section 4.1.1, existence of a unique positive global weak solution of model M1 is shown by the help of a Lyapunov functional (see section 4.1.1.2), Schaefer’s fixed point theorem (cf. theorem B.1) and the linear theory of evolution equations involving maximal regularity (cf. theorem 3.3.1). Some a-priori estimates of the solution of model M1 are obtained in sections 4.1.2.1 and 4.1.2.2. The homogenization of model M1 is conducted in section 4.1.2.3.

Model M2 is investigated in section 4.2. The global existence and uniqueness of a positive weak solution of M2 is proved in section 4.2.1. Some a-priori estimates of the solution of model M2 are obtained in sections 4.2.2.1 and 4.2.2.2. The homogenized model for model M2 is achieved in section 4.2.2.3.

In chapter 5, numerical simulations are performed. In section 5.1, simulations for model M1 at the micro scale and at the macro scale are shown. We conclude this chapter with the numerical computations for model M2 in section 5.2.
A short summary and outlook of this thesis are given in chapter 6. The appendix contains two sections. In section A, a few elementary inequalities are collected. Some classical theorems on Sobolev spaces are listed in section B.
Chapter 2

The Model

In this chapter, we introduce two different models: Model M1 and Model M2. In model M1, we consider only diffusion and reaction of chemical species inside the pore space. In this case, the species are transported via diffusion. For model M2, we consider diffusion, reaction and advection of chemical species. Here the species are transported via both advection and diffusion. The dissolution process occurs on the surface of the solid parts.

We begin with the diffusion-advection equation in section 2.1. In section 2.2, we give a short description of reaction rates. Section 2.3 deals with precipitation and dissolution. The derivations of models M1 and M2 are shown in sections 2.4.1 and 2.4.2 respectively. We conclude this chapter by obtaining the settings for M1 and M2 at the microscopic scale in sections 2.5.2 and 2.5.3 respectively.

2.1 Diffusion-Advection Equation

Let \( \Omega \subset \mathbb{R}^n \) be the given porous medium with sufficiently smooth boundary \( \partial \Omega \). Suppose that \( u(t,x) \) denotes the concentration of a chemical species, \( A \), present in the fluid and \( Q(t,x) \) is the flux, i.e., rate per unit area of the amount of species entering or leaving the domain through the boundary. Also, \( f \) denotes the rate per unit volume by which the species is either consumed or produced. Then the diffusion-advection equation for \( A \) is given by

\[
\varphi \frac{\partial}{\partial t} u - \nabla \cdot Q = f, \tag{2.1.1}
\]

where \( \varphi \in (0,1) \) is the porosity of the medium (for derivation of (2.1.1) see [Log01]). We focus on the flux \( Q \). In homogenization one considers two modes of transportation in the pore space. The first one is advection in which the substance is carried from one place to another by the bulk motion of the fluid present in the medium. The advective flux is given as

\[
Q_{\text{adv}} = \bar{q} u, \tag{2.1.2}
\]

where \( \bar{q} \) is the fluid velocity. The another process by which mass can be transported is diffusion. In this work we assume the diffusive flux \( Q_{\text{diff}} \) to be given by Fick’s law, i.e.,

\[
Q_{\text{diff}} := \text{diffusive flux} = -D_{\text{diff}} \nabla_x u, \tag{2.1.3}
\]

where \( D_{\text{diff}} \) is a positive definite symmetric matrix. Later on we restrict ourselves to the case of scalar \( D_{\text{diff}} \).

2.2 Reaction Rates

In a chemical reaction, a chemical species can either be consumed or produced. This leads us to introduce two types of reaction rates: the rate of consumption if the species
is consumed and the rate of production if the species is produced. For example, let us consider $N$ number of chemical species be involved in $J$ number of reactions which are given as

$$\tau_{ij}X_1 + \tau_{2j}X_2 + \ldots + \tau_{Nj}X_N \rightarrow \nu_{ij}Y_1 + \nu_{2j}Y_2 + \ldots + \nu_{Nj}Y_N, \quad (2.2.1)$$

where $X_i$ and $Y_i$ denote the chemical species, and $-\tau_{ij}$ and $\nu_{ij}$ are the 

**stoichiometric coefficients** for $1 \leq i \leq N$ and $1 \leq j \leq J$. The rate of consumption of $X_i$ is given by (in this setting by mass action law)

$$R_{X_i}(u) = \sum_{j=1}^{J} (-\tau_{ij}) R_j(u) = -\sum_{j=1}^{J} \tau_{ij}k_j \prod_{i=1}^{N} u_i^{\nu_{ij}},$$

where $k_j$ is the reaction rate factor. Similarly the rate of production of $Y_i$ is given as $R_{P_{Y_i}} = \sum_{j=1}^{J} k_j \nu_{ij} \prod_{i=1}^{N} u_i^{\tau_{ij}}$. If there is no confusion, from here and on we simply prefer the term reaction rate for both the rate of consumption and the rate of production. The reactions of type (2.2.1) are called the **irreversible reactions**. Now we introduce the reversible reactions.

A reversible reaction is a reaction in which reactants react to form products called the **forward reaction** and products react to give the reactants back called the **backward reaction**. When the reversible reactions reach equilibrium, it means that the reaction rates are not zero but they proceed with equal rate. For example, let us consider the following reversible reaction

$$\tau_{ij}X_1 + \tau_{2j}X_2 + \ldots + \tau_{Nj}X_N \rightleftharpoons \nu_{ij}Y_1 + \nu_{2j}Y_2 + \ldots + \nu_{Nj}Y_N, \quad 1 \leq j \leq J, \quad (2.2.2)$$

where $X_i$ and $Y_i$ denote the chemical species, and $-\tau_{ij}$ and $\nu_{ij}$ are the stoichiometric coefficients for $1 \leq i \leq N$ and $1 \leq j \leq J$. Let $u_i$ and $v_i$ denote the concentrations of $X_i$ and $Y_i$ respectively. Then the reaction rate of the species $X_i$ (by mass action law) is given as

$$R_{X_i}(u) = \sum_{j=1}^{J} (-\tau_{ij}) R_j(u) = -\sum_{j=1}^{J} \tau_{ij}(R_j^f(u) - R_j^b(u)), \quad (2.2.3)$$

where

$$R_j^f(u) = \text{forward reaction rate} = k_j^f \prod_{m=1}^{N} u_m^{\nu_{mj}} \quad \text{and} \quad (2.2.4)$$

$$R_j^b(u) = \text{backward reaction rate} = k_j^b \prod_{m=1}^{N} v_m^{\nu_{mj}}, \quad (2.2.5)$$

where $k_j^f, k_j^b > 0$ are the forward and backward reaction rate factors. Similarly, we can express the reaction rate for $Y_i$ as well for all $1 \leq i \leq N$. We note that the expression for reaction rate in (2.2.3) is motivated from the work of Kräutle (cf. [Krä08]).

### 2.3 Dissolution and Precipitation

Crystal (immobile species) dissolution is a process in which a solid substance solubilizes in a given solvent, i.e., the mass transfer from the surface of the solid parts to the liquid phase. Precipitation or adsorption is the reverse process of dissolution. When a chemical solution, containing a substance, is supersaturated or the crystals of this substance are present in the solution, precipitation occurs. Following the notion of Knabner and van Duijn (cf. [KvD96]), let $c_1$ and $c_2$ be the concentrations of two chemical species $M_1$ and $M_2$ present in the pore space. Let $c_{12}$ be the concentration of an immobile species $M_{12}$ attached to the
solid parts. We assume that \( n \) molecules of \( M_1 \) and \( m \) molecules of \( M_2 \) precipitate to give one molecule of \( M_{12} \). The reverse reaction of dissolution is also possible, i.e.,

\[
M_1 + M_2 \rightleftharpoons M_{12},
\]

then by mass action law the rate of precipitation \( R_p \) is given by

\[
R_p = k_p c_1^n c_2^m,
\]

where \( k_p \) is the precipitation rate constant. We assume that the dissolution rate \( R_d \) is constant in the presence of immobile species on \( \Gamma^* \) and has to be such that in the absence of immobile species the overall rate is zero. To achieve this, we set \( R_d \in k_d \psi(c_{12}) \), where \( k_d > 0 \) is the dissolution rate constant and \( \psi(c_{12}) \) is defined by

\[
\psi(c_{12}) = \begin{cases} 
0 & \text{if } c_{12} < 0, \\
[0, 1] & \text{if } c_{12} = 0, \\
1 & \text{if } c_{12} > 0.
\end{cases}
\]

Therefore the equation for immobile species is

\[
\frac{\partial c_{12}}{\partial t} \in (R_p - k_d \psi(c_{12})).
\]

In this work, we consider only on dissolution together with diffusion and reaction of chemical species and from here on precipitation is no longer considered. However, interested readers can look into the works of Knabner and van Duijn (cf. [Kna91], [Kna86], [KvDH95], [KvD96] etc.) and references therein for modeling and mathematical analysis of models involving precipitation and dissolution.

## 2.4 Diffusion-Reaction Models

Let \( \Omega \subset \mathbb{R}^n \) be the given porous medium. Assume that \( \Omega \) is a bounded domain. Let \( \Omega^p \) and \( \Omega^s \) denote the pore space and union of the solid parts such that \( \Omega = \Omega^p \cup \Omega^s \) and \( \Omega^p \cap \Omega^s = \emptyset \), see figure 1.1.1. Suppose \( \partial \Omega \) and \( \Gamma^* \) denote the boundary of the domain \( \Omega \) and union of the boundaries of the solid parts respectively. We define \( \partial \Omega^p := \partial \Omega \cup \Gamma^* \). Both \( \partial \Omega \) and \( \Gamma^* \) are assumed to be sufficiently smooth. For a \( T > 0 \), \([0, T]\) denotes the time interval.

### 2.4.1 Model M1

Let \( I \) number of mobile species be present in the pore space \( \Omega^p \) (see figure 2.4.1). These species diffuse and react with each other. All these reactions are reversible. We assume that the fluid velocity is 0, i.e., there is no advection. The reaction is shown below

\[
\tau_{1j} X_1 + \tau_{2j} X_2 + \ldots + \tau_{IJ} X_I \rightleftharpoons \nu_{1j} X_1 + \nu_{2j} X_2 + \ldots + \nu_{IJ} X_I, \quad \text{for } 1 \leq j \leq J,
\]

where \( X_i \), for \( 1 \leq i \leq I \), denotes the chemical species involved in \( J \) number of reactions. The stoichiometric coefficients are \( -\tau_{ij} \in \mathbb{Z}_0^- \) and \( \nu_{ij} \in \mathbb{Z}_0^+ \) respectively. Let \( u_i \) denote the concentration of \( X_i \) for \( 1 \leq i \leq I \). Then the system of diffusion-reaction equations of these species is given as\(^3\)

\[
\frac{\partial u}{\partial t} - \nabla \cdot D \nabla u = SR(u) \quad \text{in } (0, T) \times \Omega^p,
\]

\(^3\)The reaction rates (given by mass action law) are of the form \((2.4.7)\) which is motivated from the work of Kräutle in [Krä08].
2.4. Diffusion-Reaction Models

where \( u = (u_1, u_2, \ldots, u_I) \), and \( SR(u) \) is the reaction term. Here \( \bar{D} := \text{diag}(d_1, d_2, \ldots, d_I) \) is the diagonal positive definite matrix of diffusion coefficients \( d_i \) for \( 1 \leq i \leq I \) and \( S \) is the \( I \times J \)-th order stoichiometric matrix with entries \( s_{ij} = \nu_{ij} - \tau_{ij} \), i.e.,

\[
S = (s_{ij})_{1 \leq i \leq I, 1 \leq j \leq J} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1J} \\ s_{21} & s_{22} & \cdots & s_{1J} \\ \vdots & \vdots & \ddots & \vdots \\ s_{I1} & s_{I2} & \cdots & s_{IJ} \end{bmatrix}_{I \times J}, \tag{2.4.3}
\]

and \( R = (R_j)_{1 \leq j \leq J} \) is the \( J \)-th order reaction rate vector which is given as

\[
R_j(u) = R^f_j(u) - R^b_j(u), \tag{2.4.4}
\]

where

\[
R^f_j(u) = \text{forward reaction rate} = k^f_j \prod_{s_{mj} < 0}^{m=1} u_m^{-s_{mj}}, \tag{2.4.5}
\]

and

\[
R^b_j(u) = \text{backward reaction rate} = k^b_j \prod_{s_{mj} > 0}^{m=1} u_m^{s_{mj}}, \tag{2.4.6}
\]

where \( k^f_j, k^b_j > 0 \) are forward and backward reaction rate factors respectively. Therefore the reaction rate term for the \( i \)-th species is given by

\[
(SR(u))_i = \sum_{j=1}^{J} s_{ij} R_j(u) \\
= \sum_{j=1}^{J} s_{ij} (R^f_j(u) - R^b_j(u)) \\
= \sum_{j=1}^{J} s_{ij} \left( k^f_j \prod_{s_{mj} < 0}^{m=1} u_m^{-s_{mj}} - k^b_j \prod_{s_{mj} > 0}^{m=1} u_m^{s_{mj}} \right). \tag{2.4.7}
\]

We suppose that the species present in the fluid have no interaction with the boundaries \( \partial \Omega \) and \( \Gamma^* \), in other words, no flux is entering or leaving the domain \( \Omega \) through \( \partial \Omega \) and \( \Gamma^* \).
This can be mathematically written as
\[-\bar{D} \nabla u \cdot \vec{n} = 0 \text{ on } (0, T) \times \partial \Omega, \quad (2.4.8)\]
\[-\bar{D} \nabla u \cdot \vec{n} = 0 \text{ on } (0, T) \times \Gamma^*, \quad (2.4.9)\]
The BCs (2.4.8) and (2.4.9) can be rewritten as
\[-\bar{D} \nabla u \cdot \vec{n} = 0 \text{ on } (0, T) \times \partial \Omega, \quad (2.4.10)\]

Initially for \( t = 0 \), we assume \( u(0,x) = u_0(x) \) in \( \Omega^p \), where \( u_0(x) > 0 \) componentwise, i.e.,
\[ \text{for all } i = 1, 2, ..., I. \quad (2.4.11)\]

For technical reasons, we replace the matrix \( \bar{D} \) by a strictly positive constant \( D > 0 \) and from here on we assume
\[ \bar{D} := D > 0. \quad (2.4.12)\]
Therefore the diffusion-reaction model is given by
\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot D \nabla u &= SR(u) \quad \text{in } (0, T) \times \Omega^p, \quad (2.4.13) \\
-D \nabla u \cdot \vec{n} &= 0 \quad \text{on } (0, T) \times \Gamma^*, \quad (2.4.14) \\
-D \nabla u \cdot \vec{n} &= 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (2.4.15) \\
u(0, x) &= u_0(x) \quad \text{in } \Omega^p. \quad (2.4.16)
\end{align*}
\]

We are mainly interested in the global solution of the problem (2.4.13)-(2.4.16), which is shown in chapter 4. In order to prove the existence of the global solution of this problem, we need the assumption (2.4.12) and this is the price which we have to pay at here. There are existence results for the global solution of a system of diffusion-reaction equations for some special situations (see [Pie10], [PS97]), but to our knowledge the existence of the global solution for (2.4.13)-(2.4.16) with \( I (> 2) \) different diffusion coefficients is still unknown.

2.4.2 Model M2

Let \( \Omega, \Gamma^*, \partial \Omega \) and \( \partial \Omega^p \) be as in section 2.4.1. We incorporate the dissolution process, defined in section 2.3, in the previous model. Let \( \vec{q} \) be the given velocity field of the fluid which is present in the pore space of the porous medium \( \Omega \) such that
\[
\begin{align*}
\nabla \cdot \vec{q} &= 0 \quad \text{in } \Omega^p, \quad (2.4.17) \\
\vec{q} &= 0 \quad \text{on } \Gamma^*. \quad (2.4.18)
\end{align*}
\]

Let \( I_1 \) number of mobile species present in the fluid. We refer to these \( I_1 \) species as \textit{type I} species. Let \( I_2 \) number of immobile species (crystals) present on the surface of the solid parts. Due to the presence of the fluid in \( \Omega^p \), immobile species interact with the fluid on \( \Gamma^* \), i.e., the dissolution of immobile species takes place on the surface of the solid parts. Suppose that a number of \( I_2 \) mobile species is supplied by immobile species via dissolution. We call these \( I_2 \) mobile species as \textit{type II} species. Confer the figure 2.4.2. Both \textit{type I} and \textit{type II} species transport inside the domain by the effect of diffusion and advection and they react with each other under the following reaction:
\[
\tau_{1j} X_1 + \tau_{2j} X_2 + ... + \tau_{I_1j} X_{I_1} + \kappa_{1j} Y_1 + \kappa_{2j} Y_2 + ... + \kappa_{I_2j} Y_{I_2},
\]

\[
\tau_{1j} X_1 + \tau_{2j} X_2 + ... + \tau_{I_1j} X_{I_1} + \tilde{\kappa}_{1j} Y_1 + \tilde{\kappa}_{2j} Y_2 + ... + \tilde{\kappa}_{I_2j} Y_{I_2}. \quad (2.4.19)
\]
2.4. Diffusion-Reaction Models

Figure 2.4.2: Model M2 with mobile species in $\Omega^p$ and immobile species on $\Gamma^*$.

where $1 \leq j \leq J$. For all $i = 1, 2, ..., I_1$, $k = 1, 2, ..., I_2$, $X_i$ and $Y_k$ denote type I and type II species respectively. The stoichiometric coefficients $-\tau_{ij}$, $-\kappa_{ij} \in \mathbb{Z}_0^-$ and $\bar{\tau}_{ij}$, $\bar{\kappa}_{ij} \in \mathbb{Z}_0^+$ respectively. We define two stoichiometric matrix $S_1$ and $S_2$ of order $I_1 \times J$-th and $I_2 \times J$-th whose entries are $s_{ij} = \bar{\tau}_{ij} - \tau_{ij}$ and $\nu_{ij} = \bar{\kappa}_{ij} - \kappa_{ij}$ respectively, i.e.,

$$S_1 = (s_{ij})_{1 \leq i \leq I_1, 1 \leq j \leq J} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1J} \\ s_{21} & s_{22} & \cdots & s_{1J} \\ \vdots & \vdots & \ddots & \vdots \\ s_{I_11} & s_{I_12} & \cdots & s_{I_1J} \end{bmatrix}_{I_1 \times J} \tag{2.4.20}$$

and

$$S_2 = (\nu_{ij})_{1 \leq k \leq I_2, 1 \leq j \leq J} = \begin{bmatrix} \nu_{11} & \nu_{12} & \cdots & \nu_{1J} \\ \nu_{21} & \nu_{22} & \cdots & \nu_{1J} \\ \vdots & \vdots & \ddots & \vdots \\ \nu_{I_21} & \nu_{I_22} & \cdots & \nu_{I_2J} \end{bmatrix}_{I_2 \times J} \tag{2.4.21}$$

For $i = 1, 2, ..., I_1$, $k = 1, 2, ..., I_2$ and $m = 1, 2, ..., I_2$, let $u_i$, $v_k$ and $w_m$ denote the concentrations of type I, type II and immobile species. Then the systems of diffusion-reaction equations for type I and type II species are given as

$$\frac{\partial u}{\partial t} - \nabla \cdot (D_1 \nabla u - \bar{q} u) = S_1 R(u,v) \quad \text{in} \quad (0,T) \times \Omega^p \tag{2.4.22}$$

and

$$\frac{\partial v}{\partial t} - \nabla \cdot (D_2 \nabla v - \bar{q} v) = S_2 R(u,v) \quad \text{in} \quad (0,T) \times \Omega^p. \tag{2.4.23}$$

where $D_1$ and $D_2$ are diagonal positive definite matrices. The dissolution equation for immobile species is given as

$$\frac{\partial w}{\partial t} = -k_d z \quad \text{on} \quad (0,T) \times \Gamma^*, \quad (2.4.24)$$

$$z \in \psi(w) \quad \text{on} \quad (0,T) \times \Gamma^*, \quad (2.4.25)$$

where

$$\psi(w_m) = \begin{cases} \{0\} & \text{if } w_m < 0, \\ [0,1] & \text{if } w_m = 0, \\ \{1\} & \text{if } w_m > 0. \end{cases} \quad (2.4.26)$$
For reasons of mathematical necessity, like (2.4.12), we replace the matrices $D_1$ and $D_2$ by a strictly positive constant $D$ and from here on we assume $D := D_1 := D_2 > 0$. Let $\partial \Omega := \partial \Omega_{in} \cup \partial \Omega_{out}$, where on $\partial \Omega_{in}$ and $\partial \Omega_{out}$ we prescribe the inflow and outflow boundary conditions for the type I and type II species. Since type II species are supplied by the dissolution process on $\Gamma^*$, the flux for the type II species on $\Gamma^*$ is equal to the rate of change of immobile species on $\Gamma^*$, i.e., for the type II species, we have an additional boundary condition. The complete diffusion-reaction-dissolution model is given as\(^4\)

For type I species:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot (D \nabla u - \vec{q} u) &= S_1 R(u, v) & \text{in } (0, T) \times \Omega^p, \\
-(D \nabla u - \vec{q} u) \cdot \vec{n} &= d & \text{on } (0, T) \times \partial \Omega_{in}, \\
-D \nabla u \cdot \vec{n} &= 0 & \text{on } (0, T) \times \partial \Omega_{out}, \\
-D \nabla v \cdot \vec{n} &= 0 & \text{on } (0, T) \times \Gamma^*, \\
u(0, x) &= u_0(x), & \text{in } \Omega^p.
\end{align*}
\]

(2.4.27)

(2.4.28)

(2.4.29)

(2.4.30)

where $d \leq 0$ componentwise, i.e., $d_i \leq 0$ for all $1 \leq i \leq I_1$.

For type II species:

\[
\begin{align*}
\frac{\partial v}{\partial t} - \nabla \cdot (D \nabla v - \vec{q} v) &= S_2 R(u, v) & \text{in } (0, T) \times \Omega^p, \\
-(D \nabla v - \vec{q} v) \cdot \vec{n} &= d & \text{on } (0, T) \times \partial \Omega_{in}, \\
-D \nabla u \cdot \vec{n} &= 0 & \text{on } (0, T) \times \partial \Omega_{out}, \\
-D \nabla v \cdot \vec{n} &= \frac{\partial w}{\partial t} & \text{on } (0, T) \times \Gamma^*, \\
v(0, x) &= v_0(x) & \text{in } \Omega^p.
\end{align*}
\]

(2.4.32)

(2.4.33)

(2.4.34)

(2.4.35)

(2.4.36)

For immobile species:

\[
\begin{align*}
\frac{\partial w}{\partial t} &= -k_d z & \text{on } (0, T) \times \Gamma^*, \\
z &= \psi(w) & \text{on } (0, T) \times \Gamma^*, \\
w(0, x) &= w_0(x) & \text{on } \Gamma^*.
\end{align*}
\]

(2.4.37)

(2.4.38)

(2.4.39)

where $\psi(w)$ is given by (2.4.26) and the initial conditions are strictly positive, i.e., $u_0(x)$, $v_0(x)$ and $w_0(x) > 0$ componentwise. For the velocity $\vec{q}$, we assume the following conditions:

\[
\nabla \cdot \vec{q} = 0 \text{ in } \Omega^p, \quad -\vec{q} \cdot \vec{n} > 0 \text{ on } \partial \Omega_{in}, \quad -\vec{q} \cdot \vec{n} \leq 0 \text{ on } \partial \Omega_{out} \quad \text{and} \quad \vec{q} = 0 \text{ on } \Gamma^*. \quad (2.4.40)
\]

The reaction rate term for the $i$-th species of type I is given by

\[
(S_1 R_j(u, v))_i = \sum_{j=1}^{J} s_{ij} \left( R^f_j(u, v) - R^b_j(u, v) \right)
\]

\[
\begin{align*}
&= \sum_{j=1}^{J} s_{ij} \left( \prod_{r=1}^{J} u_r^{s_{rj}} \prod_{l=1}^{J} v_l^{v_{lj}} - k_b^j \prod_{r=1}^{J} u_r^{s_{rj}} \prod_{l=1}^{J} v_l^{v_{lj}} \right).
\end{align*}
\]

(2.4.41)

\(^4\)The proposed mathematical model is motivated from the works in [vDP04], [KvDH95], [Krä08].
2.5. Scaling

Similarly, the reaction rate term for the \( k \)-th species of type II is given as

\[
(S_2 R_j(u,v))_k = \sum_{j=1}^{J} \nu_{kj} \left( R^f_j(u,v) - R^b_j(u,v) \right)
\]

\[
= \sum_{j=1}^{J} \nu_{kj} \left( k^f_j \prod_{r=1}^{I_s} u_r^{-s_{rj}} \prod_{l=1}^{I_v} v_l^{-\nu_{lj}} - k^b_j \prod_{r=1}^{I_s} u_r^{-s_{rj}} \prod_{l=1}^{I_v} v_l^{\nu_{lj}} \right),
\]  

(2.4.42)

where \( k^f_j \) and \( k^b_j > 0 \) are the forward and backward reaction rate factors. In next section, we derive the models M1 and M2 at the microscopic scale.

2.5 Scaling

2.5.1 The \( \varepsilon \)-periodic Approximation of \( \Omega \)

We begin this section by making some assumptions on our porous medium \( \Omega \) introduced in section 2.4. Let \( Y = (0,1)^n \subset \mathbb{R}^n \) be the unit representative cell which is composed of a solid part \( Y^s \) and a pore part \( Y^p \) such that \( Y = Y^s \cup Y^p \) and \( Y^s \subset Y \) (see figure 2.5.1). Let \( \Gamma \) be the sufficiently smooth boundary of \( Y^s \).

Let \( \chi(y) \) be the \( Y \)-periodic characteristic (indicator) function of \( Y^p \) defined by

\[
\chi(y) = \begin{cases} 
1 & \text{for } y \in Y^p, \\
0 & \text{for } y \in Y - Y^p.
\end{cases}
\]

The domain \( \Omega \) is assumed to be periodic and is covered by a finite union of the cells \( Y \). In order to avoid the technical difficulties, we postulate that:

- solid parts do not touch the boundary \( \partial \Omega \),
- solid parts do not touch each other,
- solid parts do not touch the boundary of \( Y \).

For \( n = 2 \), the disconnectedness of solid parts does not disrupt the generality as the connection of two solid parts will imply the blocking of porous samples, see figure 2.5.2. However, for \( n \geq 3 \), the disconnectedness of the solid parts is actually an assumption, since the connection between the two solid parts is possible without violating the periodicity of the domain, see figure 2.5.3.
For any \( m = (m_1, m_2, ..., m_n) \in \mathbb{Z}^n \), we define

\[
Y_m := Y + \sum_{l=1}^{n} m_l e_l, \tag{2.5.1}
\]

\[
Y^p_m := Y^p + \sum_{l=1}^{n} m_l e_l, \tag{2.5.2}
\]

\[
Y^s_m := Y^s + \sum_{l=1}^{n} m_l e_l, \tag{2.5.3}
\]

\[
\Gamma_m := \Gamma + \sum_{l=1}^{n} m_l e_l, \tag{2.5.4}
\]

where \( e_l \) is the \( l \)-th unit vector, such that

\[
\Omega \subset \bigcup_{m \in \mathbb{Z}^n} Y_m, \tag{2.5.5}
\]

\[
\Omega^p \subset \bigcup_{m \in \mathbb{Z}^n} Y^p_m, \tag{2.5.6}
\]

\[
\Omega^s \subset \bigcup_{m \in \mathbb{Z}^n} Y^s_m, \tag{2.5.7}
\]

\[
\Gamma^* \subset \bigcup_{m \in \mathbb{Z}^n} \Gamma_m. \tag{2.5.8}
\]

Here we follow the notations introduced in [Mil92]. Let \((\varepsilon_n)_{n \in \mathbb{N}}\) be a sequence of positive real numbers converging to zero. If there is no confusion, we drop the suffix 'n'. Let \( \Omega \) is covered by a finite union of \( \varepsilon Y_m \) cells such that \( \varepsilon Y_m \subset \Omega \), where \( m \in \mathbb{Z}^n \). To be more
precise, it is assumed that there is an $\varepsilon_0 > 0$, called the natural scaling parameter, such that $\Omega$ is covered by the finite union of $\varepsilon_0 Y_m$ cells. However, for the homogenization, we consider the sequence of positive real numbers, $\varepsilon$ to converge to 0 (see fig 2.5.4).

Figure 2.5.4: A schematic representation of periodic homogenization.

We further define

\[
\begin{align*}
\Omega_\varepsilon^p &:= \bigcup_{m \in \mathbb{Z}^n} \{ \varepsilon Y_m^p : \varepsilon Y_m^p \subset \Omega \}, \\
\Omega_\varepsilon^s &:= \bigcup_{m \in \mathbb{Z}^n} \{ \varepsilon Y_m^s : \varepsilon Y_m^s \subset \Omega \}, \\
\Gamma_\varepsilon &:= \bigcup_{m \in \mathbb{Z}^n} \{ \varepsilon \Gamma_m : \varepsilon \Gamma_m \subset \Omega \}, \\
\partial \Omega_\varepsilon^p &:= \partial \Omega \cup \Gamma_\varepsilon,
\end{align*}
\]

(2.5.9) (2.5.10) (2.5.11) (2.5.12)

see figure 2.5.5.

Figure 2.5.5: $\varepsilon$-periodic scaling of the domain $\Omega$. 
We denote $dx$ and $dy$ as the volume elements in $\Omega$ and in $Y$, and $d\sigma_y$ and $d\sigma_x$ as the surface elements on $\Gamma$ and on $\Gamma_\varepsilon$ respectively. Due to $Y$-periodicity, the characteristic function of the domain $\Omega_\varepsilon$ in the domain $\Omega$ is given by

$$
\chi(\varepsilon(x)) = \chi\left(\frac{x}{\varepsilon}\right) \quad (2.5.13)
$$

and is defined as

$$
\chi(\varepsilon(x)) = 1 \text{ for } x \in \Omega_\varepsilon^p,
\quad = 0 \text{ for } x \in \Omega - \Omega_\varepsilon^p. \quad (2.5.14)
$$

### 2.5.2 Setting of Model M1 at the Micro Scale

**Nondimensionalization:** The description of model M1 at the microscopic scale using the scaling parameter $\varepsilon$ can be motivated from the nondimensionalization of the equations (2.4.13)-(2.4.16). Assume that $u_{ref}$ is the reference concentration of the mobile species which can be an upper bound of the concentration and may be given from physical considerations or maximum estimates. Let $l_{ref}$ be the reference microscopic length (e.g., a typical pore diameter) and $L_{ref}$ denote the reference macroscopic length (e.g., the diameter of the domain $\Omega$). Also assume that $T_{ref} (= L_{ref}^2 D)$ is the reference time. We set

$$
\begin{align*}
\bar{u} &= \frac{u}{u_{ref}}, \quad \bar{x} = \frac{x}{L_{ref}}, \quad \bar{t} = \frac{t}{T_{ref}}, \\
\bar{D} &= \frac{DT_{ref}}{L_{ref}^2}, \quad \varepsilon = \frac{l_{ref}}{L_{ref}}. \quad (2.5.15)
\end{align*}
$$

We denote the scaled domain $\Omega_\varepsilon^p$ and interface $\Gamma^*$ by $\Omega_\varepsilon^p$ and $\Gamma_\varepsilon$ respectively. We use the old notation $D$ for $\bar{D}$, i.e., $D = \bar{D}$. A straightforward simplification will yield the required microscopic description of (2.4.13)-(2.4.16) which is given by

$$
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot D \nabla u &= SR(u) \quad \text{in } (0,T) \times \Omega_\varepsilon^p, \quad (2.5.16) \\
u(0,x) &= u_0(x) \quad \text{in } \Omega_\varepsilon^p, \quad (2.5.17) \\
-D \nabla u \cdot \bar{n} &= 0 \quad \text{on } (0,T) \times \partial \Omega, \quad (2.5.18) \\
-D \nabla u \cdot \bar{n} &= 0 \quad \text{on } (0,T) \times \Gamma_\varepsilon. \quad (2.5.19)
\end{align*}
$$

We denote this problem by $(P_\varepsilon^1)$. In chapter 3, we give the notion of weak solution of $(P_\varepsilon^1)$ in some appropriate sense and we prove the existence of a unique positive global weak solution of this problem in chapter 4. The homogenization of $(P_\varepsilon^1)$ is also shown in chapter 4.

### 2.5.3 Setting of Model M2 at the Micro Scale

**Nondimensionalization:** In this section, we give the microscopic description of model M2. For this model, we adopt the nondimensionalization technique from [vDP04]. We nondimensionalize the equations (2.4.27)-(2.4.39) in the following way: Let $u_{ref}$, $v_{ref}$ and $w_{ref}$ be the characteristic concentrations of type I, type II and immobile species respectively which can be the upper bounds of the concentrations. Further assume that $q_{ref}$, $L_{ref}$ and $T_{ref} (= \frac{l_{ref}}{q_{ref}})$ are the characteristic velocity, length and time respectively. We set
We denote the scaled domain \( \Omega^p \) and interface \( \Gamma^* \) by \( \Omega^p_\varepsilon \) and \( \Gamma_\varepsilon \) respectively. We use the old notations \( D, k_d, \) and \( d \) for \( \tilde{D}, \tilde{k}_d, \) and \( \tilde{d} \) respectively. With the help of (2.5.20), the required microscopic description of (2.4.27)-(2.4.39) is given by

### Equations for type I species:

\[
\begin{align*}
\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot (D \nabla u_\varepsilon - \vec{q}_\varepsilon u_\varepsilon) &= S_1 R(u_\varepsilon, v_\varepsilon) \quad \text{in} \quad (0, T) \times \Omega^p_\varepsilon, \tag{2.5.21} \\
-(D \nabla u_\varepsilon - \vec{q}_\varepsilon u_\varepsilon) \cdot \vec{n} &= d \quad \text{on} \quad (0, T) \times \partial \Omega_{in}, \\
-D \nabla u_\varepsilon \cdot \vec{n} &= 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{out}, \\
u_\varepsilon(0, x) &= u_0(x) \quad \text{in} \quad \Omega^p_\varepsilon, \tag{2.5.25}
\end{align*}
\]

where \( d_i \leq 0 \) for all \( 1 \leq i \leq I_1. \)

### Equations for type II species:

\[
\begin{align*}
\frac{\partial v_\varepsilon}{\partial t} - \nabla \cdot (D \nabla v_\varepsilon - \vec{q}_\varepsilon v_\varepsilon) &= S_2 R(u_\varepsilon, v_\varepsilon) \quad \text{in} \quad (0, T) \times \Omega^p_\varepsilon, \tag{2.5.27} \\
-(D \nabla v_\varepsilon - \vec{q}_\varepsilon v_\varepsilon) \cdot \vec{n} &= 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{in}, \\
-D \nabla v_\varepsilon \cdot \vec{n} &= 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{out}, \\
-v_\varepsilon(0, x) &= v_0(x) \quad \text{in} \quad \Omega^p_\varepsilon. \tag{2.5.31}
\end{align*}
\]

### Equations for immobile species:

\[
\begin{align*}
\frac{\partial w_\varepsilon}{\partial t} &= -k_d z \quad \text{on} \quad (0, T) \times \Gamma_\varepsilon, \tag{2.5.32} \\
z &= \psi(w_\varepsilon) \quad \text{on} \quad (0, T) \times \Gamma_\varepsilon, \tag{2.5.33} \\
w_\varepsilon(0, x) &= w_0(x) \quad \text{on} \quad \Gamma_\varepsilon, \tag{2.5.34}
\end{align*}
\]

where

\[
\psi(w_{\varepsilon m}) = \begin{cases} 
\{0\} & \text{if } w_{\varepsilon m} < 0, \\
[0, 1] & \text{if } w_{\varepsilon m} = 0, \\
\{1\} & \text{if } w_{\varepsilon m} > 0.
\end{cases} \tag{2.5.35}
\]

The velocity \( \vec{q}_\varepsilon \) satisfies:

\[
\nabla \cdot \vec{q}_\varepsilon = 0 \quad \text{in} \quad \Omega^p_\varepsilon, \quad -\vec{q}_\varepsilon \cdot \vec{n} > 0 \quad \text{on} \quad \partial \Omega_{in}, \quad -\vec{q}_\varepsilon \cdot \vec{n} \leq 0 \quad \text{on} \quad \partial \Omega_{out} \text{ and } \vec{q}_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon. \tag{2.5.36}
\]

We denote the problem (2.5.21)-(2.5.35) by \( P^2_\varepsilon \). The notion of weak solution for \( (P^2_\varepsilon) \) is given in chapter 3. The existence of a unique positive global weak solution of \( (P^2_\varepsilon) \) and its homogenization are shown in chapter 4.
In this chapter, we collect some mathematical tools which are required to analyze the problems \((P_1^\varepsilon)\) and \((P_2^\varepsilon)\) in the next chapter. In section 3.1, we introduce the function spaces such as \(L^p\)-spaces, Sobolev spaces and their duals. In section 3.2, we give the notion of weak formulations for \((P_1^\varepsilon)\) and \((P_2^\varepsilon)\), respectively. We present a very short overview of maximal parabolic regularity of elliptic operators in section 3.3. Some extension and embedding theorems for the domain \(\Omega_\varepsilon^p\) are proved in section 3.4. In sections 3.5 and 3.6, we present a short overview of two-scale convergence and periodic unfolding respectively.

### 3.1 Function Spaces

#### 3.1.1 Function Spaces on \(\Omega\)
Let \(1 < p, q < \infty\) be such that \(\frac{1}{p} + \frac{1}{q} = 1\). Assume that \(\Omega \subset \mathbb{R}^n (n \geq 2)\) is a bounded domain with sufficiently smooth boundary \(\partial \Omega\). As usual, \(L^p(\Omega)\) is the set of all equivalence classes of real-valued functions \(u(\cdot)\) such that \(u(x)\) is defined for almost every \(x \in \Omega\), is measurable and \(|u(\cdot)|^p\) is Lebesgue integrable. \(L^p(\Omega)\) is a Banach space w.r.t. the norm

\[
||u||_{L^p(\Omega)} = \begin{cases} 
\left(\int_\Omega |u(x)|^p \, dx\right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\
\text{ess sup}_{x \in \Omega} |u(x)| & \text{for } p = \infty.
\end{cases}
\]  

(3.1.1)

The space \(H^{1,p}(\Omega)\) is the usual Sobolev space w.r.t. the norm

\[
||u||_{H^{1,p}(\Omega)} = \begin{cases} 
\left(||u||_{L^p(\Omega)}^p + ||\nabla u||_{L^p(\Omega)}^p\right)^{\frac{1}{p}} & \text{for } 1 \leq p < \infty, \\
\text{ess sup}_{x \in \Omega} ||u(x)| + |\nabla u(x)|| & \text{for } p = \infty.
\end{cases}
\]  

(3.1.2)

The duality pairing between \(H^{1,q}(\Omega)\) and \(H^{1,q}(\Omega)^*\) is denoted by \(\langle \cdot, \cdot \rangle_{H^{1,q}(\Omega) \times H^{1,q}(\Omega)}\). We define the continuous embedding \(L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^*\) as

\[
\langle f, v \rangle_{H^{1,q}(\Omega)^* \times H^{1,q}(\Omega)} = \langle f, v \rangle_{L^q(\Omega) \times L^q(\Omega)} \text{ for } f \in L^p(\Omega), \, v \in H^{1,q}(\Omega).
\]  

(3.1.3)

For \(k \in \mathbb{Z}_0^+\), the space \(C^k(\Omega)\) denotes the Banach space of all \(k\)-times continuously differentiable functions w.r.t. the norm

\[
||u||_{C^k(\Omega)} = \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha u(x)|.
\]  

(3.1.4)

Suppose that \(0 < \gamma \leq 1\). The space \(C^\gamma(\Omega)\) consists of all functions \(u \in C(\Omega)\) such that

\[
||u||_{C^\gamma(\Omega)} = ||u||_{C(\Omega)} + \sup_{x, y \in \Omega} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{\gamma}} \right\} < \infty.
\]  

(3.1.5)
The space \( C^\gamma(\Omega) \) is called the Hölder space. We introduce the Sobolev-Bochner space as

\[
F := F^p(\Omega) := \left\{ u \in L^p((0,T);H^{1,p}(\Omega)) : \frac{du}{dt} \in L^p((0,T);H^{1,q}(\Omega)^*) \right\}
\]

and for any \( u \in F \),

\[
||u||_F = ||u||_{L^p((0,T);H^{1,p}(\Omega))} + ||u||_{L^p((0,T);H^{1,q}(\Omega)^*)} + \left| \frac{du}{dt} \right|_{L^p((0,T);H^{1,q}(\Omega)^*)},
\]

where \( \frac{du}{dt} \) is the distributional time derivative of \( u \). For \( 0 < \theta < 1 \), let

\[
\left( H^{1,q}(\Omega)^*, H^{1,p}(\Omega) \right)_{\theta,p} \quad \text{the real-interpolation space between } H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega),
\]

\[
\left[ H^{1,q}(\Omega)^*, H^{1,p}(\Omega) \right]_\theta \quad \text{the complex-interpolation space between } H^{1,q}(\Omega)^* \text{ and } H^{1,p}(\Omega)
\]

decked with one of their usual norms (cf. [BL76], [Tri95], [Lun95], [Has06]).

**Theorem 3.1.1.** The space \( F \hookrightarrow C([0,T];(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{q},p}) \).

**Proof.** See theorem 4.10.2 in [Ama95]. See also proposition 1.2.10 and remark 1.2.11 in [Lun95].

**Theorem 3.1.2.** Let \( p > n+2 \), then \( F \hookrightarrow L^\infty((0,T) \times \Omega) \).

**Proof.**

**Step 1.:** We notice that

\[
||v(t) - v(t_0)||_{H^{1,q}(\Omega)^*} = \left| \int_0^t v'(s) \, ds \right|_{H^{1,q}(\Omega)^*} \\
\leq \int_0^t ||v'(s)||_{H^{1,q}(\Omega)^*} \, ds \\
\leq \left[ \int_0^t ||v'(s)||_{H^{1,q}(\Omega)^*}^p \, ds \right]^\frac{1}{p} \left[ \int_0^t ds \right]^\frac{1}{q} \\
\leq ||v||_{H^{1,p}((0,T);H^{1,q}(\Omega)^*)} |t - t_0|^{\frac{1}{q}}
\]

\[
\Rightarrow \frac{||v(t) - v(t_0)||_{H^{1,q}(\Omega)^*}}{|t - t_0|^{\frac{1}{q}}} \leq ||v||_{H^{1,p}((0,T);H^{1,q}(\Omega)^*)}.
\]

This implies \( H^{1,p}((0,T);H^{1,q}(\Omega)^*) \hookrightarrow C^\delta([0,T];H^{1,q}(\Omega)^*) \), where \( \delta = \frac{1}{q} = 1 - \frac{1}{p} \).

**Step 2.:** The condition \( p > n+2 \) implies \( \frac{1}{q} + \frac{n}{2p} < 1 - \frac{1}{p} \). Choose \( \lambda \in \left( \left( \frac{1}{2} + \frac{n}{2p} \right) \left( 1 - \frac{1}{p} \right)^{-1}, 1 \right) \)

and set \( \eta := \lambda(1 - \frac{1}{p}) \). Then by reiteration theorem on real-interpolation

\[
\frac{||v(t) - v(t_0)||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{\lambda,1}}}{|t - t_0|^{\delta(1-\lambda)}} = \frac{||v(t) - v(t_0)||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{\lambda(1 - \frac{1}{p}),1}}}{|t - t_0|^{\delta(1-\lambda)}} \\
= \frac{||v(t) - v(t_0)||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{(1 - \frac{1}{p}),p}}^{\lambda_1}}{|t - t_0|^{\delta(1-\lambda)}} \\
\leq C \frac{||v(t) - v(t_0)||_{H^{1,q}(\Omega)^*}^{1-\lambda}}{|t - t_0|^{\delta(1-\lambda)}} \times ||v(t) - v(t_0)||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}}^{\lambda_1}
\]

and

\[
||v(t) - v(t_0)||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{\lambda,1}} \\
\leq \frac{||v(t) - v(t_0)||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}}^{\lambda_1}}{|t - t_0|^{\delta(1-\lambda)}} \times ||v(t) - v(t_0)||_{(H^{1,q}(\Omega)^*, H^{1,p}(\Omega))_{1-\frac{1}{p},p}}^{\lambda_1}
\]

where \( \lambda_1 \) and \( \delta(1-\lambda) \) are constants dependent on \( \lambda \) and \( \delta \).
There exists a

\[ \gamma \in \mathbb{R} \]

\[ \alpha \] 20

Chapter 3. Mathematical Preliminaries

Therefore by step 1 and theorem 3.1.1, it follows that

\[ : \Omega \hookrightarrow u \]

Theorem 3.1.3.

and for \( u \in \mathbb{R}^n \) we have the following embedding (cf. theorem 1.3.3.d in [Tri95] and corollary 5.28 in [KR13])

\[ (H^1,\alpha(\Omega)^*, H^1,\alpha(\Omega))_{\eta,1} \hookrightarrow (H^1,\alpha(\Omega)^*, H^1,\alpha(\Omega))_{\eta,p} \hookrightarrow H^{2n-1,\alpha}(\Omega) \hookrightarrow C^\alpha(\tilde{\Omega}), \]

where \( \alpha = 2\eta - 1 - \frac{n}{p} > 0 \). Therefore combining the steps 2 and 3, we obtain

\[ F \hookrightarrow C^\beta([0, T]; C^\alpha(\tilde{\Omega})) \hookrightarrow C^\alpha((0, T) \times \Omega) \]

\[ \hookrightarrow L^\infty((0, T) \times \Omega), \]

where \( \sigma = \min(\alpha, \beta) \).

\[ \star \]

**Theorem 3.1.3.** Let \( p > n + 2 \). Then \( (H^1,\alpha(\Omega)^*, H^1,\alpha(\Omega))_{1-\frac{1}{p},p} \hookrightarrow L^\infty(\Omega) \).

**Proof.** Let us denote \( E_0 = H^1,\alpha(\Omega)^*, E_1 = H^1,\alpha(\Omega) \) and \( E_1_{1-\frac{1}{p},p} = (H^1,\alpha(\Omega)^*, H^1,\alpha(\Omega))_{1-\frac{1}{p},p} \). By lemma 3.4 in [GGKR00]: \( [E_0, E_1]_{\frac{1}{p}} \hookrightarrow L^p(\Omega) \). From this and reiteration theorem on real-interpolation, we obtain

\[ E_1_{1-\frac{1}{p},p} = ([E_0, E_1]_{\frac{1}{p}}, [E_0, E_1])_{1-\frac{1}{p},p} \hookrightarrow (L^p(\Omega), H^1,\alpha(\Omega))_{1-\frac{1}{p},p} = H^{1,\frac{2}{p}}(\Omega). \]

There exists a \( t > 0 \) such that \( p > n + 2 \Rightarrow 1 - \frac{n+2}{p} > t > 0 \Rightarrow 1 - \frac{2}{p} > t + \frac{n}{p} \). From theorem 4.6.1 (e) in Triebel [Tri95]: \( H^{1,\frac{2}{p}}(\Omega) \hookrightarrow C^t(\tilde{\Omega}) \). Since \( C^t(\tilde{\Omega}) \hookrightarrow L^\infty(\Omega) \), \( H^{1,\frac{2}{p}}(\Omega) \hookrightarrow C^t(\tilde{\Omega}) \hookrightarrow L^\infty(\Omega) \). Therefore \( (H^1,\alpha(\Omega)^*, H^1,\alpha(\Omega))_{1-\frac{1}{p},p} \hookrightarrow L^\infty(\Omega) \).

Now we introduce the norms on the vector-valued function spaces. Let \( I \in \mathbb{N} \) and \( u : \Omega \rightarrow \mathbb{R}^I \) be a vector-valued function. We define

\[ [L^p(\Omega)]^I := \underbrace{L^p(\Omega) \times L^p(\Omega) \times \ldots \times L^p(\Omega)}_{I-times} \]  

and for \( u \in [L^p(\Omega)]^I \) the corresponding norm is given as

\[ |||u|||_{[L^p(\Omega)]^I} := \left( \sum_{i=1}^{I} ||u_i||_{L^p(\Omega)} \right)^{\frac{1}{p}}. \]  

Similarly,

\[ |||u|||_{[L^\infty(\Omega)]^I} := \max_{1 \leq i \leq I} ||u_i||_{L^\infty(\Omega)}, \]

\[ |||u|||_{[H^1,\alpha(\Omega)]^I} := \left( \sum_{i=1}^{I} ||u_i||_{H^1,\alpha(\Omega)} \right)^{\frac{1}{p}}, \]

\[ |||u|||_{[H^1,\infty(\Omega)]^I} := \max_{1 \leq i \leq I} ||u_i||_{H^1,\infty(\Omega)}. \]

\[ \cdot \]
From section 2.5.1, we notice that the surface area of $\Gamma$ space

Similarly, the following theorem:

Proof. Cf. lemma 7.3 in [Rou05].

Furthermore, let $V$, $H$ and $V^*$ be the Gelfand triple, where $V$ a Banach space, $H$ a Hilbert space and $V^*$ is the dual of $V$. Let $H$ be identified with its own dual $(H \cong H^*)$ and $V \subset H$, then $H \subset V^*$. Denote $\Xi := \{u \in L^p((0,T); V) : \frac{du}{dt} \in L^q((0,T); V^*)\}$. We have the following theorem:

**Theorem 3.1.4.** Let $V$, $H$ and $V^*$ be as above. Then $\Xi \subset C([0,T]; H)$ and the following rule of integration holds for any $u, v \in \Xi$ and any $0 \leq t_1 \leq t_2 \leq T$:

\[
\int_{t_1}^{t_2} \frac{d}{dt}(u(t),v(t))_H dt = \int_{t_1}^{t_2} \left\langle \frac{du}{dt},v(t) \right\rangle_{V^* \times V} dt + \int_{t_1}^{t_2} \left\langle u(t),\frac{dv}{dt} \right\rangle_{V \times V^*} dt.
\]

**Proof.** Cf. lemma 7.3 in [Rou05].

### 3.1.2 Function Spaces on $\Omega^\varepsilon$

The function spaces on the domain $\Omega^\varepsilon_p$ are defined in the analogous way as in section 3.1.1 by replacing the domain $\Omega$ by $\Omega^\varepsilon$ in the definitions of the function spaces. The spaces on $\Omega^\varepsilon$ are endowed with their usual norms as given in (3.1.1)-(3.1.9).

From section 2.5.1, we notice that the surface area of $\Gamma^\varepsilon$ increases proportionally to $\frac{1}{\varepsilon}$, i.e., $|\Gamma^\varepsilon| \to \infty$ as $\varepsilon \to 0$. Keeping this in mind, the $L^p - L^q$ duality on $\Gamma^\varepsilon$ is defined as

\[
(u, v)_{L^p(\Gamma^\varepsilon) \times L^q(\Gamma^\varepsilon)} := \varepsilon \int_{\Gamma^\varepsilon} u(x)v(x) d\sigma_x \text{ for } u \in L^p(\Gamma^\varepsilon) \text{ and } v \in L^q(\Gamma^\varepsilon),
\]
and the space $L^p(\Gamma_\varepsilon)$ is furnished with the norm

$$||.||_{L^p(\Gamma_\varepsilon)} = \varepsilon \int_{\Gamma_\varepsilon} |.|^p \, d\sigma_x \quad \text{and} \quad ||.||_{L^\infty(\Gamma_\varepsilon)} = \text{ess sup}_{x \in \Gamma_\varepsilon} |.|.$$ 

(3.1.28)

The vector-valued functions and their respective norms on $\Omega^p_\varepsilon$ can be defined in the similar way as in (3.1.12)-(3.1.26). For the sake of simplicity, we use the following notations:

- $F^w_\varepsilon := F^w_1 := \left[H^{1,p}(\{(0,T);H^{1,q}(\Omega^p_\varepsilon)\} \cap L^p((0,T);H^{1,p}(\Omega^p_\varepsilon))\right]_1$, 
- $G^w_\varepsilon := G^w_1 := \left[H^{1,p}(\{(0,T);H^{1,q}(\Omega^p_\varepsilon)\} \cap L^p((0,T);H^{1,p}(\Omega^p_\varepsilon))\right]_1$, 
- $H^w_\varepsilon := \left[H^{1,p}(\{(0,T);L^p(\Gamma_\varepsilon)\}\right]_2$, 
- $M^w_\varepsilon := [L^\infty((0,T);L^\infty(\Gamma_\varepsilon))]_1^2$,
- $X^w_{p_1} := \left[H^{1,q}(\Omega^p_\varepsilon),H^{1,p}(\Omega^p_\varepsilon)\right]_1^{1-p,p} I_1$,
- $X^w_{p_2} := \left[H^{1,q}(\Omega^p_\varepsilon),H^{1,p}(\Omega^p_\varepsilon)\right]_1^{1-p,p} I_2$, 
- $X^w_{p_3} := \left[L^\infty(\Gamma_\varepsilon)\right]_2$. 

### 3.2 Weak Formulation of $(P^1_\varepsilon)$ and $(P^2_\varepsilon)$

We note that in case of $(P^1_\varepsilon)$ $I_1 = I$, since there are only $I$ mobile species present in $\Omega^p_\varepsilon$.

**Definition 3.2.1.** A function $u_\varepsilon \in F^w_\varepsilon$ is said to be a weak solution of the problem (2.5.16)-(2.5.19) if it satisfies

(i) $\int_\Omega^p \left\langle \frac{\partial u_\varepsilon(t,x)}{\partial t}, \phi \right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)]^1 \times [H^{1,q}(\Omega^p_\varepsilon)]^1} \, dt + \int_{\Omega^p} \langle D\nabla u_\varepsilon(t,x), \nabla \phi(x) \rangle_1 \, dx$

$$= \langle SR(u_\varepsilon(t),\phi)_{[H^{1,q}(\Omega^p_\varepsilon)]^1 \times [H^{1,q}(\Omega^p_\varepsilon)]^1} \, dx$$

for every $\phi \in [H^{1,q}(\Omega^p_\varepsilon)]^1$ and for a.e. $t$. 

(ii) $u_\varepsilon(0, x) = u_0(x).$ 

(3.2.1)

**Definition 3.2.2.** A quadruple $(u_\varepsilon,v_\varepsilon,w_\varepsilon,z) \in F^w_\varepsilon \times G^w_\varepsilon \times H^w_\varepsilon \times M^w_\varepsilon$ is said to be a weak solution of the problem (2.5.21)-(2.5.35) if it satisfies

(i) $\int_\Omega^p \left\langle \frac{\partial u_\varepsilon(t,x)}{\partial t}, \phi \right\rangle_{[H^{1,q}(\Omega^p_\varepsilon)]^1 \times [H^{1,q}(\Omega^p_\varepsilon)]^1} \, dt + \int_{\Omega^p} \langle D\nabla u_\varepsilon(t,x), \nabla \phi(x) \rangle_1 \, dx$

$$+ \int_{\partial \Omega_{in}} \langle (d - \vec{q} \cdot \vec{n}) u_\varepsilon(t,x), \phi(x) \rangle_1 \, ds + \int_{\Omega^p} \langle \vec{q} \cdot \nabla u_\varepsilon(t,x), \phi(x) \rangle_1 \, dx \, dt$$

$$= \langle S_1 R(u_\varepsilon(t),v_\varepsilon(t),\phi)_{[H^{1,q}(\Omega^p_\varepsilon)]^1 \times [H^{1,q}(\Omega^p_\varepsilon)]^1} \, dx,$$

(ii) $-\int_{\partial \Omega_{in}} \langle \vec{q} \cdot \vec{n} v_\varepsilon(t,x), \xi(x) \rangle_1 \, ds + \int_{\Omega^p} \langle \vec{q} \cdot \nabla v_\varepsilon(t,x), \xi(x) \rangle_1 \, dx$

$$+ \varepsilon \int_{\Gamma_\varepsilon} \left\langle \frac{\partial w_\varepsilon(t,x)}{\partial t}, \xi(x) \right\rangle_2 \, dx \quad = \langle S_2 R(u_\varepsilon(t),v_\varepsilon(t),\xi)_{[H^{1,q}(\Omega^p_\varepsilon)]^2 \times [H^{1,q}(\Omega^p_\varepsilon)]^2} \, dx,$$

(3.2.4)
3.3 Maximal Parabolic Regularity

Let $1 < p < \infty$, $X$ be a Banach space and $A : D(A) \subseteq X \to X$ be a closed, not necessarily bounded, operator. Also assume that $f : (0,T) \to X$ is measurable. Consider the following problem

\[
\begin{align*}
\frac{\partial u(t)}{\partial t} + Au(t) &= f(t) \quad \text{for } t > 0, \\
\quad u(0) &= 0. 
\end{align*}
\]

In the theory of parabolic equations, it is well known that in general the time derivative, $\frac{\partial u}{\partial t}$, of the solution of (3.3.1)-(3.3.2) is less regular than $f$. One can look for a method so that this loss of regularity no longer occurs, i.e., for every $f \in L^p((0,T);X)$, does there exist a unique solution $u \in L^p((0,T);D(A)) \cap H^{1,p}((0,T);X)$ of (3.3.1)-(3.3.2) which satisfies

\[
\|u\|_{L^p((0,T);X)} + \|u_t\|_{L^p((0,T);X)} + \|u\|_{L^p((0,T);D(A))} \leq C \|f\|_{L^p((0,T);X)},
\]

where the constant $C$ is independent of $u$. The maximal regularity property of $A$ resolves this issue. We start with the following definition.

**Definition 3.3.1.** Let $1 < p < \infty$. The operator $A$ is said to have the maximal (parabolic) $L^p$-regularity property if for every $f \in L^p((0,T);X)$, there exists a unique solution $u \in L^p((0,T);D(A)) \cap H^{1,p}((0,T);X)$ of (3.3.1)-(3.3.2) which satisfies

\[
\|u\|_{L^p((0,T);X)} + \|u_t\|_{L^p((0,T);X)} + \|u\|_{L^p((0,T);D(A))} \leq C \|f\|_{L^p((0,T);X)},
\]

where $C > 0$ is a constant.

For a detailed overview on maximal regularity, we refer the interested readers to [ACFP07], [Mon09], [Pru¨02], [RDR09], [KW04] and references therein.

3.3.1 Maximal Regularity of Differential Operators

Let $1 < p < \infty$. Set $D(A) := H^{1,p}(\Omega)$ and $X := H^{1,q}(\Omega)^*$. Clearly, $D(A) \subseteq X$. Let $\mu = (\mu_{ij})_{1 \leq i \leq n}$ be a positive definite symmetric matrix-field, where $\mu_{ij} \in C(\Omega)$ and there is a constant $C > 0$

\[
\sum_{1 \leq i \leq n} \mu_{ij}(x)\zeta_i\zeta_j \geq C|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^n \text{ and } x \in \Omega.
\]

We define a sesquilinear form $a(u,v) : H^{1,p}(\Omega) \times H^{1,q}(\Omega) \to \mathbb{R}$ by

\[
a(u,v) := \int_\Omega \mu \nabla u \cdot \nabla v \, dx + \kappa \int_\Omega uv \, dx \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega),
\]

where $\kappa > 0$. We further define an operator $A : H^{1,p}(\Omega) \to H^{1,q}(\Omega)^*$ associated with the form $a(u,v)$ by

\[
\langle Au,v \rangle := a(u,v) \quad \text{for } u \in H^{1,p}(\Omega) \text{ and } v \in H^{1,q}(\Omega).
\]
In [CL94] or [RDR09], it is shown that: (i) \( |A|^s \|_{L(X)} \leq K e^{\theta |s|} \) for some \( 0 < \theta < \frac{\pi}{2} \), \( s \in \mathbb{R} \), where \( K > 0 \) and (ii) \((0,0) \subset \rho(A) \) (resolvent of \( A \)) and \( \| (\lambda + A)^{-1} \|_{L(X)} \leq \frac{C}{|\lambda|^{1+\delta}} \) for every \( \lambda \in [0,\infty) \), where \( C > 0 \). By theorem of Dore and Venni (cf. [DV87]),

the operator \( A \) has maximal \( L^p \)-regularity on \( H^{1,q}(\Omega)^* \). \hfill (3.3.8)

**Theorem 3.3.1** (Prüss and Schnaubelt). Let \( 1 < p < \infty \) and \( A : D(A) \subseteq X \to X \) be a closed linear operator with maximal \( L^p \)-regularity on \( X \). Then for \( u_0 \in (X,D(A))_{1-\frac{1}{p}} \) and \( f \in L^p((0,T);X) \), there exists a unique solution \( u \in H^{1,p}(0,T;X) \cap L^p((0,T);D(A)) \) of the problem

\[
\frac{\partial u(t)}{\partial t} + Au(t) = f(t) \quad \text{for } t > 0, \quad u(0) = u_0
\]

and we have the estimate

\[
\|u\|_{L^p((0,T);D(A))} + \|u\|_{L^p((0,T);X)} + \|\frac{\partial u}{\partial t}\|_{L^p((0,T);X)} \leq C_p \left( \|u_0\|_{(X,D(A))_{1-\frac{1}{p}}} + \|f\|_{L^p((0,T);X)} \right),
\]

where the constant \( C_p \) is independent of \( u_0 \), \( f \) and \( u \).

**Proof.** See theorem 2.5 in [PS01]. \hfill ♦

### 3.4 Some Theorems and Lemmas

#### 3.4.1 Trace Theorems

**Lemma 3.4.1.1.** Let \( \Gamma \) be as in (2.5.11). Then

\[
\varepsilon |\Gamma_\varepsilon| = |\Gamma| \frac{|\Omega|}{|Y|}.
\]

**Proof.** \(^6\)

\[
\varepsilon |\Gamma_\varepsilon| = \varepsilon \int_{\Gamma_\varepsilon} d\sigma_x = \varepsilon^n \sum_{k \in \mathbb{Z}^n} \int_{\Gamma_k} d\sigma_y = \varepsilon^n \sum_{k \in \mathbb{Z}^n} \frac{|\Gamma_k|}{|Y_k|} \int_{Y_k} dy = \sum_{k \in \mathbb{Z}^n} \frac{|\Gamma_k|}{|Y_k|} \int_{\varepsilon Y_k} dx = \frac{|\Gamma|}{|Y|} \int_{\Omega} dx = |\Gamma| \frac{|\Omega|}{|Y|}.
\]

\hfill ♦

**Theorem 3.4.1.2.** Let \( 1 \leq p < \infty \). Suppose \( Y^p \), \( Y^s \) and \( \Gamma \) are defined as in section 2.5.1. Then there exists a bounded linear operator \( T : H^{1,p}(Y^p) \to L^p(\Gamma) \) such that

\[
(a) \quad Tu := u|_{\Gamma} \quad \text{for } u \in H^{1,p}(Y^p) \cap C(\Omega^p)
\]

and

\[
(b) \quad \|u\|^p_{L^p(\Gamma)} \leq C_1 \left[ \|u\|^p_{L^p(Y^p)} + \|\nabla u\|^p_{L^p(Y^p)} \right],
\]

where the constant \( C_1 \) depends on \( Y^p \) and \( p \) only.

\(^6\)For \( k \in \mathbb{Z}^n \), \( \Gamma_k \) and \( Y_k \) are the translated image of \( \Gamma \) and \( Y \), \( |\Gamma_k| = |\Gamma| \) and \( |Y_k| = |Y| \). Also \( x = \varepsilon y \Rightarrow d\sigma_x = \varepsilon^{n-1} d\sigma_y \) and \( dx = \varepsilon^n dy \).
3.4. Some Theorems and Lemmas

Proof. See lemma 5.3 (a) in [HJ91]. See also lemma 2.7.2 in [NR92].

3.4.2 Extension Theorems

The proof follows by a scaling argument. For details confer lemma 5.3 (b) in [HJ91].

Proof. See also lemma 2.7.2 in [NR92].

Lemma 3.4.2.1. Let \( 1 \leq p < \infty \). For \( u \in H^{1,p}(\Omega^p) \), there exists an extension \( \tilde{u} \) of \( u \) into all of \( Y \) such that

\[
\text{(a) } \tilde{u} := u \text{ in } Y^p
\]

and

\[
\text{(b) } \|\tilde{u}\|_{H^{1,p}(Y)} \leq C_3 \|u\|_{H^{1,p}(Y^p)} ^p,
\]

where the constant \( C_3 \) depends on \( p \) and \( Y^p \) only but is independent of \( u \) and \( \tilde{u} \).

Proof. Confer lemma 5 (a) in [HJ91].

Theorem 3.4.2.2. Let \( 1 \leq p \leq \infty \). Suppose that \( \Omega^p_\varepsilon \) and \( \Omega \) are defined as in section 2.5.1. For \( u \in H^{1,p}(\Omega^p_\varepsilon) \), there exists a bounded linear operator \( P^\varepsilon : H^{1,p}(\Omega^p_\varepsilon) \rightarrow H^{1,p}(\Omega) \) such that

\[
\text{(a) } P^\varepsilon u := u \text{ in } \Omega^p_\varepsilon
\]

and

\[
\text{(b) } \|P^\varepsilon u\|_{H^{1,p}(\Omega)} \leq C_4 \|u\|_{H^{1,p}(\Omega^p_\varepsilon)} ^p,
\]

where the constant \( C_4 \) is independent of \( \varepsilon \) and \( u \) but depends on \( p \).

Proof. The proof follows by a scaling argument\(^7\). For details confer theorem 5.2 in [HJ91]. See also [Tar80].

Now we prove a theorem similar to theorem 3.4.2.2 for the functions depending on both \( t \) and \( x \). Let \( 1 \leq p \leq \infty \). For \( u \in L^p((0,T);H^{1,p}(\Omega^p_\varepsilon)) \), we define an operator \( Q^\varepsilon : L^p((0,T);H^{1,p}(\Omega^p_\varepsilon)) \rightarrow L^p((0,T);H^{1,p}(\Omega)) \) such that

\[
Q^\varepsilon u(t,x) := (P^\varepsilon u(t,.))(x) \text{ for } u \in L^p((0,T);H^{1,p}(\Omega^p_\varepsilon)),
\]

where \( P^\varepsilon \) is the extension operator from theorem 3.4.2.2. Then

\[
\frac{\partial}{\partial t} [Q^\varepsilon u(t,x)] = \frac{\partial}{\partial t} [P^\varepsilon u(t,.)](x) = (P^\varepsilon \left( \frac{\partial u}{\partial t} (t,.)) \right)(x) = Q^\varepsilon \left( \frac{\partial u}{\partial t} \right)(t,x).
\]

Based on the above definition we have the following extension theorem for the functions depending on \( t \) and \( x \).

\(^7\)A more general form of this theorem is given in Miller [Mil92].
Theorem 3.4.2.3. Let $\Omega$ and $\Omega^p_\varepsilon$ be defined as in section 2.5.1 and $1 \leq p \leq \infty$. Then there exists a bounded linear operator $Q^\varepsilon : L^p((0,T);H^1,\varepsilon(\Omega^p_\varepsilon)) \cap H^1,\varepsilon((0,T);L^p(\Omega^p_\varepsilon)) \rightarrow L^p((0,T);H^1,\varepsilon(\Omega^p_\varepsilon)) \cap H^1,\varepsilon((0,T);L^p(\Omega^p_\varepsilon)))$ such that for all $u \in L^p((0,T);H^1,\varepsilon(\Omega^p_\varepsilon)) \cap H^1,\varepsilon((0,T);L^p(\Omega^p_\varepsilon)))$

$Q^\varepsilon\left(\frac{\partial u}{\partial t}\right) = \frac{\partial}{\partial t} (Q^\varepsilon u(t))$ \hfill (3.4.11)

and

$||Q^\varepsilon u||_{L^p((0,T);H^1,\varepsilon(\Omega^p_\varepsilon))} \leq C_5 ||u||_{L^p((0,T);H^1,\varepsilon(\Omega^p_\varepsilon))}$, \hfill (3.4.12)

where the constant $C_5$ is independent of $\varepsilon$ and $u$.

Proof. Here we only show the measurability of $Q^\varepsilon u$. The inequality (3.4.12) follows by scaling. Since we know that every continuous function is measurable, we show $Q^\varepsilon u$ is continuous. But by theorem 3.4.2.2 it can be shown $P^\varepsilon u(t)$ is continuous on $\Omega$. The continuity of $Q^\varepsilon u$ on $[0,T] \times \bar{\Omega}$ follows from the definition (3.4.10).

Theorem 3.4.2.4. Let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $u \in (H^{1,q}(\Omega^p_\varepsilon)^*, H^1,\varepsilon(\Omega^p_\varepsilon))_{1-\frac{1}{p}, p}$. Then there exists an extension $\bar{u}$ of $u$ such that $\bar{u} \in (H^{1,q}(\Omega)^*, H^1,\varepsilon(\Omega))_{1-\frac{1}{p}, p}$.

Proof. Let $\theta = 1 - \frac{1}{p}$. We use the K-functional definition for real interpolation space $(H^{1,q}(\Omega^p_\varepsilon)^*, H^1,\varepsilon(\Omega^p_\varepsilon))_{\theta, p}$.

To begin with, let $v \in H^{1,q}(\Omega^p_\varepsilon)$, then by theorem 3.4.2.2 there exists an extension $P^\varepsilon v$ of $v$ such that

$\begin{align*}
\text{(a)} & \quad P^\varepsilon v := v \text{ in } \Omega^p_\varepsilon \\
\text{and} & \quad \text{(b)} \quad ||P^\varepsilon v||_{H^{1,q}(\Omega)} \leq C_4 ||v||_{H^{1,q}(\Omega^p_\varepsilon)},
\end{align*}$

where $C_4$ is independent of $\varepsilon$ and $v$. Let $a_0 \in H^{1,q}(\Omega^p_\varepsilon)^*$. Since $P^\varepsilon$ is a linear operator from $H^{1,q}(\Omega^p_\varepsilon)$ into $H^{1,q}(\Omega)$, we can define a function $\bar{a}_0$ (an extension of $a_0$) by the following formula

$\langle \bar{a}_0, P^\varepsilon v \rangle_{H^{1,q}(\Omega)^*} = \langle a_0, P^\varepsilon v \rangle_{H^{1,q}(\Omega^p_\varepsilon)^*} \quad \forall v \in H^{1,q}(\Omega^p_\varepsilon).$ \hfill (3.4.15)

Therefore

$\begin{align*}
||\bar{a}_0||_{H^{1,q}(\Omega)^*} & = \sup_{||P^\varepsilon v||_{H^{1,q}(\Omega)} \leq 1} \left| \langle \bar{a}_0, P^\varepsilon v \rangle_{H^{1,q}(\Omega)^*} \right| H^{1,q}(\Omega) \\
& = \sup_{||v||_{H^{1,q}(\Omega^p_\varepsilon)} \leq 1} \left| \langle a_0, v \rangle_{H^{1,q}(\Omega^p_\varepsilon)^*} \right| H^{1,q}(\Omega^p_\varepsilon) \quad \text{by (3.4.14) and (3.15)} \\
& \leq ||a_0||_{H^{1,q}(\Omega^p_\varepsilon)^*} \quad \Rightarrow ||\bar{a}_0||_{H^{1,q}(\Omega)^*} \leq ||a_0||_{H^{1,q}(\Omega^p_\varepsilon)^*}.
\end{align*}$ \hfill (3.4.16)

Again assume that $b_0 \in H^1,\varepsilon(\Omega^p_\varepsilon)$. Let $\tilde{b}_0 \in H^1,\varepsilon(\Omega)$ denote the extension of $b_0$ such that

$||\tilde{b}_0||_{H^1,\varepsilon(\Omega)} \leq C_4 ||b_0||_{H^1,\varepsilon(\Omega^p_\varepsilon)} \quad \text{for } b_0 \in H^1,\varepsilon(\Omega^p_\varepsilon),$ \hfill (3.4.17)

where $C_4$ is independent of $\varepsilon$ and $b_0$. Let $t > 0$. Then

$\begin{align*}
||\bar{a}_0||_{H^{1,q}(\Omega)^*} + t ||\tilde{b}_0||_{H^1,\varepsilon(\Omega)} & \leq ||a_0||_{H^{1,q}(\Omega^p_\varepsilon)^*} + C_4 t ||b_0||_{H^1,\varepsilon(\Omega^p_\varepsilon)} \\
& \leq \max(1, C_4) \left( ||a_0||_{H^{1,q}(\Omega^p_\varepsilon)^*} + t ||b_0||_{H^1,\varepsilon(\Omega^p_\varepsilon)} \right).
\end{align*}$

---

\[8\text{For the definition of real-interpolation space see } [\text{Lun95}], [\text{Tri95}], [\text{BL76}].\]
Taking the infimum on both sides, we get
\[
\inf_{\bar{a}=\bar{a}_0+b_0} \left( \|\bar{a}_0\|_{H^{1,q}(\Omega)} + t \|\bar{b}_0\|_{H^{1,p}(\Omega)} \right)
\]
\[
\leq \max(1, C_4) \inf_{\bar{a}=\bar{a}_0+b_0} \left( \|a_0\|_{H^{1,q}(\Omega^p_1)} + t \|b_0\|_{H^{1,p}(\Omega^p_1)} \right),
\]
i.e.,
\[
t^{-\theta} \inf_{\bar{a}=\bar{a}_0+b_0} \left( \|\bar{a}_0\|_{H^{1,q}(\Omega)} + t \|\bar{b}_0\|_{H^{1,p}(\Omega)} \right)
\]
\[
\leq \max(1, C_4) t^{-\theta} \inf_{\bar{a}=\bar{a}_0+b_0} \left( \|a_0\|_{H^{1,q}(\Omega^p_1)} + t \|b_0\|_{H^{1,p}(\Omega^p_1)} \right),
\]
i.e.,
\[
\left| t^{-\theta} \inf_{\bar{a}=\bar{a}_0+b_0} \left( \|\bar{a}_0\|_{H^{1,q}(\Omega)} + t \|\bar{b}_0\|_{H^{1,p}(\Omega)} \right) \right|^p
\]
\[
\leq [\max(1, C_4)]^p \left\| t^{-\theta} \inf_{\bar{a}=\bar{a}_0+b_0} \left( \|a_0\|_{H^{1,q}(\Omega^p_1)} + t \|b_0\|_{H^{1,p}(\Omega^p_1)} \right) \right\|^p.
\]
Thus
\[
\int_0^\infty \left| t^{-\theta} \inf_{\bar{a}=\bar{a}_0+b_0} \left( \|\bar{a}_0\|_{H^{1,q}(\Omega)} + t \|\bar{b}_0\|_{H^{1,p}(\Omega)} \right) \right|^p \frac{dt}{t}
\]
\[
\leq [\max(1, C_4)]^p \int_0^\infty \left| t^{-\theta} \inf_{\bar{a}=\bar{a}_0+b_0} \left( \|a_0\|_{H^{1,q}(\Omega^p_1)} + t \|b_0\|_{H^{1,p}(\Omega^p_1)} \right) \right|^p \frac{dt}{t},
\]
i.e.,
\[
\int_0^\infty \left| t^{-\theta} K(t, \bar{u}, H^{1,q}(\Omega)^*, H^{1,p}(\Omega)) \right|^p \frac{dt}{t}
\]
\[
\leq [\max(1, C_4)]^p \int_0^\infty \left| t^{-\theta} K(t, u, H^{1,q}(\Omega^p_1)^*, H^{1,p}(\Omega^p_1)) \right|^p \frac{dt}{t}.
\]
Therefore

\[
\|[\ddot{u}]\|_{(H^{1,p}(\Omega),H^{1,p}(\Omega))_{1-\frac{1}{p},p}} \leq \max(1,C_4) \|u\|_{(H^{1,p}(\Omega^\circ),H^{1,p}(\Omega^\circ))_{1-\frac{1}{p},p}} = C_6 \|u\|_{(H^{1,p}(\Omega^\circ),H^{1,p}(\Omega^\circ))_{1-\frac{1}{p},p}},
\]

(3.4.18)

where the constant \(C_6 := \max(1,C_4)\) is independent of \(\varepsilon\) and \(u\).

\section{3.4.3 Embedding Theorems}

\textbf{Theorem 3.4.3.1.} Let \(\Omega\) and \(\Omega^\circ\) be as in section 2.5.1. Assume that \(1 \leq p < n\) and \(u \in H^{1,p}(\Omega^\circ)\). Then \(u \in L^q(\Omega^\circ)\) and there is a constant \(C_7\)

\[
\|u\|_{L^q(\Omega^\circ)} \leq C_7 \|u\|_{H^{1,p}(\Omega^\circ)},
\]

(3.4.19)

where \(q = \frac{np}{n-p}\) and \(C_7\) is independent of \(\varepsilon\) and \(u\). In other words, \(H^{1,p}(\Omega^\circ) \hookrightarrow L^q(\Omega^\circ)\) with embedding constant independent of \(\varepsilon\).

\textbf{Proof.} Let \(u \in H^{1,p}(\Omega^\circ)\). Then from theorem 3.4.2.2, there exists an extension \(P^\varepsilon u\) of \(u\) from \(H^{1,p}(\Omega^\circ)\) to \(H^{1,p}(\Omega)\) such that

\[
\|P^\varepsilon u\|_{H^{1,p}(\Omega)} \leq C_4 \|u\|_{H^{1,p}(\Omega^\circ)},
\]

(3.4.20)

Let \(v := P^\varepsilon u\). By assumption \(\Omega\) is a bounded domain with sufficiently smooth boundary, then from theorem 2 of section 5.6.1 in [Eva98] we get

\[
\|v\|_{L^q(\Omega)} \leq C \|v\|_{H^{1,p}(\Omega)} \text{ for } v \in H^{1,p}(\Omega),
\]

(3.4.21)

where \(q = \frac{np}{n-p}\) and \(C\) depends only on \(p, n\) and \(\Omega\) but is independent of \(v\). Therefore

\[
\|u\|_{L^q(\Omega^\circ)}^q = \int_{\Omega^\circ} |u(x)|^q \, dx = \int_{\Omega^\circ} |v(x)|^q \, dx \\
\leq \int_{\Omega} |v(x)|^q \, dx \\
\leq C^q \|v\|_{H^{1,p}(\Omega)}^q \text{ from (3.4.21)} \\
= C^q \|P^\varepsilon u\|_{H^{1,p}(\Omega)}^q \\
\leq C^q C_4^q \|u\|_{H^{1,p}(\Omega^\circ)}^q \text{ from (3.4.20)} \\
\implies \|u\|_{L^q(\Omega^\circ)} \leq C_7 \|u\|_{H^{1,p}(\Omega^\circ)},
\]

where \(C_7 := C C_4\) is independent of \(\varepsilon\) and \(u\).

\textbf{Theorem 3.4.3.2.} Suppose that \(\Omega\) and \(\Omega^\circ\) are as in section 2.5.1. Then for \(u \in H^{1,2}(\Omega^\circ)\) the following inequality holds

\[
\|u\|_{L^2(\partial\Omega)} \leq C_8 \|u\|_{H^{1,2}(\Omega^\circ)} \|u\|_{L^2(\Omega^\circ)},
\]

(3.4.22)

where the constant \(C_8\) is independent of \(\varepsilon\) and \(u\).

\textbf{Proof.} The proof follows by combining the theorems 3.4.2.2 and B.7.
Theorem 3.4.3.3. Let $1 < p, q < \infty$ be such that $p > n + 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $u \in (H^{1,q}(\Omega^p)^\ast, H^{1,p}(\Omega^p))_{1-\frac{1}{p}, \frac{1}{p}}$ such that $\sup_{\varepsilon > 0} ||u||_{(H^{1,q}(\Omega^p)^\ast, H^{1,p}(\Omega^p))_{1-\frac{1}{p}, \frac{1}{p}}} < \infty$. Then $u \in L^\infty(\Omega^p)$ and

$$\sup_{\varepsilon > 0} ||u||_{L^\infty(\Omega^p)} < \infty.$$  \hspace{1cm} (3.4.23)

Proof. From theorem 3.1.3, we know that for $u \in (H^{1,q}(\Omega)^\ast, H^{1,p}(\Omega))_{1-\frac{1}{p}, \frac{1}{p}}$, $u \in L^\infty(\Omega)$ and

$$||u||_{L^\infty(\Omega)} \leq C_9 ||u||_{(H^{1,q}(\Omega)^\ast, H^{1,p}(\Omega))_{1-\frac{1}{p}, \frac{1}{p}}}$$  \hspace{1cm} (3.4.24)

where the constant $C_9$ is independent of $\varepsilon$ and $u$. Let $u \in (H^{1,q}(\Omega^p)^\ast, H^{1,p}(\Omega^p))_{1-\frac{1}{p}, \frac{1}{p}}$, then

$$||u||_{L^\infty(\Omega^p)} = \esssup_{x \in \Omega^p} |u(x)| \leq \esssup_{x \in \Omega} |u(x)| = ||u||_{L^\infty(\Omega)} \leq C_9 ||u||_{(H^{1,q}(\Omega)^\ast, H^{1,p}(\Omega))_{1-\frac{1}{p}, \frac{1}{p}}}$$

by (3.4.24)

$$\leq C_6 C_9 \sup_{\varepsilon > 0} ||u||_{(H^{1,q}(\Omega^p)^\ast, H^{1,p}(\Omega^p))_{1-\frac{1}{p}, \frac{1}{p}}} < \infty \quad \forall \varepsilon > 0,$$

where the constants $C_6$ and $C_9$ are independent of $\varepsilon$ and $u$. Therefore $\sup_{\varepsilon > 0} ||u||_{L^\infty(\Omega^p)} < \infty$. \hfill \checkmark

Theorem 3.4.3.4. Let $p > n + 2$, then $F^u_\varepsilon \mapsto \big[ L^\infty((0,T) \times \Omega^p) \big]^I$. 

Proof. Since $F_\varepsilon : = H^{1,p}((0,T); H^{1,q}(\Omega^p)^\ast) \cap L^p((0,T); H^{1,p}(\Omega^p)) \hookrightarrow L^\infty((0,T) \times \Omega^p)$ by theorem 3.1.2. Therefore the theorem follows. \hfill \checkmark

3.5 Two-scale Convergence

Definition 3.5.1. Let $\varepsilon$ be a sequence of positive real numbers converging to 0. A sequence of functions $(u_\varepsilon)_{\varepsilon > 0}$ in $L^p(\Omega)$ is said to two-scale convergent to a limit $u \in L^p(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x, \frac{x}{\varepsilon}) \phi(x, \frac{x}{\varepsilon}) dx = \int_\Omega \int_Y u(x, y) \phi(x, y) dx dy, \quad \hfill (3.5.1)$$

for all $\phi \in L^q(\Omega; C_{per}(Y))$.\footnote{\textit{C}_{per}(Y) denotes the space of $Y$-periodic continuous functions in $y$.}

The above definition is followed from the following theorem which is proved by Nget-seng in [Ngu89].

Theorem 3.5.3. For every bounded sequence, $(u_\varepsilon)_{\varepsilon > 0}$, in $L^p(\Omega)$ there exist a subsequence and a $u \in L^p(\Omega \times Y)$ such that the subsequence two-scale converges to $u$. 

Proof. See theorem 1 in [Ngu89]. Confer also theorem 14 in [LNW02] or theorem 0.1 in [All92]. \hfill \checkmark

Remark 3.5.4. If $(u_\varepsilon)_{\varepsilon > 0}$ is two-scale convergent to $u$ then we write $u_\varepsilon \overset{2}{\rightharpoonup} u$. 

Proof. 

\hfill \checkmark
We state some theorems on two-scale convergence. The proofs of all these theorems can be found in [Ngu89], [LNW02], [All92].

**Theorem 3.5.5.** Let \((u_\varepsilon)_{\varepsilon>0}\) be strongly convergent to \(u \in L^p(\Omega)\), then \((u_\varepsilon)_{\varepsilon>0}\) is two-scale convergent to \(u_1(x,y) = u(x)\).

**Proof.** Cf. theorem 9 in [LNW02].

**Theorem 3.5.6.** Let \((u_\varepsilon)_{\varepsilon>0}\) be two-scale convergent to \(u \in L^p(\Omega \times Y)\), then \((u_\varepsilon)_{\varepsilon>0}\) is weakly convergent to \(\int_Y u(x,y) dy \) in \(L^p(\Omega)\) and \((u_\varepsilon)_{\varepsilon>0}\) is bounded.

**Proof.** Cf. theorem 19 in [LNW02].

In the definition 3.5.1, one can notice that the space of test functions is chosen as \(L^q(\Omega; C_\text{per}(Y))\), but we can replace the space of test functions by \(C_0^\infty(\Omega; C_\text{per}(Y))\), if \((u_\varepsilon)_{\varepsilon>0}\) satisfies certain condition which is given in the following theorem:

**Theorem 3.5.7.** Let \((u_\varepsilon)_{\varepsilon>0}\) be bounded in \(L^p(\Omega)\) such that
\[
\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x) \phi(x, x/\varepsilon) dx = \int_\Omega \int_Y u(x,y) \phi(x,y) dx dy \quad \text{for all } \phi \in C_0^\infty(\Omega; C_\text{per}(Y)).
\] (3.5.2)

Then \((u_\varepsilon)_{\varepsilon>0}\) is two-scale convergent to \(u\).

**Proof.** See proposition 13 in [LNW02].

**Theorem 3.5.8.** Let \((u_\varepsilon)_{\varepsilon>0}\) be a sequence in \(H^{1,p}(\Omega)\) such that \(u_\varepsilon \to u\) in \(H^{1,p}(\Omega)\). Then \((u_\varepsilon)_{\varepsilon>0}\) two-scale converges to \(u\) and there exist a subsequence \(\varepsilon\), still denoted by same symbol, and a \(u_1 \in L^p(\Omega; H^{1,p}_\text{per}(Y))\) such that \(\nabla_x u_\varepsilon \overset{2}{\to} \nabla u + \nabla_y u_1\).

**Proof.** Cf. theorem 20 in [LNW02].

**Theorem 3.5.9.** Let \((u_\varepsilon)_{\varepsilon>0}\) and \((\varepsilon \nabla_x u_\varepsilon)_{\varepsilon>0}\) be bounded in \(L^p(\Omega)\) and \([L^p(\Omega)]^n\) respectively. Then there exists \(u \in L^p(\Omega; H^{1,p}_\text{per}(Y))\) such that up to a subsequence, still denoted by \(\varepsilon\), we have
\[
u_\varepsilon \overset{2}{\to} u \\
\text{and } \varepsilon \nabla_x u_\varepsilon \overset{2}{\to} \nabla_y u
\]
as \(\varepsilon \to 0\).

**Proof.** Cf. theorem 3.16 in [Zie09].

Since in this work we will only consider evolution equations which introduces time as an additional parameter, we transfer the definition 3.5.1 to the functions depending on \(t\) and \(x\).

**Definition 3.5.10.** A sequence of functions \((u_\varepsilon)_{\varepsilon>0}\) in \(L^p((0,T) \times \Omega)\) is said to two-scale convergent to a limit \(u \in L^p((0,T) \times \Omega \times Y)\) if
\[
\lim_{\varepsilon \to 0} \int_0^T \int_\Omega u_\varepsilon(t,x) \phi(t,x, x/\varepsilon) dx dt = \int_0^T \int_\Omega \int_Y u(t,x,y) \phi(t,x,y) dx dy dt
\] (3.5.3)
for all \(\phi \in L^q((0,T) \times \Omega; C_\text{per}(Y))\).

All the above theorems on two-scale convergence can be generalized for the functions depending on \(t\) and \(x\). Here we only give the statements of such theorems. For the proof of these theorems, see in [Cla98], [Pet03], [Zie09], [NR92].
Theorem 3.5.11. For every bounded sequence, \((u_\varepsilon)_{\varepsilon>0}\), in \(L^p((0,T) \times \Omega)\) there exist a subsequence and a \(u \in L^p((0,T) \times \Omega \times Y)\) such that the subsequence two-scale converges to \(u\).

Theorem 3.5.12. Let \((u_\varepsilon)_{\varepsilon>0}\) be strongly convergent to \(u \in L^p((0,T) \times \Omega)\), then \((u_\varepsilon)_{\varepsilon>0}\) is two-scale convergent to \(u_1(t,x,y) = u(t,x)\).

Theorem 3.5.13. Let \((u_\varepsilon)_{\varepsilon>0}\) be a sequence in \(L^p((0,T); H^{1,p}(\Omega))\) such that \(u_\varepsilon \to u\) weakly in \(L^p((0,T); H^{1,p}(\Omega))\). Then \((u_\varepsilon)_{\varepsilon>0}\) two-scale converges to \(u\) and there exist a subsequence \(\varepsilon\), still denoted by same symbol, and a \(u_1 \in L^p((0,T) \times \Omega; H^{1,p}_{\text{per}}(Y))\) such that \(\nabla_x u_\varepsilon \overset{\Delta}{\rightharpoonup} \nabla_y u + \nabla_y u_1\).

Theorem 3.5.14. Let \((u_\varepsilon)_{\varepsilon>0}\) and \((\varepsilon \nabla_x u_\varepsilon)_{\varepsilon>0}\) be bounded in \(L^p((0,T) \times \Omega)\) and \(L^p((0,T) \times \Omega)^n\) respectively. Then there exists \(u \in L^p((0,T) \times \Omega; H^{1,p}_{\text{per}}(Y))\) such that up a subsequence, still denoted by \(\varepsilon\), we have

\[
\nabla_x u_\varepsilon \overset{\Delta}{\rightharpoonup} u
\]

and

\[
\varepsilon \nabla_x u_\varepsilon \overset{\Delta}{\rightharpoonup} \nabla_y u
\]

as \(\varepsilon \to 0\).

Next we define the notion of two-scale convergence on the \((n-1)\) dimensional surface \(\Gamma_\varepsilon\). We follow the notations of section 2.5.1.

Definition 3.5.15 (cf. [ADH96], [NR96]). Let \(1 \leq p < \infty\). A sequence \((u_\varepsilon)_{\varepsilon>0}\) in \(L^p((0,T) \times \Gamma_\varepsilon)\) is said to two-scale converge to a limit \(u \in L^p((0,T) \times \Omega \times \Gamma)\) if

\[
\lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} u_\varepsilon(t,x)\phi(t,x,\frac{x}{\varepsilon}) \, d\sigma_x \, dt = \int_0^T \int_{\Omega} \int_{\Gamma} u(t,x,y)\phi(t,x,y) \, dx \, dy \, dt \tag{3.5.4}
\]

for all \(\phi \in C([0,T] \times \bar{\Omega}; C_{\text{per}}(Y))\).

Theorem 3.5.16. Let \((u_\varepsilon)_{\varepsilon>0}\) be a sequence in \(L^p((0,T) \times \Gamma_\varepsilon)\) such that

\[
\varepsilon \int_0^T \int_{\Gamma_\varepsilon} |u_\varepsilon(t,x)|^p \, d\sigma_x \, dt \leq C, \tag{3.5.5}
\]

where \(C\) is independent of \(\varepsilon\). Then there exists a subsequence (still denoted by \(\varepsilon\)) and a two-scale limit \(u \in L^p((0,T) \times \Omega \times \Gamma)\) such that \(u_\varepsilon\) is two-scale convergent to \(u\) in the sense of (3.5.4).

Proof. Confer theorem 2.1 in [ADH96].

### 3.6 Periodic Unfolding

Arbogast, Douglas, and Hornung in [ADH90] introduced the concept of *dilation operator* to study the homogenization on periodic domains with double porosity. This method is further used in [BLM96], [NRJ07], [ACP08] etc. Later on the idea of dilation operator is extended by Cioranescu, Damlamian and Griso (cf. [CDG02], [CDG08]) to examine the homogenization problems on periodic domains under the name of *periodic unfolding*. We continue our discussion with the definition of periodic unfolding on fixed domains.

Let \(\Omega, Y, m, k\) and \(\Gamma_\varepsilon\) be defined as in section 2.5.1. For any \(z \in \mathbb{R}^n\), suppose \([z]\) denotes the unique integer combination \(\sum_{j=1}^n k_j e_j\) of \(e_j\) such that \(z - [z]\) lies in \(Y\) (see figure 3.6.1) and we set

\[
[z] = z - [z] \quad \text{for a.e.} \ z \in \mathbb{R}^n.
\]
Thus for any $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we have
\[ x = \varepsilon \left( \left\lceil \frac{x}{\varepsilon} \right\rceil + \left\{ \frac{x}{\varepsilon} \right\} \right) \quad \text{a.e. } x \in \mathbb{R}^n. \tag{3.6.1} \]

Setting
\[
\Xi_{\varepsilon} = \{ \xi \in \mathbb{Z}^n : \varepsilon (\xi + Y) \subset \Omega \}, \\
\hat{\Omega}_{\varepsilon} = \text{int} \left( \bigcup_{\xi \in \Xi_{\varepsilon}} \varepsilon (\xi + Y) \right) \\
\Lambda_{\varepsilon} = \Omega - \hat{\Omega}_{\varepsilon}.
\]

**Definition 3.6.1.** Assume that $1 \leq p \leq \infty$. Let $u \in L^p((0, T) \times \Omega)$ such that for every $t$, $u(t)$ is extended by zero outside of $\Omega$. We define the unfolding operator $T^\varepsilon : L^p((0, T) \times \Omega) \to L^p((0, T) \times \Omega \times Y)$ as
\[
T^\varepsilon u(t, x, y) = u \left( t, \varepsilon \left\lceil \frac{x}{\varepsilon} \right\rceil + \varepsilon y \right) \quad \text{for a.e. } (t, x, y) \in (0, T) \times \hat{\Omega}_{\varepsilon} \times Y, \\
= 0 \quad \text{for a.e. } (t, x, y) \in (0, T) \times \Lambda_{\varepsilon} \times Y. \tag{3.6.2}
\]

We collect some properties of $T^\varepsilon$.

**Theorem 3.6.2.** Let $1 < p < \infty$. Then the unfolding operator $T^\varepsilon$ has the following properties:

(i) $T^\varepsilon$ is linear.

(ii) If $u \in L^p((0, T) \times \Omega)$, then for a.e. $t$ and $x$, $T^\varepsilon u(t, x, \{ \frac{x}{\varepsilon} \}) = u(t, x)$.

(iii) Let $u, v \in L^p((0, T) \times \Omega)$, then $T^\varepsilon (uv) = T^\varepsilon (u) T^\varepsilon (v)$.

(iv) Let $u \in L^1((0, T) \times \Omega)$, then $\int_0^T \int_{\hat{\Omega}_{\varepsilon}} u(t, x) \, dt \, dx = \frac{1}{|Y|} \int_0^T \int_{\Omega \times Y} T^\varepsilon (u(t, x, y)) \, dx \, dy \, dt$.

(v) Let $u \in L^p((0, T) \times \Omega)$, then $\| T^\varepsilon u \|_{L^p((0, T) \times \Omega \times Y)} \leq |Y|^{\frac{1}{p}} \| u \|_{L^p((0, T) \times \Omega)}$.

(vi) Let $u \in L^p((0, T) \times \Omega)$, then $(T^\varepsilon u)_{\varepsilon > 0}$ is strongly convergent to $u$ in $L^p((0, T) \times \Omega \times Y)$.

(vii) Let $(u_{\varepsilon})_{\varepsilon > 0} \subset L^p((0, T) \times \Omega)$ be such that $(T^\varepsilon u_{\varepsilon})_{\varepsilon > 0}$ is weakly convergent to $\tilde{u}$ in $L^p((0, T) \times \Omega \times Y)$, then $(u_{\varepsilon})_{\varepsilon > 0}$ weakly converges to $u$ in $L^p((0, T) \times \Omega)$, where $u = \frac{1}{|Y|} \int_Y \tilde{u} \, dy$.

\[^{10}\text{This figure is provided to the author by Prof. Alan Damlamian via personal communication.}\]
(viii) Let \((u_\varepsilon)_{\varepsilon>0}\) be a bounded sequence in \(L^p((0,T) \times \Omega)\). Then the following statements are equivalent:

(a) \((T^\varepsilon(u_\varepsilon))_{\varepsilon>0}\) weakly converges to \(u\) in \(L^p((0,T) \times \Omega \times Y)\).

(b) \((u_\varepsilon)_{\varepsilon>0}\) two-scale converges to \(u\).

**Proof.** For the proofs of (i)-(viii), confer [CDG02], [CDG08] and [CDZ06].

Next we define the concept of boundary unfolding operator on \(\Gamma_\varepsilon\) (cf. [CDZ06]).

**Definition 3.6.3.** Let \(1 \leq p \leq \infty\). For any \(u \in L^p((0,T) \times \Gamma_\varepsilon)\), the boundary unfolding operator \(T^\varepsilon_b : L^p((0,T) \times \Gamma_\varepsilon) \to L^p((0,T) \times \Omega \times \Gamma)\) is defined as

\[
T^\varepsilon_b u(t,x,y) := u\left(t,\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)
\]

for every \((t,x,y) \in (0,T) \times \Omega \times \Gamma\). (3.6.3)

**Theorem 3.6.4.** Let \(1 < p < \infty\). Then the operator \(T^\varepsilon_b\) has the following properties:

(i) \(T^\varepsilon_b\) is linear.

(ii) If \(u \in L^p((0,T) \times \Gamma_\varepsilon)\), then \(T^\varepsilon_b u(t,x,\{\frac{x}{\varepsilon}\}) = u(t,x)\), for every \(t \in (0,T)\) and \(x \in \Omega\).

(iii) Let \(u,v \in L^p((0,T) \times \Gamma_\varepsilon)\), then \(T^\varepsilon_b (uv) = T^\varepsilon_b (u)T^\varepsilon_b (v)\).

(iv) Let \(u \in L^1((0,T) \times \Gamma_\varepsilon)\), then

\[
\int_0^T \int_{\Gamma_\varepsilon} u(t,x) dt d\sigma_x = \frac{1}{|\Omega|} \int_0^T \int_{\Omega \times Y} T^\varepsilon_b (u(t,x,y)) dx d\sigma_y dt.
\]

(v) Let \(u_\varepsilon \in L^p((0,T) \times \Gamma_\varepsilon)\), then

\[
\int_0^T \int_{\Omega} \int_{\Gamma} |T^\varepsilon_b u(t,x,y)|^p dx dy dt = \varepsilon |Y| \int_0^T \int_{\Gamma_\varepsilon} |u(t,x)|^p d\sigma_x dt.
\]

(vi) Let \(u \in L^p((0,T) \times \Omega)\), then \((T^\varepsilon_b u)_{\varepsilon>0}\) is strongly convergent to \(u\) in \(L^p((0,T) \times \Omega \times \Gamma)\).

(vii) Let \((u_\varepsilon)_{\varepsilon>0}\) be a bounded sequence in \(L^p((0,T) \times \Gamma_\varepsilon)\). Then the following statements are equivalent:

(a) \((T^\varepsilon_b (u_\varepsilon))_{\varepsilon>0}\) weakly converges to \(u\) in \(L^p((0,T) \times \Omega \times \Gamma)\).

(b) \((u_\varepsilon)_{\varepsilon>0}\) two-scale converges to \(u\) in the sense of (3.5.4).

**Proof.** The proofs of (i)-(vii) can be found in [CDG02], [CDG08] and [CDZ06].
Existence of a Unique Positive Global Weak Solution of a System of Diffusion – Reaction Equations and Homogenization

This chapter is the main body of this work and investigates the models I and II, introduced in chapter 2. Section 4.1 deals with the model M1. In section 4.1.1 we prove the positivity, existence and uniqueness of the solution of the problem \((P^1_\varepsilon)\) which is global in time. We obtain some \(\varepsilon\)-independent a-priori estimates of this solution in section 4.1.2.1 and we upscale the model from the micro scale to the macro scale in section 4.1.2.3. Next we treat the model M2 in section 4.2. We discuss the positivity, existence and uniqueness of the global solution of \((P^2_\varepsilon)\) in section 4.2.1. In section 4.2.2.1 some \(\varepsilon\)-independent a-priori estimates are obtained. Finally we conclude this chapter with the homogenization of model M2.

4.1 Model M1

4.1.1 Existence and Uniqueness of the Global Solution of \((P^1_\varepsilon)\)

Let the following assumptions be satisfied:\(^\text{11}\)

\[(i) \ p > n + 2.\]  
\[(ii) \ u_0 \geq 0, \ i.e., \ u_0_i \geq 0 \ for \ all \ i = 1, 2, \ldots, I. \]  
\[(iii) \ u_0_i \in (H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega_\varepsilon))_{1-\frac{1}{p'},p'} \ for \ i = 1, 2, \ldots, I. \]  
\[(iv) \ All \ reactions \ are \ linearly \ independent \ such \ that \ the \ stoichiometric \ matrix \ S = (s_{ij})_{1\leq i \leq I, 1\leq j \leq J} \ has \ maximal \ column \ rank, \ i.e., \ rank(S) = J. \]  
\[(v) \ sup_{\varepsilon>0} ||u_0_i||_{(H^{1,q}(\Omega^p_\varepsilon)^*, H^{1,p}(\Omega_\varepsilon))_{1-\frac{1}{p'},p'}} < \infty \ for \ all \ i = 1, 2, \ldots, I. \]

**Theorem 4.1.1.1 (Existence theorem).** Suppose that the assumptions (4.1.1)-(4.1.5) are satisfied, then there exists a unique positive global weak solution \(u_\varepsilon \in F^w_\varepsilon\) of the problem \((P^1_\varepsilon)\).

**Strategy of the proof:** We adopt the methodology of Kräutle (cf. [Krä08], [Krä11]) in order to prove the positivity, existence and uniqueness of the global solution of the problem \((P^1_\varepsilon)\). As already mentioned in chapter 1 that for \(p > n + 1\), on the macroscopic level, Kräutle has shown the existence of a unique positive global weak solution in \([H^{1,p}((0,T);L^p(\Omega)) \cap L^p((0,T);H^{2,p}(\Omega))]^J\) for the problem (2.5.16)-(2.5.19). Here we will show that with a little stronger condition on \(p\), i.e., for \(p > n + 2\), there exists a unique

\(^{11}\)Note that in case of model M1, we only have \(I\) number of mobile species. We choose \(I_1 = I\) in the function spaces introduced in section 3.1.
positive global weak solution of the problem \((P_1^\varepsilon)\) in \(F^\varepsilon\).

Before dealing with the problem \((P_1^\varepsilon)\), we consider a slightly modified problem and introduce the rate function \(\bar{R} : \mathbb{R}^J \to \mathbb{R}^J\) as

\[
\bar{R}(u_\varepsilon) := R(u_\varepsilon^+),
\]

(4.1.6)

where \(u_\varepsilon^+\) is the positive part of \(u_\varepsilon\) defined componentwise as

\[
\begin{align*}
  u_{\varepsilon i}^+ &:= \max(u_{\varepsilon i}, 0), \\
  u_{\varepsilon i}^- &:= \max(-u_{\varepsilon i}, 0) = -\min(u_{\varepsilon i}, 0)
\end{align*}
\]

(4.1.7)

This gives

\[
\begin{align*}
  \frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot D\nabla u_\varepsilon &= S\bar{R}(u_\varepsilon) \quad &\text{in } (0, T) \times \Omega^\varepsilon, \\
  u_\varepsilon(0, x) &= u_0(x) \quad &\text{in } \Omega^\varepsilon, \\
  -D\nabla u_\varepsilon \cdot \vec{n} &= 0 \quad &\text{on } (0, T) \times \partial\Omega, \\
  -D\nabla u_\varepsilon \cdot \vec{n} &= 0 \quad &\text{on } (0, T) \times \Gamma^\varepsilon.
\end{align*}
\]

(4.1.8)-(4.1.11)

Let us denote this problem by \((P_1^\varepsilon)^+\). We will prove the existence of a global solution of \((P_1^\varepsilon)^+\). Since we show that the solution of \((P_1^\varepsilon)^+\) is non-negative, it solves \((P_1^\varepsilon)\). We conclude this section by proving the uniqueness of the solution of \((P_1^\varepsilon)\). We commence our investigation of the positivity of the solution of \((P_1^\varepsilon)^+\).

**Lemma 4.1.1.2.** Let (4.1.1)-(4.1.5) hold and a function \(u_\varepsilon \in F^\varepsilon\) be the solution of \((P_1^\varepsilon)^+\). Then \(u_{\varepsilon i} \geq 0\) on \((0, T) \times \Omega^\varepsilon\) for all \(i\).

**Proof.** The proof follows exactly as the one for lemma 3.2 given in [Krä08]. Let \(\Omega_{\varepsilon i}^-\) be the support of \(u_{\varepsilon i}^-(t)\). We multiply the \(i\)-th PDE of (4.1.8) by \(-u_{\varepsilon i}^-(t)\) and integrate over \(\Omega_{\varepsilon i}^-\). The rest follows by Gronwall’s inequality.

Now we show the existence of a global weak solution of \((P_1^\varepsilon)^+\). The basic ingredients are a Lyapunov functional, Schaefer’s fixed point theorem (see appendix theorem B.1) and a result from [PS01] (cf. theorem 3.3.1). For technical reasons, we add an extra term on both sides of \((P_1^\varepsilon)^+\), i.e., for a constant \(\kappa > 0\) we have

\[
\begin{align*}
  \frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot D\nabla u_\varepsilon + \kappa u_\varepsilon &= S\bar{R}(u_\varepsilon) + \kappa u_\varepsilon \quad &\text{in } (0, T) \times \Omega^\varepsilon, \\
  u_\varepsilon(0, x) &= u_0(x) \quad &\text{in } \Omega^\varepsilon, \\
  -D\nabla u_\varepsilon \cdot \vec{n} &= 0 \quad &\text{on } (0, T) \times \partial\Omega, \\
  -D\nabla u_\varepsilon \cdot \vec{n} &= 0 \quad &\text{on } (0, T) \times \Gamma^\varepsilon.
\end{align*}
\]

(4.1.12)-(4.1.15)

We denote the problem (4.1.12)-(4.1.15) by \((P_1^\varepsilon)^{+\varepsilon}\). We see that a solution of \((P_1^\varepsilon)^{+\varepsilon}\) is also a solution of \((P_{1\varepsilon M})^+\). We prove the global existence of a weak solution of \((P_{1\varepsilon M})^+\).

**4.1.1.1 Schaefer’s Fixed Point Operator**

Let us define a fixed point operator \(Z_1 : F^\varepsilon \to F^\varepsilon\) via

\[
Z_1(u_\varepsilon) = u_\varepsilon,
\]

(4.1.16)
where \( u_\varepsilon \) is the solution of the linear problem
\[
\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot D\nabla u_\varepsilon + \kappa u_\varepsilon = S\bar{R}(v_\varepsilon) + \kappa v_\varepsilon \\
u_\varepsilon(0, x) = u_0(x) \\
- D\nabla u_{\varepsilon_i} \cdot \vec{n} = 0 \\
- D\nabla u_{\varepsilon_i} \cdot \vec{n} = 0
\]
in \((0, T) \times \Omega^p_\varepsilon\), \((0, T) \times \Omega^p_\varepsilon\), \((0, T) \times \partial \Omega\), \((0, T) \times \Gamma_\varepsilon\). (4.1.17) for \( i = 1, 2, \ldots, I \).

**Remark 4.1.1.1.1.** The reformulation of (4.1.17)-(4.1.20) is given by
\[
\frac{\partial u_\varepsilon}{\partial t} + Au_\varepsilon = f(v_\varepsilon), \\
u_\varepsilon(0, x) = u_0(x),
\]
where \( f(v_\varepsilon) = S\bar{R}(v_\varepsilon) + \kappa v_\varepsilon \) and the operator \( A : H^{1,p}(\Omega^p_\varepsilon)^I \to [H^{1,q}(\Omega^p_\varepsilon)^*]^I \) is defined as
\[
A_{\varepsilon_i} := (A_1 u_{\varepsilon_1}, A_2 u_{\varepsilon_2}, \ldots, A_I u_{\varepsilon_I})
\]
such that for \( 1 \leq i \leq I \),
\[
\langle A_{\varepsilon_i}, w_{\varepsilon_i} \rangle := \int_{\Omega^p_\varepsilon} D\nabla u_{\varepsilon_i}(x) \cdot \nabla w_{\varepsilon_i}(x) \, dx \\
+ \kappa \int_{\Omega^p_\varepsilon} u_{\varepsilon_i}(x) w_{\varepsilon_i}(x) \, dx \\
\text{for } u_{\varepsilon_i} \in H^{1,p}(\Omega^p_\varepsilon) \text{ and } w_{\varepsilon_i} \in H^{1,q}(\Omega^p_\varepsilon),
\]
where \( \kappa > 0 \). Let us call this reformulated problem as (AP). The assumption (4.1.3) guarantees \( u_0 \in L^p_{\varepsilon,M} \). By theorem 3.4.3.3: Since \( v_\varepsilon \in F^p_\varepsilon \), \( v_\varepsilon \in L^\infty((0, T) \times \Omega^p_\varepsilon)^I \). This shows that \( f(v_\varepsilon) = S\bar{R}(v_\varepsilon) + \kappa v_\varepsilon \in [L^p((0, T); H^{1,q}(\Omega^p_\varepsilon)^*)]^I \). Moreover section 3.3.1 ensures the maximal regularity of \( A \) on \([H^{1,q}(\Omega^p_\varepsilon)^*]^I \). Therefore theorem 3.3.1 gives the existence of a unique solution \( u_\varepsilon \in F^p_\varepsilon \) of the problem (AP). Thus the operator \( Z_1 \) is well-defined.

**Remark 4.1.1.2.** Every fixed point of \( Z_1 \) is a solution of the problem \((P^{1+}_{\varepsilon,M})\).

In order to use Schaefer’s fixed point theorem, we need to verify the following conditions:

(i) The operator \( Z_1 \) is continuous and compact.

(ii) The set \( \{ u_\varepsilon \in F^p_\varepsilon | \exists \lambda \in [0, 1] : u_\varepsilon = \lambda Z_1(u_\varepsilon) \} \) is bounded, i.e., there exists a constant \( C_{10} > 0 \) such that any arbitrary solution \( u_\varepsilon \in F^p_\varepsilon \) of the equation
\[
u_\varepsilon = \lambda Z_1(u_\varepsilon)
\]
satisfies
\[
\|u_\varepsilon\|_{F^p_\varepsilon} \leq C_{10},
\]
where \( C_{10} \) is independent of \( \lambda, \varepsilon, u_\varepsilon \) and \( t \). Equations (4.1.17)-(4.1.20) and (4.1.22) imply
\[
\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot D\nabla u_\varepsilon + \kappa u_\varepsilon = \lambda S\bar{R}(u_\varepsilon) + \kappa u_\varepsilon \\
u_\varepsilon(0, x) = \lambda u_0(x) \\
- D\nabla u_{\varepsilon_i} \cdot \vec{n} = 0 \\
- D\nabla u_{\varepsilon_i} \cdot \vec{n} = 0
\]
in \((0, T) \times \Omega^p_\varepsilon\), \((0, T) \times \Omega^p_\varepsilon\), \((0, T) \times \partial \Omega\), \((0, T) \times \Gamma_\varepsilon\). (4.1.24) for \( 1 \leq i \leq I \).

\[\text{We denote the problem (4.1.24)-(4.1.27) as (}P^{1+}_{\varepsilon,M})\]
4.1.1.2 Introduction of the Lyapunov Functions

Let \( \mu^0 \in \mathbb{R}^I \) be a solution of the linear system
\[
S^T \mu^0 = -\log K,
\] (4.1.28)
where \( K \in \mathbb{R}^J \) is the vector of equilibrium constants \( k_f^j / k_b^j \) related to the \( J \) kinetic reactions. Due to assumption (4.1.4), the system (4.1.24) has a solution \( \mu^0 \). Using (4.1.28), we define the following functions:

Let \( g_i : \mathbb{R}_0^+ \to \mathbb{R} \) and \( g : \mathbb{R}_0^+I \to \mathbb{R} \) be defined as\(^{14}\)
\[
  g_i(u_{\varepsilon_i}) = (\mu^0_i - 1 + \log u_{\varepsilon_i})u_{\varepsilon_i} + e^{(1-\mu^0_i)} \quad \text{for each } i = 1, 2, ..., I
\]
and
\[
  g(u_{\varepsilon}) = \sum_{i=1}^I g_i(u_{\varepsilon_i}).
\]

Also for \( r \in \mathbb{N} \), we define \( f_r : \mathbb{R}_0^+I \to \mathbb{R} \) and \( F_r : L^\infty(\Omega_\varepsilon^p)^I \to \mathbb{R} \) as
\[
  f_r(u_{\varepsilon}) = \left[ g(u_{\varepsilon}) \right]^r
\]
and
\[
  F_r(u_{\varepsilon}) = \int_{\Omega_\varepsilon^p} f_r(u_{\varepsilon}(x)) \, dx.
\]

Proposition 4.1.1.2.1. For all \( i = 1, 2, ..., I \) and \( \varepsilon > 0 \),
\[
  g_i(u_{\varepsilon_i}) \geq g_i(u_{\varepsilon_i}) \geq u_{\varepsilon_i} \quad \text{(4.1.29)}
\]
and
\[
  F_r(u_{\varepsilon}) \geq \| u_{\varepsilon_i} \|_{L^r(\Omega_\varepsilon^p)}^r \quad \text{(4.1.30)}
\]

Proof. The inequality (4.1.29) is straightforward. For (4.1.30) see that
\[
  F_r(u_{\varepsilon}) = \int_{\Omega_\varepsilon^p} f_r(u_{\varepsilon}(x)) \, dx = \int_{\Omega_\varepsilon^p} [g(u_{\varepsilon}(x))]^r \, dx \geq \int_{\Omega_\varepsilon^p} |u_{\varepsilon_i}(x)|^r \, dx.
\]

Proposition 4.1.1.2.2. Let \( \alpha > 0 \). There exist constants \( C_{11}, C_{12}, C_{13} > 0 \) depending on \( \alpha \) and \( \mu_i \) but independent of \( \varepsilon \) and \( u_{\varepsilon_i} \) such that
\[
  g_i(u_{\varepsilon_i}) \leq C_{11}(1 + u_{\varepsilon_i}^{1+\alpha}) \quad \text{for all } i, \quad \text{(4.1.31)}
\]
\[
  g(u_{\varepsilon}) \leq C_{12}(1 + |u_{\varepsilon}|^{1+\alpha}) \quad \text{(4.1.32)}
\]
and
\[
  f_r(u_{\varepsilon}) \leq C_{13}(1 + |u_{\varepsilon}|_{L^I}^{r(1+\alpha)}) \quad \text{(4.1.33)}
\]

\(^{14}\)Here we have considered the natural logarithm, i.e. \( \log u_{\varepsilon_i} \).
Proof. The proof follows from the definitions of $g_t$, $g$ and $f_r$. ♦

From (4.1.30) it is clear that the $L^r$ - norm of $u_{\varepsilon}$, will be finite if we can obtain an upper bound of $F_r(u_{\varepsilon})$. This is the main concern of the following theorem:

**Theorem 4.1.1.2.3.** Let $r \in \mathbb{N}$ ($r \geq 2$), $0 \leq t \leq T$ and $0 \leq \lambda \leq 1$. Further assume that $u_{\varepsilon} \in \mathcal{F}^u_\varepsilon$ is a solution of $(\text{P}^{1+}_{\varepsilon, M})$. Then the following inequality holds good:

$$F_r(u_{\varepsilon}(t)) \leq e^{\text{tr}(e(1-\lambda))} F_r(u_{\varepsilon}(0)) \quad \text{for a.e. } t \text{ and for all } r. \quad (4.1.34)$$

To prove this theorem, we need the following lemmas as basic ingredients. For $p > n + 1$ and $\zeta \in [H^{1,p}(0,T); L^p(\Omega)] \cap L^p([0,T); H^{2,p}(\Omega))]$, these lemmas have been proved in [Krâ08] but they can be adapted for the functions in $\mathcal{F}^u_\varepsilon$ with $p > n + 2$.

**Lemma 4.1.1.2.4.** Let $p > n + 2$. The map $F_r : L^\infty(\Omega^p_\varepsilon)^I \rightarrow \mathbb{R}$ is continuous.

**Proof.** The proof is analogous to the proof of the theorem 3.4 in [Krâ08]. ♦

Let us consider the derivative (in the classical sense) of $f_r : \mathbb{R}_0^I \rightarrow \mathbb{R}^I$ which is given as

$$\partial f_r(v_{\varepsilon}) = \nabla v_{\varepsilon} f_r(v_{\varepsilon})$$

$$= r[g(v_{\varepsilon})]^{-1} \nabla v_{\varepsilon} g(v_{\varepsilon})$$

$$= r f_{r-1}(v_{\varepsilon}) \left( \mu^0 + \log v_{\varepsilon} \right).$$

We see that $\partial f_r(v_{\varepsilon})$ is undefined for $v_{\varepsilon} = 0$ whereas $f_{r-1}(v_{\varepsilon})$ is defined for all $v_{\varepsilon} \geq 0$. Since we only know the nonnegativity of $v_{\varepsilon}$, for any $\delta > 0$, we define

$$v_{\varepsilon}^{\delta} := v_{\varepsilon} + \delta. \quad (4.1.35)$$

Clearly, $v_{\varepsilon}^{\delta} \geq \delta > 0$ and $v_{\varepsilon}^{\delta} \in \mathcal{F}^u_\varepsilon$. From here on we work with the function $v_{\varepsilon}^{\delta}$ unless stated otherwise. We aim to prove that for $v_{\varepsilon}^{\delta} \in \mathcal{F}^u_\varepsilon$,

$$\partial f_r(v_{\varepsilon}^{\delta}) \in L^q((0,T); H^{1,q}(\Omega^p_\varepsilon))^I. \quad (4.1.36)$$

To prove (4.1.36), our point of departure is the following lemma which deals with the continuity of $\partial f_r$.

**Lemma 4.1.1.2.5.** Let $p > n + 2$ and $\delta > 0$, then the map

$$v_{\varepsilon}^{\delta} \mapsto \partial f_r(v_{\varepsilon}^{\delta}), \text{ i.e., } \partial f_r : \mathcal{F}^u_\varepsilon \rightarrow L^\infty((0,T) \times \Omega^p_\varepsilon)^I$$

is continuous.

**Proof.** Let $v_{\varepsilon}^{\delta} \in \mathcal{F}^u_\varepsilon$. For $p > n + 2$, from theorem 3.4.3.3 it follows that $v_{\varepsilon}^{\delta} \in [L^\infty((0,T) \times \Omega^p_\varepsilon)]^I$. The rest follows as in lemma 3.6 in [Krâ08]. ♦

**Lemma 4.1.1.2.6.** (Derivative of the vector function $x \mapsto \partial f_r(v_{\varepsilon}^{\delta}(t,x))$ w.r.t. $x \in \Omega^p_\varepsilon$) Let $p > n + 2$, $r \in \mathbb{N}$ ($r \geq 2$) and $v_{\varepsilon}^{\delta} \in \mathcal{F}^u_\varepsilon$. We define the mapping $w(v_{\varepsilon}^{\delta}) : (0,T) \times \Omega^p_\varepsilon \rightarrow \mathbb{R}^{I \times n}$ by

$$w(v_{\varepsilon}^{\delta})(t,x) := \{ (r-1)f_{r-2}(v_{\varepsilon}^{\delta}) - f_{r-1}(v_{\varepsilon}^{\delta}) \} \nabla v_{\varepsilon}^{\delta}(t,x), \quad (4.1.37)$$

where $M_{\mu}(v_{\varepsilon}^{\delta})$ is the $I \times I$-th order symmetric matrix with entries $(\mu^0 + \log v_{\varepsilon}^{\delta})$ and $\Lambda_{\frac{1}{v_{\varepsilon}^{\delta}}}$ is the $I \times I$-th order diagonal matrix with entries $\frac{1}{v_{\varepsilon}^{\delta}}$. Then

$$\nabla_x (\partial f_r(v_{\varepsilon}^{\delta})) = w(v_{\varepsilon}^{\delta}) \in L^q((0,T); H^{1,q}(\Omega^p_\varepsilon))^I \times n, \quad (4.1.38)$$

i.e.,

$$\partial f_r(v_{\varepsilon}^{\delta}) \in L^q((0,T); H^{1,q}(\Omega^p_\varepsilon))^I. \quad (4.1.39)$$
Proof. Let \( v_{\delta} \in \mathcal{F}^n_u \). For \( p > n + 2 \), theorem 3.4.3.3 implies \( v_{\delta} \in L^\infty((0,T) \times \Omega^n_p)^I \). Since \( v_{\delta} \geq \delta \), from the definitions of \( f_r(v_{\delta}) \), \( M_\mu(v_{\delta}) \) and \( \Lambda_{v_{\delta}} \), we have

\[
r(r - 1)f_{r-2}(v_{\delta})M_\mu(v_{\delta}) + r f_{r-1}(v_{\delta}) \Lambda_{v_{\delta}} \in L^\infty((0,T) \times \Omega^n_p)^I. \tag{4.1.40}
\]

Also note that for \( p > n + 2 \) and \( v_{\delta} \in \mathcal{F}^u_x \), \( \nabla_x v_{\delta} \in L^q((0,T); L^q(\Omega^n_p))^{I \times n} \). Therefore \( w(v_{\delta}) \in L^q((0,T); L^q(\Omega^n_p))^{I \times n} \). Next we prove that \( \nabla_x (\partial f_r(v_{\delta})) = w(v_{\delta}) \). This follows from the density of \( C^\infty((0,T) \times \Omega^n_p)^I \) in \( \mathcal{F}^u_x \) (for details confer lemma 3.6 in [Krä08]).

Lemma 4.1.1.2.7. Let \( u_\delta \in \mathcal{F}^u_x \) be the solution of the problem \((P^{1+}_{\epsilon,M})\) and \( \delta > 0 \) be such that \( u_{\delta} := u_\epsilon + \delta \). Then we have the following inequality

\[
\int_0^t \left\langle \frac{\partial u_\delta}{\partial \theta}, \partial f_r(u_{\delta}) \right\rangle_{[H^{1,q}(\Omega^n_p)^I \times [H^{1,q}(\Omega^n_p)]^I]} d\theta 
- \int_0^t \left\langle \nabla D\nabla u_\delta, \partial f_r(u_{\delta}) \right\rangle_{[H^{1,q}(\Omega^n_p)^I \times [H^{1,q}(\Omega^n_p)]^I]} d\theta + \kappa \int_0^t \int_{\Omega^n_p} \langle u_\epsilon, \partial f_r(u_{\delta}) \rangle_I dx d\theta 
= \lambda \int_0^t \left\langle \mathcal{S} \mathcal{R}(u_\epsilon), \partial f_r(u_{\delta}) \right\rangle_{[H^{1,q}(\Omega^n_p)^I \times [H^{1,q}(\Omega^n_p)]^I]} d\theta + \lambda \kappa \int_0^t \int_{\Omega^n_p} \langle u_\epsilon, \partial f_r(u_{\delta}) \rangle_I dx d\theta,
\]

i.e.,

\[
\int_0^t \left\langle \frac{\partial u_\delta}{\partial \theta}, \partial f_r(u_{\delta}) \right\rangle_{[H^{1,q}(\Omega^n_p)^I \times [H^{1,q}(\Omega^n_p)]^I]} d\theta 
- \int_0^t \langle D \nabla u_\delta, \nabla_x \partial f_r(u_{\delta}) \rangle_{[L^p(\Omega^n_p)]^{I \times n} \times [L^q(\Omega^n_p)]^{I \times n}} d\theta 
+ \lambda \int_0^t \left\langle \mathcal{S} \mathcal{R}(u_\epsilon), \partial f_r(u_{\delta}) \right\rangle_{[H^{1,q}(\Omega^n_p)^I \times [H^{1,q}(\Omega^n_p)]^I]} d\theta - (1 - \lambda) \kappa \int_0^t \int_{\Omega^n_p} \langle u_\epsilon, \partial f_r(u_{\delta}) \rangle_I dx d\theta,
\]

i.e.,

\[
\int_0^t \left\langle \frac{\partial u_\delta}{\partial \theta}, \partial f_r(u_{\delta}) \right\rangle_{[H^{1,q}(\Omega^n_p)^I \times [H^{1,q}(\Omega^n_p)]^I]} d\theta = I^{(t)}_{\text{diff}} + I^{(t)}_{\text{reac}} + I^{(t)}_{\text{Ex}} \quad \text{for a.e.} \ t,
\]

where

\[
I^{(t)}_{\text{diff}} := - \sum_{k=1}^n \int_0^t \int_{\Omega^n_p} \left\langle D \frac{\partial}{\partial x_k} u_\delta, \frac{\partial}{\partial x_k} (\partial f_r(u_{\delta})) \right\rangle_I dx d\theta,
\]

\[
I^{(t)}_{\text{reac}} := \lambda \int_0^t \left\langle \mathcal{S} \mathcal{R}(u_\epsilon), \partial f_r(u_{\delta}) \right\rangle_{[H^{1,q}(\Omega^n_p)^I \times [H^{1,q}(\Omega^n_p)]^I]} d\theta.
\]
and

\[ I_{Ex}^{(t)} := -(1 - \lambda)\kappa \int_0^t \int_{\Omega^p_\varepsilon} (u_{\varepsilon}, \partial f_r(u_{\varepsilon})) dx \, d\theta. \]

Now we simplify the terms \( I_{diff}^{(t)} \), \( I_{reac}(t) \) and \( I_{Ex}(t) \) one by one.\(^{15}\)

\[ I_{reac}^{(t)} = \lambda \int_0^t \int_{\Omega^p_\varepsilon} \langle SR(u_{\varepsilon}), \partial f_r(u_{\varepsilon}) \rangle_I \, dx \, d\theta \]
\[ = \lambda \int_0^t \int_{\Omega^p_\varepsilon} \langle rf_{r-1}(u_{\varepsilon}), (\mu^0 + \log u_{\varepsilon}), S\bar{R}(u_{\varepsilon}) \rangle_I \, dx \, d\theta \]
\[ = \lambda r \int_0^t \int_{\Omega^p_\varepsilon} f_{r-1}(u_{\varepsilon}) (\mu^0 + \log u_{\varepsilon}, S\bar{R}(u_{\varepsilon}))_I \, dx \, d\theta \quad \text{for a.e. } t. \quad (4.1.43) \]

Following the steps of lemma 3.7 in [Krä08], we can estimate the integral on the r.h.s. of (4.1.43), i.e.,

\[ I_{reac}^{(t)} \leq \lambda r C \sum_{i=1}^I \left( \int_0^t \int_{\Omega^p_\varepsilon} \left( \delta |\mu_i^0| + T|\Omega^p_\varepsilon| \delta |\log \delta| \right) \, dx \, d\theta + \delta \int_0^t \int_{\Omega^p_\varepsilon} (u_{\varepsilon}, + \delta) \, dx \, d\theta \right) =: h(t, u_{\varepsilon}, \delta) \quad \text{for a.e. } t, \]

where \( C \) is independent of \( \lambda \) and \( u_{\varepsilon} \), and all the other factors of \( h(t, u_{\varepsilon}, \delta) \) are bounded and tending to zero as \( \delta \to 0 \) for a.e. \( t \), i.e.,

\[ I_{reac}^{(t)} \leq h(t, u_{\varepsilon}, \delta) \to 0 \quad \text{as } \delta \to 0 \quad \text{for a.e. } t. \quad (4.1.44) \]

From lemma 5.8 in [Krä08] we get

\[ I_{diff}^{(t)} = -\sum_{k=1}^n \int_0^t \int_{\Omega^p_\varepsilon} \langle D \frac{\partial}{\partial x_k} u_{\varepsilon}, \frac{\partial}{\partial x_k} (\partial f_r(u_{\varepsilon}))_I \rangle \, dx \, d\theta \]
\[ = -r(r - 1)D \int_0^t \int_{\Omega^p_\varepsilon} f_{r-2}(u_{\varepsilon}) \sum_{k=1}^n \langle \mu^0 + \log u_{\varepsilon}, \partial_x u_{\varepsilon} \rangle_I^2 \, dx \, d\theta \]
\[ -rD \int_0^t \int_{\Omega^p_\varepsilon} f_{r-1}(u_{\varepsilon}) \sum_{i=1}^I \sum_{k=1}^n \frac{1}{u_{\varepsilon,i}} \left( \frac{\partial u_{\varepsilon,k}}{\partial x_k} \right)^2 \, dx \, d\theta \quad \text{for a.e. } t. \quad (4.1.45) \]

Both the terms of (4.1.45) are nonpositive, hence

\[ I_{diff}^{(t)} \leq 0 \quad \text{for a.e. } t. \quad (4.1.46) \]

\(^{15}\)We have \( p > n + 2 \). Then \( u_{\varepsilon} \in \mathcal{F}^p_\varepsilon \) implies that \( u_{\varepsilon} \in L^\infty((0, T) \times \Omega^p_\varepsilon) \). This gives \( SR(u_{\varepsilon}) \in L^p((0, T); L^p(\Omega^p_\varepsilon))^I \to L^p((0, T); H^{3,q}(\Omega^p_\varepsilon))^I \). Recall the definition (3.1.3) for the continuous embedding \( L^p(\Omega^p_\varepsilon) \to H^{3,q}(\Omega^p_\varepsilon)^* \) as

\[ \langle f, \zeta \rangle_{H^{3,q}(\Omega^p_\varepsilon)^*} = \langle f, \zeta \rangle_{L^p(\Omega^p_\varepsilon) \times L^q(\Omega^p_\varepsilon)}, \text{ for } f \in L^p(\Omega^p_\varepsilon) \text{ and } \zeta \in H^{3,q}(\Omega^p_\varepsilon). \]
\[ I_{Ex}^{(t)} = -\kappa (1-\lambda) \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} u_{\varepsilon_i} \partial f_{r}(u_{\varepsilon_i}) \, dx \, d\theta \]

\[ = \kappa (1-\lambda) \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} r(\delta - u_{\varepsilon_i}) f_{r-1}(u_{\varepsilon_i}) (\mu_i^0 + \log u_{\varepsilon_i}) \, dx \, d\theta \quad \text{since } u_{\varepsilon_i} = u_{\varepsilon_i} + \delta \]

\[ = \delta \kappa (1-\lambda) \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} r(\mu_i^0 + \log u_{\varepsilon_i}) f_{r-1}(u_{\varepsilon_i}) \, dx \, d\theta \]

\[ + r \kappa (1-\lambda) \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} -u_{\varepsilon_i} (\mu_i^0 + \log u_{\varepsilon_i}) f_{r-1}(u_{\varepsilon_i}) \, dx \, d\theta \quad (4.1.47) \]

It can be shown that

\[ -u_{\varepsilon_i} (\mu_i^0 + \log u_{\varepsilon_i}) \leq e^{-(1+\mu_i^0)} \quad \forall i. \quad (4.1.48) \]

We have \( \log u_{\varepsilon_i} \leq u_{\varepsilon_i} \leq g_i(u_{\varepsilon_i}) \) and \( g_i(u_{\varepsilon_i}) \geq (e-1)e^{-\mu_i^0} \). Choosing a constant \( C = \max_{1 \leq i \leq l} (1+|\mu_i^0|e^{-\mu_i^0}(e-1)) \), we obtain

\[ \mu_i^0 + \log u_{\varepsilon_i} \leq \mu_i^0 + g_i(u_{\varepsilon_i}) \leq |\mu_i^0| + g_i(u_{\varepsilon_i}) \leq C g_i(u_{\varepsilon_i}) \quad (4.1.49) \]

Combining (4.1.47), (4.1.48) and (4.1.49), we get

\[ I_{Ex}^{(t)} \leq (1-\lambda) \left[ r \delta \kappa \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} C g_i(u_{\varepsilon_i}) f_{r-1}(u_{\varepsilon_i}) \, dx \, d\theta \right] \]

\[ + \kappa (1-\lambda) \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} r(e(e-1))^{-1} g_i(u_{\varepsilon_i}) f_{r-1}(u_{\varepsilon_i}) \, dx \, d\theta \]

\[ \leq r \delta \kappa (1-\lambda) C \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} g(u_{\varepsilon_i}) f_{r-1}(u_{\varepsilon_i}) \, dx \, d\theta \]

\[ + \kappa (1-\lambda) \sum_{i=1}^{l} \int_{0}^{t} \int_{\Omega^p_{\delta}} r(e(e-1))^{-1} g(u_{\varepsilon_i}) f_{r-1}(u_{\varepsilon_i}) \, dx \, d\theta \quad \text{since } g_i(u_{\varepsilon_i}) \leq g(u_{\varepsilon_i}) \]

\[ \leq I r \kappa \delta C \int_{0}^{t} \int_{\Omega^p_{\delta}} f_r(u_{\varepsilon_i}) \, dx \, d\theta + I r \kappa (e(e-1))^{-1} \int_{0}^{t} \int_{\Omega^p_{\delta}} f_r(u_{\varepsilon_i}) \, dx \, d\theta \quad \text{since } 0 \leq \lambda \leq 1 \]

and \( f_r = f_{r-1}g \) for a.e. \( t \). \quad (4.1.50) \]

As \( \delta \to 0 \), \( f_r(u_{\varepsilon_i}) \) is bounded in \( L^1((0,T) \times \Omega) \). Therefore for a.e. \( t \) the first term in (4.1.50) tends to zero as \( \delta \to 0 \). Denote the first term by \( l(t,u_{\varepsilon_i},\delta) \), then

\[ I_{Ex}^{(t)} \leq l(t,u_{\varepsilon_i},\delta) + I r \kappa (e(e-1))^{-1} \int_{0}^{t} \int_{\Omega^p_{\delta}} f_r(u_{\varepsilon_i}) \, dx \, d\theta \quad \text{for a.e. } t. \quad (4.1.51) \]

Therefore combining (4.1.42), (4.1.44), (4.1.46) and (4.1.51) we obtain

\[ \int_{0}^{t} \left( \frac{\partial u_{\varepsilon_i}}{\partial \theta}, \partial f_r(u_{\varepsilon_i}) \right)_{[H^{1,q}([\Omega^p_{\delta}]),H^{-1,q}([\Omega^p_{\delta}])]^t} \, d\theta \]

\[ = I_{diff}^{(t)} + I_{reac}^{(t)} + I_{Ex}^{(t)} \]

\[ \leq 0 + h(t,u_{\varepsilon_i},\delta) + l(t,u_{\varepsilon_i},\delta) + I r \kappa (e(e-1))^{-1} \int_{0}^{t} \int_{\Omega^p_{\delta}} f_r(u_{\varepsilon_i}) \, dx \, d\theta \]

\[ \leq h(t,u_{\varepsilon_i},\delta) + l(t,u_{\varepsilon_i},\delta) + I r \kappa (e(e-1))^{-1} \int_{0}^{t} F_r(u_{\varepsilon_i}) \, d\theta \quad \text{for a.e. } t, \]
where \( h(t, \delta, u_{\varepsilon \delta}) \) and \( l(t, u_{\varepsilon \delta}, \delta) \to 0 \) as \( \delta \to 0 \) for a.e. \( t \).

\[ \]

**Proof of theorem 4.1.1.2.** Let \( u \in \mathcal{F}_u \) be the solution of \((P^{1+}_{\varepsilon \lambda M})\). Due to lemma 4.1.1.2 we know that \( u \geq 0 \). For any fixed \( \delta > 0 \), let

\[
u_{\varepsilon \delta} := u_{\varepsilon} + \delta.
\]

Let us choose a positive constant \( \eta \) and a smooth function \( \bar{u}_{\varepsilon \delta} \in C^\infty ([0, T] \times \Omega_t^\varepsilon) \) sufficiently close to \( u_{\varepsilon \delta} \) such that

\[
\bar{u}_{\varepsilon \delta} \geq \frac{\delta}{2},
\]

\[
||| \partial_t \bar{u}_{\varepsilon \delta} - \bar{u}_{\varepsilon \delta} |||_{L^p((0,T);H^{1,q}(\Omega_t^\varepsilon)^r)} \leq \eta,
\]

\[
||\mathcal{F}_r(u_{\varepsilon \delta}(t)) - \mathcal{F}_r(u_{\varepsilon}(0))|| - ||\mathcal{F}_r(\bar{u}_{\varepsilon \delta}(t)) - \mathcal{F}_r(\bar{u}_{\varepsilon}(0))|| \leq \delta,
\]

\[
||\partial_t \mathcal{F}_r(u_{\varepsilon \delta}) - \partial_t \mathcal{F}_r(\bar{u}_{\varepsilon \delta}) || _{L^\infty((0,T) \times \Omega_t^\varepsilon)^r} \leq \eta,
\]

and

\[
\eta ||\partial_t \mathcal{F}_r(u_{\varepsilon \delta}) || _{L^q((0,T);H^{1,q}(\Omega_t^\varepsilon)^r)} + \eta ||\partial_t \bar{u}_{\varepsilon \delta} || _{L^1((0,T) \times \Omega_t^\varepsilon)^r} \leq \delta.
\]

Then

\[
\left| \int_0^t \langle \partial_t \mathcal{F}_r(u_{\varepsilon \delta}), \partial_t u_{\varepsilon \delta} \rangle _{H^{1,q}(\Omega_t^\varepsilon)^r \times [H^{1,q}(\Omega_t^\varepsilon)^r]^t} d \theta - \int_0^t \langle \partial_t \mathcal{F}_r(\bar{u}_{\varepsilon \delta}), \partial_t \bar{u}_{\varepsilon \delta} \rangle _{H^{1,q}(\Omega_t^\varepsilon)^r \times [H^{1,q}(\Omega_t^\varepsilon)^r]^t} d \theta \right|
\]

\[
= \left| \int_0^t \langle \partial_t \mathcal{F}_r(u_{\varepsilon \delta}) - \partial_t \mathcal{F}_r(\bar{u}_{\varepsilon \delta}), \partial_t \bar{u}_{\varepsilon \delta} \rangle _{H^{1,q}(\Omega_t^\varepsilon)^r \times [H^{1,q}(\Omega_t^\varepsilon)^r]^t} d \theta \right|
\]

\[
\leq \sum_{i=1}^I \int_0^T \left| \langle \partial_t \mathcal{F}_r(u_{\varepsilon \delta}), \partial_t \bar{u}_{\varepsilon \delta} \rangle _{H^{1,q}(\Omega_t^\varepsilon)^r \times H^{1,q}(\Omega_t^\varepsilon)^r} \right| d \theta
\]

\[
\leq \sum_{i=1}^I \left| \langle \partial_t \mathcal{F}_r(u_{\varepsilon \delta}), \partial_t \bar{u}_{\varepsilon \delta} \rangle _{L^q(\Omega_t^\varepsilon)^r \times L^p(\Omega_t^\varepsilon)} \right| d \theta
\]

\[
\leq \sum_{i=1}^I \left| \langle \partial_t \mathcal{F}_r(u_{\varepsilon \delta}), \partial_t \bar{u}_{\varepsilon \delta} \rangle _{H^{1,q}(\Omega_t^\varepsilon)^r \times H^{1,q}(\Omega_t^\varepsilon)^r} \right| d \theta
\]

\[
\leq \sum_{i=1}^I \left| \langle \partial_t \mathcal{F}_r(u_{\varepsilon \delta}), \partial_t \bar{u}_{\varepsilon \delta} \rangle _{L^q(\Omega_t^\varepsilon)^r \times L^p(\Omega_t^\varepsilon)} \right| d \theta
\]

\[
\leq \sum_{i=1}^I \left| \langle \partial_t \mathcal{F}_r(u_{\varepsilon \delta}), \partial_t \bar{u}_{\varepsilon \delta} \rangle _{H^{1,q}(\Omega_t^\varepsilon)^r \times H^{1,q}(\Omega_t^\varepsilon)^r} \right| d \theta
\]

\[
\leq \sum_{i=1}^I \left[ \eta \left| \partial_t \bar{u}_{\varepsilon \delta} \right| _{L^q((0,T) \times \Omega_t^\varepsilon)} + \eta \left| \partial_t \mathcal{F}_r(u_{\varepsilon \delta}) \right| _{L^q((0,T);H^{1,q}(\Omega_t^\varepsilon)^r)} \right]
\]

\[
\leq \sum_{i=1}^I \delta = \delta I \quad \text{by (4.1.56)}. \quad (4.1.57)
\]
For the smooth function \( \bar{u}_{\varepsilon, \delta} \), we have

\[
F_r(\bar{u}_{\varepsilon, \delta}(t)) - F_r(\bar{u}_{\varepsilon, \delta}(0)) = \int_0^t \frac{d}{d\theta} (F_r(\bar{u}_{\varepsilon, \delta}(\theta))) \, d\theta \\
= \int_0^t \frac{d}{d\theta} \int_{\Omega^r} f_r(\bar{u}_{\varepsilon, \delta}) \, dx \, d\theta \\
= \int_0^t \int_{\Omega^r} \frac{\partial}{\partial \theta} f_r(\bar{u}_{\varepsilon, \delta}) \, dx \, d\theta \\
= \sum_{i=1}^l \int_0^t \int_{\Omega^r} \frac{\partial f_r(\bar{u}_{\varepsilon, \delta})}{\partial \theta} \frac{\partial \bar{u}_{\varepsilon, \delta}}{\partial \theta} \, dx \, d\theta \\
= \sum_{i=1}^l \int_0^t \int_{\Omega^r} \left( \frac{\partial f_r(\bar{u}_{\varepsilon, \delta})}{\partial \theta} \frac{\partial \bar{u}_{\varepsilon, \delta}}{\partial \theta} \right)_{H^1, \alpha(\Omega^r) \times H^1, \alpha(\Omega^r)} \, d\theta \\
= \int_0^t \left\langle \frac{\partial f_r(\bar{u}_{\varepsilon, \delta})}{\partial \theta}, \frac{\partial \bar{u}_{\varepsilon, \delta}}{\partial \theta} \right\rangle_{[H^1, \alpha(\Omega^r)]^t \times [H^1, \alpha(\Omega^r)]^t} \, d\theta. \tag{4.1.58}
\]

This implies

\[
\left| F_r(u_{\varepsilon, \delta}(t)) - F_r(u_{\varepsilon, \delta}(0)) - \int_0^t \left\langle \frac{\partial f_r(u_{\varepsilon, \delta})}{\partial \theta}, \frac{\partial \bar{u}_{\varepsilon, \delta}}{\partial \theta} \right\rangle_{[H^1, \alpha(\Omega^r)]^t \times [H^1, \alpha(\Omega^r)]^t} \, d\theta \right| \\
\leq \left| [F_r(u_{\varepsilon, \delta}(t)) - F_r(u_{\varepsilon, \delta}(0))] - [F_r(\bar{u}_{\varepsilon, \delta}(t)) - F_r(\bar{u}_{\varepsilon, \delta}(0))] \right| \\
+ \int_0^t \left| \frac{\partial f_r(u_{\varepsilon, \delta})}{\partial \theta} \frac{\partial \bar{u}_{\varepsilon, \delta}}{\partial \theta} \right|_{[H^1, \alpha(\Omega^r)]^t \times [H^1, \alpha(\Omega^r)]^t} \, d\theta \\
- \int_0^t \left| \frac{\partial f_r(u_{\varepsilon, \delta})}{\partial \theta} \frac{\partial \bar{u}_{\varepsilon, \delta}}{\partial \theta} \right|_{[H^1, \alpha(\Omega^r)]^t \times [H^1, \alpha(\Omega^r)]^t} \, d\theta \right| \text{ by (4.1.58)} \\
\leq \delta + \delta I \text{ by (4.1.54) and (4.1.57)} \\
\leq (I + 1)\delta.
\]

This gives

\[
|F_r(u_{\varepsilon, \delta}(t)) - F_r(u_{\varepsilon, \delta}(0))| \\
\leq (I + 1)\delta + \int_0^t \left| \frac{\partial f_r(u_{\varepsilon, \delta})}{\partial \theta} \frac{\partial \bar{u}_{\varepsilon, \delta}}{\partial \theta} \right|_{[H^1, \alpha(\Omega^r)]^t \times [H^1, \alpha(\Omega^r)]^t} \, d\theta \\
\leq (I + 1)\delta + h(t, u_{\varepsilon, \delta}, \delta) + l(t, u_{\varepsilon, \delta}, \delta) + I r \kappa (e(c - 1))^1 \int_0^t F_r(u_{\varepsilon, \delta}) \, d\theta \text{ by lemma 4.1.1.2.7, } \tag{4.1.59}
\]

where \( h(t, u_{\varepsilon, \delta}, \delta), l(t, u_{\varepsilon, \delta}, \delta) \to 0 \) as \( \delta \to 0 \) for a.e. \( t \). Therefore from the continuity of \( F_r \) (4.1.59) reduces to

\[
F_r(u_{\varepsilon}(t)) \leq F_r(u_{\varepsilon}(0)) + I r \kappa (e(c - 1))^1 \int_0^t F_r(u) \, d\theta \text{ for a.e. } t.
\]

Gronwall’s inequality gives

\[
F_r(u_{\varepsilon}(t)) \leq e^{I r \kappa (e(c - 1))^1 t} F_r(u_{\varepsilon}(0)) \text{ for all } r \text{ and for a.e. } t.
\]

This completes the proof. ♦

An immediate consequence of theorem 4.1.1.2.3 is the following corollary which gives the a-priori estimates (global in time) of the solution of \((P_{\varepsilon,M}^1)\). For all \( r \in \mathbb{N} \), let us define
\[ C_{14} := C_{14}(r) := \left[ i \text{ess sup}_{t \in (0,T)} \text{sup}_{i} \text{sup}_{e>0} C_{13} e^{\frac{1}{2}(e(-1))^{-1}t \|u\|_{L^\infty(\Omega_{e}^p),r}^{(1+\alpha)}} \right]^{\frac{1}{\gamma}} \]

and \[ C_{15} := \text{sup}_{e>0} \left[ 1 + \left( \frac{1}{2} \|u\|_{L^\infty(\Omega_{e}^p),r}^{1+\alpha} \right) \right]. \]

**Corollary 4.1.1.2.8.** Let \( p > n+2, r \in \mathbb{N} \) \((2 \leq r < \infty)\) and \( 0 \leq \lambda \leq 1. \) Suppose that \( u_\varepsilon \in F^u_\varepsilon \) is the solution of the problem \((P_{\lambda_M}^+)\), then the following estimates holds:

\[
\sup_{e>0} \left\| u_\varepsilon(t) \right\|_{L^r(\Omega_{e}^p),r} \leq C_{14} < \infty \quad \text{for all } r \text{ and for a.e. } t, \quad (4.1.60)
\]

and

\[
\sup_{e>0} \left\| u_\varepsilon(t) \right\|_{L^\infty(\Omega_{e}^p),r} \leq C_{15} < \infty \quad \text{for a.e. } t. \quad (4.1.61)
\]

**Proof.** From theorem 3.4.3.3, it follows that for \( p > n+2, u_0 \in L^\infty(\Omega_{e}^p),r \). For the problem \((P_{\lambda_M}^+)\), \( u_\varepsilon(0) = \lambda u_0 \) and \( \sup_{e>0} \left\| u_\varepsilon(t) \right\|_{L^\infty(\Omega_{e}^p),r} < \infty. \) Therefore from theorem 4.1.1.2.3, for \( 0 \leq t \leq T \), we have

\[
F_r(u_\varepsilon(t)) \leq e^{\frac{1}{2}(e(-1))^{-1}t} F_r(u_\varepsilon(0)) \quad \text{for all } r \text{ and for a.e. } t
\]

\[
\implies \int_{\Omega_{e}^p} f_r(u_\varepsilon(t,x)) dx \leq e^{\frac{1}{2}(e(-1))^{-1}t} F_r(\lambda u_0) \quad \text{for all } r \text{ and for a.e. } t
\]

\[
\implies \int_{\Omega_{e}^p} u_\varepsilon(t,x) dx \leq e^{\frac{1}{2}(e(-1))^{-1}t} \int_{\Omega_{e}^p} f_r(\lambda u_0(x)) dx \quad \text{for all } r \text{ and for a.e. } t. \quad (4.1.62)
\]

From proposition 4.1.1.2.2, we have

\[
f_r(\lambda u_0) \leq C_{13} \left( 1 + |\lambda u_0| r^{(1+\alpha)} \right), \quad (4.1.63)
\]

where \( \alpha > 0 \) and \( C_{13} \) is independent of \( \varepsilon, \delta, \lambda \) and \( u_\varepsilon \). Combining (4.1.62) and (4.1.63), we obtain

\[
\left\| u_\varepsilon(t) \right\|_{L^r(\Omega_{e}^p),r} \leq C_{13} e^{\frac{1}{2}(e(-1))^{-1}t} \left[ 1 + \left( \frac{1}{2} \|u\|_{L^\infty(\Omega_{e}^p),r}^{1+\alpha} \right) \right], \quad (4.1.64)
\]

\[
\sum_{i=1}^{I} \left\| u_\varepsilon(t) \right\|_{L^r(\Omega_{e}^p),r} \leq I \text{ess sup}_{t \in (0,T)} \text{sup}_{i} \text{sup}_{e>0} C_{13} e^{\frac{1}{2}(e(-1))^{-1}t} \left[ 1 + \left( \frac{1}{2} \|u\|_{L^\infty(\Omega_{e}^p),r}^{1+\alpha} \right) \right] = C_{14}'.
\]
where \( C_{14} \) is independent of \( i, \varepsilon \) and \( t \) but depends on \( r \). For every \( r \in \mathbb{N} \ (2 \leq r < \infty) \), \( C_{14}^{r} \leq \infty \) for all \( i, \varepsilon \) and \( t \). Therefore

\[
|||u_{\varepsilon}(t)|||_{L^{r}(\Omega_{\varepsilon}^{r})}^{r} \leq C_{14} < \infty \quad \forall \ \varepsilon, \ r \text{ and for a.e. } t
\]

This establishes the inequality (4.1.60). Again from theorem 4.1.1.2.3, for \( 0 \leq t \leq T \), we have

\[
F_{r}(u_{\varepsilon}(t)) \leq e^{r\varepsilon(e(-1))^{-1}t} F_{r}(\lambda u_{0}) \quad \text{for all } r \text{ and for a.e. } t
\]

Proceeding as above, we obtain

\[
|||u_{\varepsilon}(t)|||_{L^{r}(\Omega_{\varepsilon})}^{r} \leq C_{13} e^{r\varepsilon(e(-1))^{-1}t} \int_{\Omega_{\varepsilon}} \left(1 + |u_{0}|_{I}^{1+\alpha}\right) dx
\]

\[
\leq C_{13} e^{r\varepsilon(e(-1))^{-1}t} \int_{\Omega_{\varepsilon}} \left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r} dx
\]

\[
= C_{13} e^{r\varepsilon(e(-1))^{-1}t} \left(\left|\left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r}\right|_{L^{r}(\Omega_{\varepsilon})}\right)
\]

\[
\Rightarrow |||u_{\varepsilon}(t)|||_{L^{r}(\Omega_{\varepsilon})}^{r} \leq C_{13} e^{r\varepsilon(e(-1))^{-1}t} \left(\left|\left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r}\right|_{L^{r}(\Omega_{\varepsilon})}\right)^{\frac{1}{r}}
\]

\[
\leq \sup_{r \in \mathbb{N}} \left(C_{13} e^{r\varepsilon(e(-1))^{-1}t}\right)^{\frac{1}{r}} \left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{\frac{1}{r}} |||u_{\varepsilon}(t)|||_{L^{r}(\Omega_{\varepsilon})} \quad \forall \ i, r \text{ and for a.e. } t.
\]

(4.1.65)

Taking limit sup as \( r \to \infty \) on both sides, we obtain

\[
|||u_{\varepsilon}(t)|||_{L^{\infty}(\Omega_{\varepsilon})} \leq \left(\left|\left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r}\right|_{L^{\infty}(\Omega_{\varepsilon})}\right)^{\frac{1}{r}} \leq \left(\left|\left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r}\right|_{L^{\infty}(\Omega_{\varepsilon})}\right)^{\frac{1}{r}} \leq \left(\left|\left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r}\right|_{L^{\infty}(\Omega_{\varepsilon})}\right)^{\frac{1}{r}}
\]

\[
\leq \sup_{r \in \mathbb{N}} \left(1 + \left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r}\right)^{\frac{1}{r}} \leq C_{15} < \infty \quad \forall \ \varepsilon, \ i \text{ and for a.e. } t,
\]

i.e.,

\[
\max_{1 \leq i \leq I} |||u_{\varepsilon}(t)|||_{L^{\infty}(\Omega_{\varepsilon})} \leq C_{15} < \infty \quad \forall \ \varepsilon \text{ and for a.e. } t,
\]

i.e.,

\[
\sup_{r \in \mathbb{N}} \left(1 + \left(1 + |u_{0}|_{I}^{1+\alpha}\right)^{r}\right)^{\frac{1}{r}} \leq C_{15} < \infty \quad \text{for a.e. } t.
\]

\[\Box\]

**Corollary 4.1.1.2.9.** Let \( p > n+2, \ r \in \mathbb{N} \) and \( 0 \leq \lambda \leq 1 \). Then there exists a positive constant \( C \) (depending only on \( r \in \mathbb{N}, \ T, \ |\Omega| \) and \( I \) but independent of \( \varepsilon, \lambda \) and \( u_{\varepsilon} \)) such that any arbitrary solution \( u_{\varepsilon} \in F_{\varepsilon}^{r} \) of the problem \( P_{1_{\lambda}M}^{r+} \) satisfies

\[
|||u_{\varepsilon}|||_{F_{\varepsilon}} \leq C.
\]
The reformulation of (4.1.24)-(4.1.27) is given by

\[
\frac{\partial u_\varepsilon(t)}{\partial t} + A u_\varepsilon(t) = f(t),
\]
\[
u_\varepsilon(0, x) = \lambda u_0(x),
\]
where \(f(t) = \lambda S \overline{R}(u_\varepsilon(t)) + \lambda \kappa u_\varepsilon(t)\) and \(\kappa > 0\). The operator \(A\) is defined as in remark 4.1.1.1 and has the maximal parabolic regularity on \([H^{1,q}(\Omega_\varepsilon^*)]^I\). \(f\) is in \(L^p((0,T); H^{1,q}(\Omega_\varepsilon^*))^I\). Moreover, by assumption (4.1.3), \(u_0 \in X^w_1\). Therefore by theorem 3.3.1, there exists a \(C > 0\) such that

\[
\|u_\varepsilon\|_{\mathcal{F}_\varepsilon^w} \leq \hat{C} \left( \|\lambda u_0\|_{X^w_1} + \|\lambda S \overline{R}(u_\varepsilon) + \lambda \kappa u_\varepsilon\|_{L^p((0,T); H^{1,q}(\Omega_\varepsilon^*))^I} \right)
\]
\[
\leq \hat{C} \sup_{\varepsilon, \lambda > 0} \left( \|u_0\|_{X^w_1} + \|S \overline{R}(u_\varepsilon)\|_{L^p((0,T); H^{1,q}(\Omega_\varepsilon^*))^I} + \kappa \|u_\varepsilon\|_{L^p((0,T); H^{1,q}(\Omega_\varepsilon^*))^I} \right)
\]
\[
=: C < \infty,
\]
where \(C\) is independent of \(\varepsilon, \lambda\) and \(u_\varepsilon\).

### 4.1.1.3 Compactness and Continuity of \(Z_1\)

**Lemma 4.1.1.3.1.** The fixed point operator \(Z_1\) is continuous and compact.

**Proof.** Here we will only show the continuity of the operator \(Z_1\) as the compactness follows with similar arguments. Let \((v_{\varepsilon,n})_{n \geq 1}\) be a sequence in \(\mathcal{F}_\varepsilon^w\) converging to a limit \(v_\varepsilon \in \mathcal{F}_\varepsilon^w\). From theorem 3.4.3.4, \((v_{\varepsilon,n})_{n \geq 1}\) is convergent to \(v_\varepsilon\) in \([L^\infty((0,T) \times \Omega_\varepsilon^*)]^I\). This implies that \((SR(v_{\varepsilon,n}) + \kappa v_{\varepsilon,n})_{n \geq 1}\) is convergent to \(SR(v_\varepsilon) + \kappa v_\varepsilon\) in \([L^p((0,T) \times \Omega_\varepsilon^*)]^I\). Due to the continuous embedding \(L^p(\Omega_\varepsilon^*) \hookrightarrow H^{1,q}(\Omega_\varepsilon^*), (SR(v_{\varepsilon,n}) + \kappa v_{\varepsilon,n})_{n \geq 1}\) is convergent to \(SR(v_\varepsilon) + \kappa v_\varepsilon\) in \([L^p((0,T); H^{1,q}(\Omega_\varepsilon^*))^I]\). From theorem 3.3.1, we conclude that the map \(Z_1\) is continuous.

### 4.1.1.4 Existence and Uniqueness of the Solution

**Proof of theorem 4.1.1.1.** Applying Schaefer’s fixed point theorem, thanks to corollary 4.1.1.2.9 and lemma 4.1.1.3.1, we get the existence of at least one fixed point, i.e., the existence of at least one solution of the problem \((P_{\varepsilon}^1)\). This solution is also a solution of \((P_{\varepsilon}^{1+})\). Due to lemma 4.1.1.2, the solution of \((P_{\varepsilon}^{1+})\) solves \((P_1^1)\). Now we prove the uniqueness of the solution of \((P_1^1)\). Let \(u_{\varepsilon,1}\) and \(u_{\varepsilon,2}\) be two solutions of the problem \((P_1^1)\), where \(u_{\varepsilon,1} \neq u_{\varepsilon,2}\). Set \(\overline{u}_\varepsilon = u_{\varepsilon,1} - u_{\varepsilon,2}\). Then we have

\[
\frac{\partial u_{\varepsilon,1}}{\partial t} - \nabla \cdot D \nabla u_{\varepsilon,1} = SR(u_{\varepsilon,1}) \quad \text{in } (0,T) \times \Omega_\varepsilon^p,
\]
\[
u_{\varepsilon,1}(0, x) = u_0(x) \quad \text{in } \Omega_\varepsilon^p,
\]
\[\begin{align*}
-D \nabla u_{\varepsilon,1} \cdot \vec{n} &= 0 \quad &\text{on } (0,T) \times \partial \Omega, \\
-D \nabla u_{\varepsilon,1} \cdot \vec{n} &= 0 \quad &\text{on } (0,T) \times \Gamma_\varepsilon,
\end{align*}
\]
\[\text{Note that } 0 \leq \lambda \leq 1.\]
for \( k = 1, 2 \). Taking the difference and using \( \bar{u}_{\varepsilon,i} \) as the test function in the \( i \)-th PDE, we obtain
\[
\frac{1}{2} \int_0^t \frac{d}{d\theta} ||\bar{u}_{\varepsilon,i}(\theta)||^2_{L^2(\Omega_{\varepsilon}^p)} \, d\theta + D \int_0^t ||\nabla \bar{u}_{\varepsilon,i}(\theta)||^2_{L^2(\Omega_{\varepsilon}^p)} \, d\theta \\
\leq \frac{1}{2} \int_0^t \left[ ||SR(u_{\varepsilon_1}(\theta))_i - SR(u_{\varepsilon_2}(\theta))_i||^2_{L^2(\Omega_{\varepsilon}^p)} + ||\bar{u}_{\varepsilon,i}(\theta)||^2_{L^2(\Omega_{\varepsilon}^p)} \right] \, d\theta.
\]
Expanding the term \( R_j(u_{\varepsilon_1}) - R_j(u_{\varepsilon_2}) \), each term in \( R_j(u_{\varepsilon_1}) - R_j(u_{\varepsilon_2}) \) contains a factor of the type \( u_{\varepsilon_1,i} - u_{\varepsilon_2,i} \), whereas all the other factors are bounded in \( L^\infty((0,T) \times \Omega_{\varepsilon}^p) \), therefore we obtain
\[
||\bar{u}_{\varepsilon,i}(t)||^2_{L^2(\Omega_{\varepsilon}^p)} \leq C \int_0^t \sum_{i=1}^l ||\bar{u}_{\varepsilon,i}(\theta)||^2_{L^2(\Omega_{\varepsilon}^p)} \, d\theta,
\]
i.e.,
\[
||\bar{u}_{\varepsilon}(t)||^2_{L^2(\Omega_{\varepsilon}^p)} \leq C \int_0^t ||\bar{u}_{\varepsilon}(\theta)||^2_{L^2(\Omega_{\varepsilon}^p)} \, d\theta.
\]
Gronwall’s inequality gives
\[
||\bar{u}_{\varepsilon}(t)||^2_{L^2(\Omega_{\varepsilon}^p)} = 0 \quad \text{for a.e. } t,
\]
\[
\Rightarrow \quad u_{\varepsilon_1} = u_{\varepsilon_2}.
\]
Hence the solution exists uniquely.

**Conclusion:** We have shown the existence of a unique positive global weak solution of the problem \((P_1)\) in the first section 4.1.1. This has also provided us some very useful *a-priori* estimates (see (4.1.60) and (4.1.61)) with the help of a Lyapunov functional. These estimates will be further used during the homogenization of model M1, which is the main concern of the next section.

4.1.2 Homogenization of the Problem \((P_{\varepsilon}^1)\)

4.1.2.1 *A-priori* Estimates

The aim of this section is to obtain \( \varepsilon \)-independent *a-priori* estimates for the solution \( u_{\varepsilon} \) of the micromodel in the domain \((0,T) \times \Omega_{\varepsilon}^p\) and then to extend these estimates to all of \((0,T) \times \Omega\). The major theorem of this section reads as:

**Theorem 4.1.2.1.1.** There exists an extension of the solution \( u_{\varepsilon} \) to all of \((0,T) \times \Omega\) such that
\[
||u_{\varepsilon}||_{L^r((0,T);L^r(\Omega))} + ||u_{\varepsilon}||_{L^\infty((0,T);L^\infty(\Omega))} + ||\nabla u_{\varepsilon}||_{L^2((0,T);L^2(\Omega))} \leq C_{16}, \quad (4.1.72)
\]
where \( C_{16} \) is independent of \( \varepsilon \) but depends only on \( r \).

We start with the following lemma:

**Lemma 4.1.2.1.2.** Let \( p > n + 2 \) be fixed and \( r \in \mathbb{N} \). Assume further that \( u_{\varepsilon} \in F_{\varepsilon}^n \) is the solution of the problem \((P_{\varepsilon}^1)\), then we have the following estimate
\[
||u_{\varepsilon}||_{L^r((0,T);L^r(\Omega_{\varepsilon}^p))} + ||u_{\varepsilon}||_{L^\infty((0,T);L^\infty(\Omega_{\varepsilon}^p))} + ||\nabla u_{\varepsilon}||_{L^2((0,T);L^2(\Omega_{\varepsilon}^p))} \leq C_{17}, \quad (4.1.73)
\]
where \( C_{17} \) is independent of \( \varepsilon \) but depends only on \( r \).
Proof. The proof of this lemma consists of several steps.

(a) For $\lambda = 1$, the \textit{a-priori} estimates obtained in (4.1.60) and (4.1.61) correspond to the \textit{a-priori} estimates of the solution of the problem ($P_\varepsilon$)$^{17}$. Therefore from (4.1.60), we have

$$
\|\|u_\varepsilon(t)\|\|_{L^r(\Omega^\varepsilon_t)} \leq C_{14} \quad \text{for all } r \text{ and for a.e. } t
$$

$$
\sum_{i=1}^l \int_0^T \|u_{\varepsilon i}(t)\|_{L^r(\Omega^\varepsilon_t)} dt \leq \int_0^T C_{14}^r dt
$$

$$
\|\|u_\varepsilon\|\|_{L^r((0,T); L^r(\Omega^\varepsilon_t))} \leq C_{18} \quad \text{for all } r,
$$

where $C_{18} := (C_{14}^r T)\frac{1}{r}$ is independent of $\varepsilon$. Next,

$$
\|\|u_\varepsilon\|\|_{L^\infty((0,T); L^\infty(\Omega^\varepsilon_t))} = \max_{1 \leq i \leq l} \|\|u_{\varepsilon i}\|\|_{L^\infty((0,T); L^\infty(\Omega^\varepsilon_t))}
$$

= \max_{1 \leq i \leq l} \text{ess sup}_{t \in (0,T)} \text{ess sup}_{x \in \Omega^\varepsilon_t} |u_{\varepsilon i}(t,x)|

= \text{ess sup}_{t \in (0,T)} \max_{1 \leq i \leq l} \text{ess sup}_{x \in \Omega^\varepsilon_t} |u_{\varepsilon i}(t,x)|

= \text{ess sup}_{t \in (0,T)} \|\|u_\varepsilon(t)\|\|_{L^\infty(\Omega^\varepsilon_t)}

\leq \text{ess sup}_{t \in (0,T)} C_{15} \quad \text{by (4.1.61)}

= C_{15},
$$

where $C_{15}$ is independent of $\varepsilon$.

(b) Testing the $i$-th PDE of (2.5.16) with $u_{\varepsilon i}(t)$, we obtain$^{18}$

$$
\int_0^T \left\langle \frac{\partial u_{\varepsilon i}(t)}{\partial t}, u_{\varepsilon i}(t) \right\rangle_{H^{1,q}(\Omega^\varepsilon_t)^* \times H^{1,q}(\Omega^\varepsilon_t)} dx dt
$$

$$
- \int_0^T \langle \nabla \cdot Du_{\varepsilon i}(t), u_{\varepsilon i}(t) \rangle_{H^{1,q}(\Omega^\varepsilon_t)^* \times H^{1,q}(\Omega^\varepsilon_t)} dx dt
$$

$$
= \int_0^T \langle SR(u_\varepsilon(t))_i, u_{\varepsilon i}(t) \rangle_{H^{1,q}(\Omega^\varepsilon_t)^* \times H^{1,q}(\Omega^\varepsilon_t)} dt,
$$

i.e.,

$$
\frac{1}{2} \int_0^T \frac{d}{dt} \|u_{\varepsilon i}(t)\|_{L^2(\Omega^\varepsilon_t)}^2 dt + \int_0^T D \|\nabla u_{\varepsilon i}(t)\|_{L^2(\Omega^\varepsilon_t)}^2 dt
$$

$$
= \int_0^T \langle SR(u_\varepsilon(t))_i, u_{\varepsilon i}(t) \rangle_{L^p(\Omega^\varepsilon_t)^* \times L^q(\Omega^\varepsilon_t)} dt
$$

$$
\leq \frac{1}{p} \int_0^T \|SR(u_\varepsilon(t))_i\|_{L^p(\Omega^\varepsilon_t)}^p dt + \frac{1}{q} \int_0^T \|u_{\varepsilon i}(t)\|_{L^q(\Omega^\varepsilon_t)}^q dt,
$$

i.e.,

$$
\frac{1}{2} \|u_{\varepsilon i}(t)\|_{L^2(\Omega^\varepsilon_t)}^2 + \int_0^T D \|\nabla u_{\varepsilon i}(t)\|_{L^2(\Omega^\varepsilon_t)}^2 dt
$$

$$
\leq \frac{1}{p} \int_0^T \|SR(u_\varepsilon(t))_i\|_{L^p(\Omega^\varepsilon_t)}^p dt + \frac{1}{q} \int_0^T \|u_{\varepsilon i}(t)\|_{L^q(\Omega^\varepsilon_t)}^q dt.
$$

$^{17}$See the remark after the theorem 9.2.2.4 in [Eva98].

$^{18}$From (4.1.60), we have $\|u_{\varepsilon i}(t)\|_{L^r(\Omega^\varepsilon_t)} \leq C$ for all $i$ and for a.e. $t$, where $C$ is independent of $\varepsilon$. This gives $\|SR(u_\varepsilon)(t)\|_{L^p(\Omega^\varepsilon_t)} \leq C$. Since $L^p(\Omega^\varepsilon_t) \hookrightarrow H^{1,q}(\Omega^\varepsilon_t)^*$, from the definition (3.1.3) we get $\langle SR(u_\varepsilon)_i, \phi_i \rangle_{H^{1,q}(\Omega^\varepsilon_t)^* \times H^{1,q}(\Omega^\varepsilon_t)} = \langle SR(u_\varepsilon)_i, \phi_i \rangle_{L^p(\Omega^\varepsilon_t)^* \times L^q(\Omega^\varepsilon_t)}$ for $\phi_i \in H^{1,q}(\Omega^\varepsilon_t)$. 


4.1. Model M1

Choosing \( r \) in (4.1.60) sufficiently large such that \( \sup_{\varepsilon > 0} ||u_{\varepsilon}(t)||_{L^q(\Omega^\varepsilon)} < \infty \) and \( \sup_{\varepsilon > 0} ||SR(u_{\varepsilon}(t))||_{L^p(\Omega^\varepsilon)} < \infty \). Also from theorem 3.4.3.3, it follows that \( \sup_{\varepsilon > 0} ||u_0||_{L^2(\Omega^\varepsilon)} < \infty \). Therefore the r.h.s. of (4.1.76) is bounded by a constant independent of \( \varepsilon, i \) and \( t \). Let us call this constant by \( \tilde{C} \). This gives

\[
\int_0^T D||\nabla u_{\varepsilon}(t)||_{L^2(\Omega^\varepsilon)}^2 dt \leq \tilde{C} \quad \text{for all } \varepsilon, i \text{ and for a.e. } t
\]

\[
\Rightarrow \sum_{i=1}^I \int_0^T ||\nabla u_{\varepsilon}(t)||_{L^2(\Omega^\varepsilon)}^2 dt \leq \sum_{i=1}^I \frac{\tilde{C}}{D}
\]

\[
\Rightarrow \sup_{\varepsilon > 0} ||\nabla u_{\varepsilon}||_{L^2((0,T);L^2(\Omega^\varepsilon)'')} \leq C_{19},
\]

\[
C_{19} := \left( \frac{\tilde{C}}{D} \right)^{\frac{1}{2}}
\]

where \( C_{19} \) is independent of \( \varepsilon \). Note that \( D > 0 \) is a constant. Adding (4.1.74), (4.1.75) and (4.1.77) will yield

\[
||u_{\varepsilon}||_{L^\infty((0,T);L^\infty(\Omega^\varepsilon)'')} + ||u_{\varepsilon}||_{L^\infty((0,T);L^\infty(\Omega^\varepsilon)'')} + ||\nabla u_{\varepsilon}||_{L^2((0,T);L^2(\Omega^\varepsilon)'')} \leq C_{18} + C_{15} + C_{19}
\]

for all \( r \)

\[
C_{17} := C_{18} + C_{15} + C_{19}
\]

where \( C_{17} \) is independent of \( \varepsilon \) but depends only on \( r \).

\[\textbf{Proof of theorem 4.1.2.1.1:} \text{ The estimate (4.1.73) from the lemma 4.1.2.1.2 and the theorem 3.4.2.3 accomplish the proof.}\]

\[\textbf{4.1.2.2 Convergence of the Micro Solution}\]

In this subsection we show the weak, strong and two-scale convergences of the solution of the micropbem (\( P^1 \)).

**Theorem 4.1.2.2.1.** There exists a constant \( C_{20} \) such that the solution, \( u_{\varepsilon} \), of the problem (\( P^1 \)) satisfies the following estimate:

\[
||u_{\varepsilon}||_{L^\infty((0,T);L^2(\Omega))}^2 + ||u_{\varepsilon}||_{L^2((0,T);H^{1,2}(\Omega))}^2 + \left|\left| \chi \frac{\partial u_{\varepsilon}}{\partial t} \right|\right|_{L^2((0,T);H^{1,2}(\Omega)'')} \leq C_{20},
\]

where \( C_{20} \) is independent of \( \varepsilon \) but depends only on \( r \).

\[\text{by theorem 3.4.3.3}\]

\[
\sup_{\varepsilon > 0} ||u_0||_{L^2(\Omega^\varepsilon)}^2 \leq C|\Omega| \quad \text{by theorem 3.4.3.3}
\]

\[
\leq C|\Omega| \sup_{\varepsilon > 0} ||u_0||_{H^{1,2}(\Omega^\varepsilon)'ankan(\Omega^\varepsilon)}, H^{1,2})(\Omega^\varepsilon))_{1-\frac{1}{p},p}.
\]

\[
\leq C|\Omega| < \infty \forall \varepsilon
\]

\[
\Rightarrow \sup_{\varepsilon > 0} ||u_0||_{L^2(\Omega^\varepsilon)}^2 \leq C|\Omega| < \infty.
\]
The proof consists of several steps.

(a) \[ \|u_{\varepsilon}\|_{L^\infty((0,T);L^2(\Omega))}^2 = \max_{1 \leq i \leq I} \text{ess sup}_{t \in (0,T)} \|u_{\varepsilon_i}(t)\|_{L^2(\Omega)}^2 \]
\[ = \max_{1 \leq i \leq I} \text{ess sup}_{t \in (0,T)} \int_{\Omega} |u_{\varepsilon_i}(t,x)|^2 \, dx \]
\[ \leq \max_{1 \leq i \leq I} \text{ess sup} \text{ ess sup}_{x \in \Omega} |u_{\varepsilon_i}(t,x)|^2 |\Omega| \]
\[ = |\Omega| \|u_{\varepsilon}\|_{L^\infty((0,T);L^2(\Omega))}^2 \]
\[ \leq |\Omega| C_{16}^2 \quad \text{by (4.1.72)}, \]

i.e., \[ \|u_{\varepsilon}\|_{L^\infty((0,T);L^2(\Omega))}^2 \leq C_{21}, \tag{4.1.79} \]

where \( C_{21} := (C_{16}^2 |\Omega|)^{\frac{1}{2}} \) is independent of \( \varepsilon \).

(b) \[ \|u_{\varepsilon}\|_{L^2((0,T);H^{1,2}(\Omega))}^2 \]
\[ = \sum_{i=1}^{I} \|u_{\varepsilon_i}\|_{L^2((0,T);H^{1,2}(\Omega))}^2 \]
\[ = \sum_{i=1}^{I} \left( \|\nabla u_{\varepsilon_i}\|_{L^2((0,T);L^2(\Omega))}^2 + \|u_{\varepsilon_i}\|_{L^2((0,T);L^2(\Omega))}^2 \right) \]
\[ \leq \sup_{\varepsilon > 0} \sum_{i=1}^{I} \left( \|\nabla u_{\varepsilon_i}\|_{L^2((0,T);L^2(\Omega))}^2 + (T|\Omega|)^{1/2-1/2} \|u_{\varepsilon_i}\|_{L^2((0,T);L^2(\Omega))}^2 \right) \]
\[ =: C_{22} < \infty, \quad \text{by (4.1.72)}, \tag{4.1.80} \]

i.e., \[ \|u_{\varepsilon}\|_{L^2((0,T);H^{1,2}(\Omega))} \leq C_{22}, \]

where \( C_{22} \) is independent of \( \varepsilon \) but depends only on \( r \).

(c) Let \( \phi \in H^{1,2}_0(0,T) \) and \( \psi \in H^{1,2}(\Omega) \). Then the weak formulation of the \( i \)-th PDE of the problem (2.5.16)-(2.5.19) is given by
\[ \int_{0}^{T} \left( \chi^{\varepsilon} \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi(t) \psi \right)_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} \, dt \]
\[ + \int_{0}^{T} \int_{\Omega} \phi(t) \chi^{\varepsilon}(x) \nabla u_{\varepsilon_i}(t,x) \nabla \psi(x) \, dx \, dt \]
\[ = \int_{0}^{T} \left( \chi^{\varepsilon} \mathbf{SR}(u_{\varepsilon}(t)), \phi(t) \psi \right)_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} \, dt, \]

i.e.,
\[ \left| \int_{0}^{T} \left( \chi^{\varepsilon} \frac{\partial u_{\varepsilon_i}(t)}{\partial t}, \phi(t) \psi \right)_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} \, dt \right| \]
\[ \leq \int_{0}^{T} \int_{\Omega} |\chi^{\varepsilon}(x)| \|\nabla u_{\varepsilon_i}(t,x)||\nabla \psi(x)||\phi(t)| \, dx \, dt \]
\[ + \frac{1}{2} \int_{0}^{T} \left[ \|\chi^{\varepsilon} \mathbf{SR}(u_{\varepsilon}(t))\|_{L^2(\Omega)}^2 + \|\phi(t) \psi\|_{L^2(\Omega)}^2 \right] \, dt. \]
Note that $|\chi'(x)| \leq 1$. From (4.1.72) the terms sup $\epsilon > 0 \|\nabla u_\epsilon\|_{L^2((0,T);L^2(\Omega))}$ and sup $\epsilon > 0 \|SR(u_\epsilon)\|_{L^2((0,T);L^2(\Omega))}$ are finite. This gives

\[
\left| \int_0^T (\chi^\varepsilon \frac{\partial u_\varepsilon(t)}{\partial t}, \phi(t)\psi)_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} \, dt \right| \leq C + \frac{1}{2} \|\phi(t)\|_{L^2(0,T)}^2 \left( \|\nabla \psi\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)} \right)
\]

\[
= C + \|\phi\|_{L^2(0,T)}^2 \|\psi\|_{H^{1,2}(\Omega)}^2 .
\]

$\phi \in H^{1,2}_0(0,T)$ implies $\|\phi\|_{L^2(0,T)} \leq \tilde{C} \|\phi\|_{H^{1,2}_0(0,T)}$, i.e., $\|\phi\|_{L^2(0,T)} \leq \|\phi\|_{H^{1,2}_0(0,T)}$, where $\tilde{C} > 0$ is the embedding constant. Taking the supremum on both sides,

\[
\tilde{C} \sup_{\frac{\phi}{C} \in L^2(0,T)} \sup_{\psi \in H^{1,2}(\Omega)} \left| \int_0^T (\chi^\varepsilon \frac{\partial u_\varepsilon(t)}{\partial t}, \frac{\phi(t)}{C} \psi)_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} \, dt \right|
\]

\[
\leq C + \tilde{C}^2 \sup_{\frac{\phi}{C} \in L^2(0,T)} \sup_{\psi \in H^{1,2}(\Omega)} \|\phi\|_{H^{1,2}(\Omega)}^2 \left( \|\psi\|_{L^2(0,T)} \right)^2 .
\]

This implies

\[
\left\| \chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq C_{23}
\]

\[
\Rightarrow \sum_{i=1}^I \left\| \chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq I C_{23}^2
\]

\[
\Rightarrow \left\| \chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq C_{24},
\]

(4.1.81)

where $C_{24} := (I C_{23}^2)^{\frac{1}{2}}$ is independent of $\varepsilon$ but depends only on $r$. Adding (4.1.79), (4.1.80) and (4.1.81), we obtain

\[
\|u_\varepsilon\|_{L^\infty((0,T);L^2(\Omega))} + \|u_\varepsilon\|_{L^2((0,T);H^{1,2}(\Omega))} + \left\| \chi^\varepsilon \frac{\partial u_\varepsilon}{\partial t} \right\|_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq C_{21} + C_{22} + C_{24}
\]

\[
= C_{20},
\]

where $C_{20} := C_{21} + C_{22} + C_{24}$ is independent of $\varepsilon$ but depends only on $r$.

The next statement is very crucial. It gives the strong convergence of the subsequence of the sequence $(u_\varepsilon)_{\varepsilon > 0}$. This is the main result of Meirmanov & Zimin in [MZ11].
Lemma 4.1.2.2.2. Let \((c_\varepsilon)_{\varepsilon > 0}\) be a bounded sequence in \(L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);H^{1,2}(\Omega))\) and weakly convergent in \(L^2((0,T);L^2(\Omega)) \cap L^2((0,T);H^{1,2}(\Omega))\) to a function \(c\). Suppose further that the sequence \((\frac{\partial}{\partial t} \chi^\varepsilon c_\varepsilon)_{\varepsilon > 0}\) is bounded in \(L^2((0,T);H^{1,2}(\Omega))\). Then the sequence \((c_\varepsilon)_{\varepsilon > 0}\) is strongly convergent to the function \(c\) in \(L^2((0,T);L^2(\Omega))\).

Proof. See theorem 2.1 in Meirmanov & Zimin [MZ11].

Theorem 4.1.2.2.3. Let \((u_\varepsilon)_{\varepsilon > 0}\) satisfies the estimates (4.1.72) and (4.1.78). Then there exists a function \(u \in L^2((0,T);H^{1,2}(\Omega))\) and a function \(u^1 \in L^2((0,T) \times \Omega; H^{1,2}_\text{ped}(Y)/\mathbb{R})\) such that up to a subsequence, still denoted by the same subscript, the following convergence results hold:

\[
\begin{align*}
(i) & \quad (u_\varepsilon)_{\varepsilon > 0} \text{ is weakly convergent to } u \text{ in } L^2((0,T);H^{1,2}(\Omega))^I. \\
(ii) & \quad (u_\varepsilon)_{\varepsilon > 0} \text{ is strongly convergent to } u \text{ in } L^2((0,T);L^2(\Omega))^I. \\
(iii) & \quad (u_\varepsilon)_{\varepsilon > 0} \text{ and } (\nabla_x u_\varepsilon)_{\varepsilon > 0} \text{ are two-scale convergent to } u \text{ and } \nabla_x u + \nabla_y u^1 \text{ in the sense of (3.5.3) respectively.}
\end{align*}
\]

Proof. (i) From the estimate (4.1.78), we note that the sequence \((u_\varepsilon)_{\varepsilon > 0}\) is bounded in \(L^2((0,T);H^{1,2}(\Omega))^I\). This implies that, up to a subsequence, still indexed by the same subscript, \((u_\varepsilon)_{\varepsilon > 0}\) is weakly convergent to a function \(u\) in \(L^2((0,T);H^{1,2}(\Omega))^I\).

(ii) From (4.1.78), it follows that, up to a subsequence, still denoted by the same subscript, \((u_\varepsilon)_{\varepsilon > 0}\) is weakly convergent to \(u\) in \(L^2((0,T);L^2(\Omega))^I \cap L^2((0,T);H^{1,2}(\Omega))^I\) and is bounded in \(L^\infty((0,T);L^2(\Omega))^I \cap L^2((0,T);H^{1,2}(\Omega))^I\). Also from (4.1.78) note that \((\frac{\partial}{\partial t} \chi^\varepsilon u_\varepsilon)_{\varepsilon > 0}\) is bounded in \(L^2((0,T);H^{1,2}(\Omega))^I\). Therefore the subsequence \((u_\varepsilon)_{\varepsilon > 0}\), still denoted by the same subscript, is strongly convergent to \(u\) in \(L^2((0,T);L^2(\Omega))^I\).

(iii) The proof follows from the estimate (4.1.78) and theorem 3.5.13.

Theorem 4.1.2.2.4. The limit function \(u\) belongs to \(L^\infty((0,T) \times \Omega \times Y)^I\). \(^{20}\)

Proof. Since \((u_\varepsilon)_{\varepsilon > 0}\) is strongly convergent to \(u\) in \(L^2((0,T);L^2(\Omega))^I\), there exists a subsequence \((u_{\varepsilon'})_{\varepsilon' > 0}\) which is pointwise convergent\(^{21}\) to \(u\) almost everywhere in \((0,T) \times \Omega\), i.e.,

\[
\lim_{\varepsilon' \to 0} u_{\varepsilon'}(t,x) = u(t,x) \quad \text{a.e.} \quad (t,x) \in (0,T) \times \Omega.
\]

By theorem 4.1.2.1.1, we have \(\|u_{\varepsilon_i}\|_{L^\infty((0,T);L^\infty(\Omega))} \leq C_{16}\) for all \(i\), therefore

\[
|u_i(t,x)|^2 \leq |u(t,x)|^2 = \sum_{i=1}^I |u_i(t,x)|^2 = \lim_{\varepsilon' \to 0} \sum_{i=1}^I |u_{\varepsilon'}(t,x)|^2 \\
\leq \sum_{i=1}^I \limsup_{\varepsilon' \to 0} \esssup_{t \in (0,T)} \esssup_{x \in \Omega} |u_{\varepsilon'}(t,x)|^2 \\
\leq \sum_{i=1}^I \limsup_{\varepsilon' \to 0} C_{16}^2 \\
= C_{16}^2 \ I \text{ for a.e. } t \text{ and } x \Rightarrow \esssup_{t \in (0,T)} \esssup_{x \in \Omega} |u_i(t,x)|^2 \leq C_{16}^2 \ I < \infty \quad \text{for all } i.
\]

\(^{20}\)Note that the function \(u\) is independent of the variable \(y\).

\(^{21}\)Cf. corollary on page 53 in [Yos70].
This gives
\[ \| u \|_{L^\infty((0,T)\times \Omega)'}^2 = \max_{1 \leq i \leq I} \| u_i \|_{L^\infty((0,T)\times \Omega)}^2 = \max_{1 \leq i \leq I} \text{ess sup}_{(t,x,y) \in (0,T)\times \Omega} |u_i(t,x)|^2 \]
\[ \leq \max_{1 \leq i \leq I} \text{ess sup}_{t \in (0,T)} \text{ess sup}_{x \in \Omega} \text{ess sup}_{y \in \Omega} |u_i(t,x)|^2 \]
\[ \leq \text{ess sup}_{y \in \Omega} C_{16}^2 I, \]
i.e., \( \| u \|_{L^\infty((0,T)\times \Omega)'} \leq C_{25} \), where \( C_{25} := \left( C_{16}^2 I \right)^\frac{1}{2} \) is independent of \( \varepsilon \) but depends only on \( r \).

**Corollary 4.1.2.2.5.** For all \( 2 \leq p < \infty \), \( (u_\varepsilon)_\varepsilon > 0 \) is strongly convergent to \( u \) in \( L^p((0,T) \times \Omega)' \).

**Proof.** This follows from the straightforward application of Lyapunov’s interpolation inequality (cf. lemma A.6) and \( L^\infty \) - estimates of \( u_\varepsilon \) and \( u \). See lemma 3.2.20 in [Pet06] for details.

**Theorem 4.1.2.2.6.** The sequence \( (SR(u_\varepsilon))_\varepsilon > 0 \) is strongly convergent to \( SR(u) \) in \( L^2((0,T) \times \Omega)' \) as \( \varepsilon \to 0 \).

**Proof.** Note that
\[ \| SR(u_\varepsilon) - SR(u) \|_{L^2((0,T) \times \Omega)'}^2 = \sum_{i=1}^{I} \| SR(u_\varepsilon)_i - SR(u)_i \|_{L^2((0,T) \times \Omega)}^2 \quad (4.1.85) \]
From (2.4.7), we have
\[ SR(u_\varepsilon)_i = \sum_{j=1}^{J} \sum_{i=1}^{I} s_{ij} \left( k^f_j \prod_{m=1}^{I} u_{\varepsilon_m}^{-s_{mj}} - k^b_j \prod_{m=1}^{I} u_{m}^{-s_{mj}} \right) \quad (4.1.86) \]
and
\[ SR(u)_i = \sum_{j=1}^{J} \sum_{i=1}^{I} s_{ij} \left( k^f_j \prod_{m=1}^{I} u_{m}^{-s_{mj}} - k^b_j \prod_{m=1}^{I} u_{m}^{-s_{mj}} \right) \quad (4.1.87) \]
From (4.1.86) and (4.1.87),
\[ \| SR(u_\varepsilon)_i - SR(u)_i \|_{L^2((0,T) \times \Omega)} = \sum_{j=1}^{J} \sum_{i=1}^{I} s_{ij} \left( k^f_j \prod_{m=1}^{I} u_{\varepsilon_m}^{-s_{mj}} - k^b_j \prod_{m=1}^{I} u_{m}^{-s_{mj}} \right) \]
\[ - \sum_{j=1}^{J} \sum_{i=1}^{I} s_{ij} \left( k^b_j \prod_{m=1}^{I} u_{\varepsilon_m}^{-s_{mj}} - k^f_j \prod_{m=1}^{I} u_{m}^{-s_{mj}} \right) \|_{L^2((0,T) \times \Omega)} \]
such that $\phi_i$, i.e.,

It can be easily shown that the terms $\left\| \prod_{m=1}^{I} u_{\varepsilon m}^{s_{m_j}} - \prod_{m=1}^{I} u_{m}^{s_{m_j}} \right\|_{L^2((0,T) \times \Omega)}$ and $\left\| \prod_{m=1}^{I} u_{\varepsilon m}^{s_{m_j}} - \prod_{m=1}^{I} u_{m}^{s_{m_j}} \right\|_{L^2((0,T) \times \Omega)}$ are strongly convergent to 0 as $\varepsilon \to 0$. Therefore $\|SR(u_{\varepsilon i}) - SR(u_i)\|_{L^2((0,T) \times \Omega)} \to 0$ as $\varepsilon \to 0$. From (4.1.85), the theorem follows.

**Remark 4.1.2.2.7.** The strong convergence of $(SR(u_{\varepsilon}))_{\varepsilon > 0}$ implies that it is two-scale convergent to $SR(u)$ in the sense of (3.5.3).

### 4.1.2.3 Passage to the Limit as $\varepsilon \to 0$

Let us consider the functions $\phi_0 \in C_0^\infty((0,T) \times \Omega)^l$ and $\phi_1 \in C_0^\infty((0,T) \times \Omega); C_{per}(Y))^l$ such that $\phi(t,x,\frac{x}{\varepsilon}) := \phi_0(t,x) + \varepsilon \phi_1(t,x,\frac{x}{\varepsilon}) \in C_0^\infty((0,T) \times \Omega); C_{per}(Y))^l$. Using $\phi$ as test function in the weak formulation of (2.5.16)-(2.5.19) one obtains

\[
\int_0^T \left\langle \frac{\partial u_{\varepsilon}(t)}{\partial t}, \phi(t) \right\rangle_{[H^{1,2}(\Omega^\varepsilon)]^l} dt - \int_0^T \left\langle \nabla \cdot D\nabla u_{\varepsilon}(t), \phi(t) \right\rangle_{[H^{1,2}(\Omega^\varepsilon)]^l} dt = \int_0^T \left\langle SR(u_{\varepsilon}(t)), \phi(t) \right\rangle_{[H^{1,2}(\Omega^\varepsilon)]^l} dt,
\]

i.e.,

\[
\sum_{i=1}^{I} \int_0^T \left\langle \frac{\partial u_{\varepsilon}(t)}{\partial t}, \phi_i(t) \right\rangle_{H^{1,2}(\Omega^\varepsilon)} dt - \sum_{i=1}^{I} \int_0^T \left\langle \nabla \cdot D\nabla u_{\varepsilon}(t), \phi_i(t) \right\rangle_{H^{1,2}(\Omega^\varepsilon)} dt = \sum_{i=1}^{I} \int_0^T \left\langle SR(u_{\varepsilon}(t)), \phi_i(t) \right\rangle_{H^{1,2}(\Omega^\varepsilon)} dt,
\]

i.e.,

\[
\sum_{i=1}^{I} \int_0^T \left\langle \frac{\partial u_{\varepsilon}(t)}{\partial t}, \phi_i(t) \right\rangle_{H^{1,2}(\Omega^\varepsilon)} dt + \sum_{i=1}^{I} \int_0^T \nabla \cdot D\nabla u_{\varepsilon}(t,x) \nabla \phi_i(t,x,\frac{x}{\varepsilon}) dx dt = \sum_{i=1}^{I} \int_0^T \left\langle SR(u_{\varepsilon}(t)), \phi_i(t) \right\rangle_{H^{1,2}(\Omega^\varepsilon)} dt.
\]

Now we pass the two-scale limit in (4.1.89) term by term.

\[
\lim_{\varepsilon \to 0} \sum_{i=1}^{I} \int_0^T \left\langle \frac{\partial u_{\varepsilon}(t)}{\partial t}, \phi_i(t) \right\rangle_{H^{1,2}(\Omega^\varepsilon)} dt = \lim_{\varepsilon \to 0} \sum_{i=1}^{I} \int_0^T u_{\varepsilon}(t,x) \left( \frac{\partial \phi_0(t,x)}{\partial t} + \varepsilon \frac{\partial \phi_1(t,x,\frac{x}{\varepsilon})}{\partial t} \right) dx dt
\]

Since this is just a mere calculation, we intended not to include here.
Again,

\[ \lim_{\varepsilon \to 0} \sum_{i=1}^{I} \int_{0}^{T} \int_{\Omega_{\varepsilon}} D\nabla_{x} u_{\varepsilon_{i}}(t,x) \nabla_{x} \phi_{0}(t,x, \frac{x}{\varepsilon}) \, dx \, dt = 0 \]

\[ \lim_{\varepsilon \to 0} \sum_{i=1}^{I} \int_{0}^{T} \int_{\Omega_{\varepsilon}} D\nabla_{x} u_{\varepsilon_{i}}(t,x) \nabla_{x} \left( \phi_{0}(t,x) + \varepsilon \phi_{1}(t,x, \frac{x}{\varepsilon}) \right) \, dx \, dt \]

\[ \lim_{\varepsilon \to 0} \left[ \sum_{i=1}^{I} \int_{0}^{T} \int_{\Omega_{\varepsilon}} \chi(\frac{x}{\varepsilon}) D\nabla_{x} u_{\varepsilon_{i}}(t,x) \left( \nabla_{x} \phi_{0}(t,x) + \nabla_{y} \phi_{1}(t,x, \frac{x}{\varepsilon}) \right) \, dx \, dt 
+ \varepsilon \sum_{i=1}^{I} \int_{0}^{T} \int_{\Omega_{\varepsilon}} \chi(\frac{x}{\varepsilon}) D\nabla_{x} u_{\varepsilon_{i}}(t,x) \nabla_{x} \phi_{1}(t,x, \frac{x}{\varepsilon}) \, dx \, dt \right] \]

\[ \lim_{\varepsilon \to 0} \sum_{i=1}^{I} \int_{0}^{T} \int_{\Omega_{\varepsilon}} \chi(\frac{x}{\varepsilon}) D\nabla_{x} u_{\varepsilon_{i}}(t,x) \left( \nabla_{x} \phi_{0}(t,x) + \nabla_{y} \phi_{1}(t,x, \frac{x}{\varepsilon}) \right) \, dx \, dt 
+ \lim_{\varepsilon \to 0} \varepsilon \sum_{i=1}^{I} \int_{0}^{T} \int_{\Omega_{\varepsilon}} \chi(\frac{x}{\varepsilon}) D\nabla_{x} u_{\varepsilon_{i}}(t,x) \nabla_{x} \phi_{1}(t,x, \frac{x}{\varepsilon}) \, dx \, dt = 0 \]

\[ \sum_{i=1}^{I} \int_{0}^{T} \int_{Y} \chi(y) D \left( \nabla_{x} u_{i}(t,x) + \nabla_{y} u_{i_{1}}(t,x,y) \right) \left( \nabla_{x} \phi_{0}(t,x) + \nabla_{y} \phi_{1}(t,x,y) \right) \, dx \, dy \, dt \]

\[ \sum_{i=1}^{I} \int_{0}^{T} \int_{Y_{p}} D \left( \nabla_{x} u_{i}(t,x) + \nabla_{y} u_{i_{1}}(t,x,y) \right) \left( \nabla_{x} \phi_{0}(t,x) + \nabla_{y} \phi_{1}(t,x,y) \right) \, dx \, dy \, dt. \]

\[ \text{Finally,} \]

\[ \lim_{\varepsilon \to 0} \sum_{i=1}^{I} \int_{0}^{T} \left\langle SR(u_{\varepsilon}(t)), \phi_{1}(t) \right\rangle_{H^{1,2}(\Omega_{\varepsilon})^{*} \times H^{1,2}(\Omega_{\varepsilon})} \, dt \]
Since L 56
Chapter 4. Existence of a Unique Global Solution and Homogenization

Let us choose is any arbitrary function of

\[(4.1.94)\] is satisfied if

\[\text{Combining (4.1.90), (4.1.91) and (4.1.92), we get}
\]

\[\left\langle Y^p \right\rangle \int_0^T \left\langle \frac{\partial u(t)}{\partial t}, \phi_0(t) \right\rangle \left[ H^{1,2}(\Omega)^* \right] \times \left[ H^{1,2}(\Omega)^* \right] dt
\]

\[\text{Now choosing } \phi_0(t, x) \equiv 0, \text{ i.e., } \phi_0(t, x) \equiv 0 \text{ for all } i = 1, 2, ..., I, \text{ then } \phi(t, x) = \phi_1(t, x, \frac{x}{\varepsilon}) \text{ and the equation (4.1.93) reduces to}
\]

\[\sum_{i=1}^I \int_0^T \int_{Y^p} D(\nabla_x u_i(t, x) + \nabla_y u_1(t, x, y)) \nabla_y \phi_1(t, x, y) dxdy dt = 0. \tag{4.1.94}\]

Let us choose

\[u_1(t, x, y) = \sum_{j=1}^n \frac{\partial u_i(t, x)}{\partial x_j} a_j(t, x, y) + c_i(t, x), \text{ for all } i = 1, 2, ..., I, \text{ where } c(x) \text{ is any arbitrary function of } x. \text{ The equation (4.1.94) is satisfied by each } u_1, \text{ if } a_j, \text{ for } j = 1, 2, ..., n, \text{ is the solution of the Cell-Problem}
\]

\[-\nabla_y \cdot (D(\nabla_y a_j(t, x, y) + e_j)) = 0 \text{ for } (t, x, y) \in (0, T) \times \Omega \times Y^p, \tag{4.1.95}\]

\[-D(\nabla_y a_j(t, x, y) + e_j) \cdot \vec{n} = 0 \text{ for } (t, x, y) \in (0, T) \times \Omega \times \Gamma, \tag{4.1.96}\]

\[y \mapsto a_j(y) \text{ is } Y \text{ - periodic.} \tag{4.1.97}\]

On the other hand, if \(a_j\) is the solution of the cell-problem (4.1.95)-(4.1.97), the equation (4.1.94) is satisfied if

\[u_1(t, x, y) = \sum_{j=1}^n \frac{\partial u_i(t, x)}{\partial x_j} a_j(t, x, y) + c_i(t, x). \text{ Setting } \phi_1(t, x, \frac{x}{\varepsilon}) \equiv 0, \text{ i.e.,}
\]

\[\text{By (4.1.60), } \sup_{\varepsilon > 0} \|u_1(t)\|_{L^\infty(\Omega^\varepsilon_t)} \leq C_{16} \forall t \text{ and for a.e. } t. \text{ This gives } \sup_{\varepsilon > 0} \|SR(u_\varepsilon)\|_{L^2(\Omega^\varepsilon_t)} \leq C. \text{ Since } L^2(\Omega^\varepsilon_t) \hookrightarrow H^{1,2}(\Omega^\varepsilon)^*, \text{ from (3.1.3) } \langle SR(u_\varepsilon), \phi_i \rangle_{H^{1,2}(\Omega^\varepsilon)^*} \times H^{1,2}(\Omega^\varepsilon) = (SR(u_\varepsilon), \phi_i)_{L^2(\Omega^\varepsilon_t) \times L^2(\Omega^\varepsilon_t)}, \phi_i \in H^{1,2}(\Omega^\varepsilon_t).
\[ \phi_i(t, x, \frac{\bar{z}}{\text{t}}) = 0 \text{ for all } i. \] Then the equation (4.1.93) reduces to

\[
|Y^p| \int_0^T \left. \left\langle \frac{\partial u(t)}{\partial t}, \phi_0(t) \right\rangle \right|_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt + \sum_{i=1}^l \int_0^T \int_{\Omega} \int_{Y^p} D \left( \nabla_x u_i(t, x) + \nabla_y u_1(t, x, y) \right) \nabla_x \phi_0(t, x) dx dy dt = |Y^p| \int_0^T \left\langle \langle SR(u(t)), \phi_0(t) \rangle \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt,
\]

i.e.,

\[
\sum_{i=1}^l \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_0(t) \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt + \sum_{i=1}^l \int_0^T \int_{\Omega} \int_{Y^p} \frac{D}{|Y^p|} \left( \nabla_x u_i(t, x) + \nabla_y u_1(t, x, y) \right) \nabla_x \phi_0(t, x) dx dy dt = \sum_{i=1}^l \int_0^T \left\langle \langle SR(u(t)), \phi_0(t) \rangle \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt.
\]

Substituting \( u_1(t, x, y) = \bar{a}(t, x, y) \cdot \nabla_x u_i(t, x) + c(x) \), i.e., \( \nabla_y u_1 = \sum_{j=1}^n \nabla_y a_{ij} \frac{\partial u_i}{\partial x_j} \) in (4.1.98), then we obtain

\[
\sum_{i=1}^l \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_0(t) \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt + \sum_{i=1}^l \int_0^T \int_{\Omega} \int_{Y^p} \frac{D}{|Y^p|} \left( \nabla_x u_i(t, x) + \sum_{j=1}^n \nabla_y a_{ij} \frac{\partial u_i(t, x, y)}{\partial x_j} \right) \nabla_x \phi_0(t, x) dx dy dt = \sum_{i=1}^l \int_0^T \left\langle \langle SR(u(t)), \phi_0(t) \rangle \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt,
\]

i.e.,

\[
\sum_{i=1}^l \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_0(t) \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt + \sum_{i=1}^l \int_0^T \int_{\Omega} \sum_{j,k=1}^n \left\langle \frac{D}{|Y^p|} \int_{Y^p} \left( \delta_{jk} + \frac{\partial a_{ij}}{\partial y_k} \right) dy \right\rangle \frac{\partial u_i(t, x)}{\partial x_j} \frac{\partial \phi_0(t, x)}{\partial x_k} dx dt = \sum_{i=1}^l \int_0^T \left\langle \langle SR(u(t)), \phi_0(t) \rangle \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt,
\]

i.e.,

\[
\sum_{i=1}^l \int_0^T \left\langle \frac{\partial u_i(t)}{\partial t}, \phi_0(t) \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt + \sum_{i=1}^l \int_0^T \int_{\Omega} P \nabla_x u_i(t, x) \nabla \phi_0(t, x) dx dt = \sum_{i=1}^l \int_0^T \left\langle \langle SR(u(t)), \phi_0(t) \rangle \right\rangle_{H^{1.2}(\Omega)^* \times H^{1.2}(\Omega)} dt.
\]
where \( P \) is a second order tensor whose components are given as

\[
p_{jk} = \int_{y \in Y_P} \frac{D}{|Y_P|} \left( \delta_{jk} + \frac{\partial u_j}{\partial y_k} \right) dy \text{ for all } j, k = 1, 2, ..., n. \tag{4.1.100}
\]

Similarly the boundary condition simplifies to

\[
P \nabla u \cdot \vec{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega. \tag{4.1.101}
\]

Therefore the strong form of the complete homogenized problem is

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot P \nabla u &= SR(u) \quad \text{in} \quad (0, T) \times \Omega, \tag{4.1.102} \\
-P \nabla u \cdot \vec{n} &= 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \tag{4.1.103} \\
u(0, x) &= u_0(x) \quad \text{in} \quad \Omega. \tag{4.1.104}
\end{align*}
\]

Let us denote this problem by \((P^1)\).

**Proposition 4.1.2.3.1.** The tensor \( P = (p_{jk}) \) is a second order positive definite symmetric tensor.

**Proof.** This follows from the definition of \( P \). For details see in [HJ91] or lemma 5.2 in [Pet03].

**Theorem 4.1.2.3.2.** There exists a unique solution \( u \in F^u_P \cap L^\infty((0, T); L^\infty(\Omega))^I \) of the homogenized problem \((4.1.102)-(4.1.104)\).

**Proof.** From \((4.1.72)\) and \((4.1.78)\), it follows that the two-scale limit \( u \in [H^{1,2}((0, T); H^{1,2}(\Omega)) \cap L^2((0, T); H^{1,2}(\Omega))] \cap L^\infty((0, T) \times \Omega)) \). We still have two things to prove:

- Uniqueness of the solution of \((P^1)\)
- \( u \in F^u_P \)

We start by proving the uniqueness of the solution. Let \( u_1 \) and \( u_2 \) be the solutions of \((4.1.102)-(4.1.104)\) such that \( u_1 \neq u_2 \) and \( u_1(0, x) = u_2(0, x) \). Proceeding in a similar fashion as in the section 4.1.1.4 we obtain

\[
\int_0^t \left\langle \frac{\partial}{\partial \theta} (u_1(\theta) - u_2(\theta)), (u_1(\theta) - u_2(\theta)) \right\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} d\theta \\
+ \int_0^t \int_{\Omega} \nabla (u_1(\theta, x) - u_2(\theta, x)) P \nabla (u_1(\theta, x) - u_2(\theta, x)) dxd\theta \\
= \int_0^t \left\langle SR(u_1(\theta)), (u_1(\theta) - u_2(\theta)) \right\rangle_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} d\theta,
\]

i.e.,

\[
\frac{1}{2} \int_0^t \int_{\Omega} \|u_1(\theta) - u_2(\theta)\|_{L^2(\Omega)}^2 d\theta + C \int_0^t \int_{\Omega} \|\nabla (u_1(\theta, x) - u_2(\theta, x))\|^2 dxd\theta \\
\leq \int_0^t \int_{\Omega} |SR(u_1(\theta, x)) - SR(u_2(\theta, x))|\|u_1(\theta, x) - u_2(\theta, x)\| dxd\theta
\]

Arguing as in section 4.1.1.4, we get

\[
||u(t) - u_2(t)||_{L^2(\Omega)}^2 \leq C \int_0^t ||u_1(\theta) - u_2(\theta)||_{L^2(\Omega)}^2 d\theta.
\]

\[24\]Here we have used the positive definiteness of the tensor \( P \).
Gronwall’s inequality yields
\[ \|(u_1(t) - u_2(t))\|_{L^2(\Omega)}^2 = 0 \text{ for a.e. } t \]
\[ \implies u_1 = u_2, \]

i.e., the solution of (4.1.102)-(4.1.104) is unique.

The abstract formulation of the problem (4.1.102)-(4.1.104) is given by
\[ \frac{\partial u(t)}{\partial t} + Au(t) = f(t), \quad t \in [0, T] \]
\[ u(0, x) = u_0(x), \]
where \( f(t) = SR(u(t)) + \kappa u(t), \kappa > 0 \), and the operator \( A : H^{1,p}(\Omega) \rightarrow [H^{1,q}(\Omega)^*] \) is defined as in remark 4.1.1.1 which has maximal parabolic regularity on \( u \) in \( H^{1,q}(\Omega) \). The embedding \( L^p(\Omega) \hookrightarrow H^{1,q}(\Omega)^* \) implies \( SR(u) + \kappa u \in L^p((0, T); L^p(\Omega))^\ast \). The embedding \( \mathcal{F}_p^u(H^{1,q}(\Omega)^*; H^{1,q}(\Omega)^*) \) is in \( \mathcal{F}_p^u((0, T); H^{1,q}(\Omega)^*) \). Therefore by theorem 3.3.1, there exists a unique solution \( u \) in \( \mathcal{F}_p^u \) of the problem (4.1.105)-(4.1.106) such that
\[ |||u|||_{\mathcal{F}_p^u} \leq \tilde{C} \left( |||u_0|||_{[H^{1,q}(\Omega)^*; H^{1,q}(\Omega)]} + |||f|||_{L^p((0, T); H^{1,q}(\Omega)^*)} \right), \]
where \( \tilde{C} > 0 \) depends only on \( p \) but is independent of \( u, u_0 \) and \( f \). In other words, the problem \((P^1)\) has a unique positive global weak solution \( u \) in \( \mathcal{F}_p^u \).

### 4.2 Model M2

#### 4.2.1 Existence and Uniqueness of the Global Solution of \((P^2_{\varepsilon})\)

Suppose that the following assumptions hold:

\[ (i) \quad p > n + 2, \quad (4.2.1) \]
\[ (ii) \quad u_0, v_0 \text{ and } w_0 \geq 0, \text{ i.e., } u_{0i}, v_{0i} \text{ and } w_{0i} \geq 0 \text{ for all } i = 1, 2, \ldots, I_1, \quad k = 1, 2, \ldots, I_2, \quad (4.2.2) \]
\[ (iii) \quad u_{0i}, v_{0i} \in (H^{1,q}(\Omega_\varepsilon^p)^*; H^{1,q}(\Omega_\varepsilon^p))_{1-p}, \text{ for all } i = 1, 2, \ldots, I_1, \quad k = 1, 2, \ldots, I_2. \quad (4.2.3) \]
\[ (iv) \quad \text{All the reactions are linearly independent such that the stoichiometric matrices}_ \]
\[ \quad \text{rank}(S_1) = J \text{ and rank}(S_2) = J. \quad (4.2.4) \]
\[ (v) \quad \sup_{\varepsilon > 0} |||u_0|||_{(H^{1,q}(\Omega_\varepsilon^p)^*; H^{1,q}(\Omega_\varepsilon^p))_{1-p}} < \infty \text{ for all } i = 1, 2, \ldots, I_1, \quad (4.2.5) \]
\[ (vi) \quad \tilde{q}_\varepsilon \text{ is the given fluid velocity which satisfies}_ \]
\[ \quad \nabla \cdot \tilde{q}_\varepsilon = 0 \text{ in } \Omega_\varepsilon^p, \quad -\tilde{q}_\varepsilon \cdot \tilde{n} > 0 \text{ on } \partial \Omega_{\text{in}}, \quad -\tilde{q}_\varepsilon \cdot \tilde{n} \leq 0 \text{ on } \partial \Omega_{\text{out}} \text{ and } \tilde{q}_\varepsilon = 0 \text{ on } \Gamma_\varepsilon. \quad (4.2.6) \]
\[ (vii) \quad \tilde{q}_\varepsilon \in L^\infty((0, T) \times \Omega_\varepsilon^p) \text{ such that } Q := \sup_{\varepsilon > 0} |||\tilde{q}_\varepsilon|||_{L^\infty((0, T) \times \Omega_\varepsilon^p)} < \infty \text{ and}_ \]
\[ \quad \tilde{q}_\varepsilon \cdot \tilde{n} \in L^\infty((0, T) \times \partial \Omega_{\text{in}}). \quad (4.2.7) \]
\[ (viii) \quad d_i \leq 0 \text{ and } d_i \in L^\infty((0, T) \times \partial \Omega_{\text{in}}) \text{ for all } i = 1, 2, \ldots, I_1. \quad (4.2.8) \]
We call this regularized function as $\psi$ and

\[ (\text{4.2.10}) \]

Remark 4.2.1.1. The suffix $\delta$ above is a regularization parameter (see section 4.2.1.1). $\mu^0$ and $\bar{\mu}$ are defined in (4.2.53) and (4.2.113) respectively. The assumptions (4.2.9) and (4.2.10) are very strong. The proofs of the inequalities (4.2.9) and (4.2.10) are still open, however, we believe that (4.2.9) and (4.2.10) can be proven.

The assumption (vii) implies

\[
\int_0^T \int_{\Omega^\varepsilon} |\bar{q}_\varepsilon|^p \, dx \, dt \leq \text{ess sup}_{(0,T) \times \Omega^\varepsilon} |\bar{q}_\varepsilon|^p \int_0^T \int_{\Omega} dx \, dt \leq \sup_{\varepsilon > 0} \|\bar{q}\|_{L^\infty((0,T) \times \Omega^\varepsilon)}^p \|\bar{q}_\varepsilon\|_{L^\infty((0,T) \times \Omega^\varepsilon)} \leq \infty.
\]

Let $\bar{q}_\varepsilon$ be the extension of $q_\varepsilon$ defined as follows:

\[
\bar{q}_\varepsilon := \begin{cases} 
q_\varepsilon & \text{in } (0,T) \times \Omega^\varepsilon \\
0 & \text{in } (0,T) \times \Omega^s.
\end{cases}
\]

For the sake of brevity, we still denote the extension of $q_\varepsilon$ by $\bar{q}_\varepsilon$. We see that the extended velocity is bounded in $L^p((0,T);L^p(\Omega))$, hence in $L^2((0,T);L^2(\Omega))$. Therefore $\bar{q}_\varepsilon$ is two-scale convergent to the limit $\bar{q}_1$ in $L^2((0,T);L^2(\Omega \times Y))$ and weakly convergent to $\bar{q} = \int Y \bar{q}_1 \, dy$ in $L^2((0,T);L^2(\Omega))$.

### 4.2.1.1 Regularization of the Function $\psi(w_{\varepsilon m})$

We can notice that there is a discontinuity in the ODE (2.5.32). Therefore in order to prove the existence of the global weak solution of the problem (2.5.21)-(2.5.35), we introduce a regularization of the function $\psi(w_{\varepsilon m})$. Let us choose $0 < \delta < 1$ such that $\varepsilon < \delta^p < \delta^2 < 1$. We call this regularized function as $\psi_\delta(w_{\varepsilon s m})$ and is defined by

\[
\psi_\delta(w_{\varepsilon s m}) = \begin{cases} 
0 & \text{if } w_{\varepsilon s m} \leq 0, \\
\frac{w_{\varepsilon s m}}{\delta} & \text{if } 0 < w_{\varepsilon s m} < \delta, \\
1 & \text{if } w_{\varepsilon s m} \geq \delta.
\end{cases}
\]

Our regularized problem is given as:

\[
\begin{align*}
\frac{\partial w_{\varepsilon s}}{\partial t} - \nabla \cdot (D \nabla w_{\varepsilon s} - \bar{q}_\varepsilon w_{\varepsilon s}) &= S_1 R(u_{\varepsilon s}, v_{\varepsilon s}) & \text{in } (0,T) \times \Omega^p, \\
-(D \nabla w_{\varepsilon s} - \bar{q}_\varepsilon w_{\varepsilon s}) \cdot \bar{n} &= d & \text{on } (0,T) \times \partial \Omega_m, \\
-D \nabla w_{\varepsilon s} \cdot \bar{n} &= 0 & \text{on } (0,T) \times \partial \Omega_{out}, \\
-D \nabla w_{\varepsilon s} \cdot \bar{n} &= 0 & \text{on } (0,T) \times \Gamma_z, \\
w_{\varepsilon s}(0,x) &= u_0(x) & \text{in } \Omega^p.
\end{align*}
\]

The function $\psi_\delta(w_{\varepsilon s m})$ is Lipschitz and monotonically increasing on $[0,\delta]$. We sometimes also use the notation $\psi_\delta(w_{\varepsilon s m}) = \psi_\delta(w_{\varepsilon s m})$. 

---

\(25\) The function $\psi_\delta(w_{\varepsilon s m})$ is Lipschitz and monotonically increasing on $[0,\delta]$. We sometimes also use the notation $\psi_\delta(w_{\varepsilon s m}) = \psi_\delta(w_{\varepsilon s m})$. 

---
are satisfied, then there exists a unique positive global weak solution

\( \frac{\partial w_{\varepsilon}}{\partial t} - \Delta (Dv_{\varepsilon} - \bar{q}v_{\varepsilon}) = R(u_{\varepsilon}, v_{\varepsilon}) \) in \( (0, T) \times \Omega_{\varepsilon}^p \),

\( -(Dv_{\varepsilon} - \bar{q}v_{\varepsilon}) \cdot \tilde{n} = d \) on \( (0, T) \times \partial \Omega_{in} \),

\( -Dv_{\varepsilon} \cdot \tilde{n} = 0 \) on \( (0, T) \times \partial \Omega_{out} \),

\( -Dv_{\varepsilon} \cdot \tilde{n} = \varepsilon \frac{\partial w_{\varepsilon}}{\partial t} \) on \( (0, T) \times \Gamma_{\varepsilon} \),

\( w_{\varepsilon}(0, x) = w_0(x) \) on \( \Gamma_{\varepsilon} \).

Let us call this problem by \( (P^2_{\varepsilon}) \).

**Theorem 4.2.1.1 (Existence theorem).** Suppose that the assumptions (4.2.1)-(4.2.10) are satisfied, then there exists a unique positive global weak solution \( (u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon}) \in F_{\varepsilon}^v \times G_{\varepsilon}^v \times H_{\varepsilon}^w \) of the problem \( (P^2_{\varepsilon}) \).

In case of problem \( (P^2_{\varepsilon}) \) too\(^{26}\), we first solve a modified problem. We introduce the rate function \( R : \mathbb{R}^t \rightarrow \mathbb{R}^t \) as

\( R(u_{\varepsilon}, v_{\varepsilon}) := R(u_{\varepsilon}^+, v_{\varepsilon}^+) \),

where \( u_{\varepsilon}^+ \) and \( v_{\varepsilon}^+ \) are the positive parts of \( u_{\varepsilon} \) and \( v_{\varepsilon} \) respectively, defined componentwise as (4.1.7). Then the problem \( (P^2_{\varepsilon}) \) reduces to:

(i) Equations for type I species:

\( \frac{\partial u_{\varepsilon}}{\partial t} - \Delta (Dv_{\varepsilon} - \bar{q}u_{\varepsilon}) = S_1 R(u_{\varepsilon}, v_{\varepsilon}) \) in \( (0, T) \times \Omega_{\varepsilon}^p \),

\( -(Dv_{\varepsilon} - \bar{q}u_{\varepsilon}) \cdot \tilde{n} = d \) on \( (0, T) \times \partial \Omega_{in} \),

\( -Dv_{\varepsilon} \cdot \tilde{n} = 0 \) on \( (0, T) \times \partial \Omega_{out} \),

\( u_{\varepsilon}(0, x) = u_0(x) \) in \( \Omega_{\varepsilon}^p \),

where \( \tilde{d}_i \leq 0 \) for all \( 1 \leq i \leq I_1 \).

(ii) Equations for type II species:

\( \frac{\partial v_{\varepsilon}}{\partial t} - \Delta (Dv_{\varepsilon} - \bar{q}v_{\varepsilon}) = S_2 R(u_{\varepsilon}, v_{\varepsilon}) \) in \( (0, T) \times \Omega_{\varepsilon}^p \),

\( -(Dv_{\varepsilon} - \bar{q}v_{\varepsilon}) \cdot \tilde{n} = 0 \) on \( (0, T) \times \partial \Omega_{in} \),

\( -Dv_{\varepsilon} \cdot \tilde{n} = 0 \) on \( (0, T) \times \partial \Omega_{out} \),

\( -Dv_{\varepsilon} \cdot \tilde{n} = \varepsilon \frac{\partial w_{\varepsilon}}{\partial t} \) on \( (0, T) \times \Gamma_{\varepsilon} \),

\( v_{\varepsilon}(0, x) = v_0(x) \) in \( \Omega_{\varepsilon}^p \).

(iii) Equations for immobile species:

\( \frac{\partial w_{\varepsilon}}{\partial t} = -k_d \psi_3(w_{\varepsilon}) \) on \( (0, T) \times \Gamma_{\varepsilon} \),

\( w_{\varepsilon}(0, x) = w_0(x) \) on \( \Gamma_{\varepsilon} \).

\(^{26}\) We adopted the idea of [Krä08], [vDP04] and [CHK07].
Let us denote the problem (4.2.25)-(4.2.37) by \((P_{\varepsilon}^{2+})\). We will prove the existence of the global solution of the problem \((P_{\varepsilon}^{2+})\). Since we show that the solution of \((P_{\varepsilon}^{2+})\) is non-negative, it solves the problem \((P_{\varepsilon}^{2})\). We conclude this section by showing the uniqueness of the solution of \((P_{\varepsilon}^{2})\). We commence our investigation with the proof of the positivity of the solution of \((P_{\varepsilon}^{2})\).

**Lemma 4.2.1.1.2.** Let (4.2.1)-(4.2.10) be satisfied. Assume that \((u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon}) \in F_u \times G_v \times H_w^0\) is a solution of the problem \((P_{\varepsilon}^{2})\). Then \(u_{\varepsilon}, v_{\varepsilon}, w_{\varepsilon} \geq 0\) componentwise, i.e., \(u_{\varepsilon i}, v_{\varepsilon k}\) and \(w_{\varepsilon m} \geq 0\) for all \(i = 1, 2, ..., I_1, k = 1, 2, ..., I_2\) and \(m = 1, 2, ..., I_2\) in \((0, T) \times \Omega^\varepsilon\).

**Proof.** (a) **Positivity of type I species:** Let \(1 \leq i \leq I_1\). Since \(u_{\varepsilon i}(t) \in H^{1,p}(\Omega^\varepsilon)\) for a.e. \(0 < t < T\), we have \(u_{\varepsilon i}^-(t) \in H^{1,p}(\Omega^\varepsilon)\). Testing the \(i\)-th PDE of (4.2.25) by \(-u_{\varepsilon i}^-(t)\), we obtain\(^{27}\)

\[
\frac{1}{2} \frac{d}{dt} \left\| u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 + D \left\| \nabla u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 + \int_{\Omega^\varepsilon} q_\varepsilon \cdot \nabla u_{\varepsilon i}^-(t) u_{\varepsilon i}^-(t) dx + \int_{\partial \Omega^\varepsilon} \left(-d_1 u_{\varepsilon i}^- - q_\varepsilon \cdot \vec{n} \right) u_{\varepsilon i}^-(t) ds = -\int_{\Omega^\varepsilon} (S_1 R(u_{\varepsilon}(t), v_{\varepsilon}(t)))_{i} u_{\varepsilon i}^-(t) dx,
\]

i.e.,

\[
\frac{1}{2} \frac{d}{dt} \left\| u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 + D \left\| \nabla u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 + \int_{\partial \Omega^\varepsilon} \left(-d_1 u_{\varepsilon i}^- - q_\varepsilon \cdot \vec{n} \right) u_{\varepsilon i}^-(t) ds 
\geq \int_{\Omega^\varepsilon} \left| q_\varepsilon \right| \left\| \nabla u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 dx + \int_{\Omega^\varepsilon} \left(-S_1 R(u_{\varepsilon i}^+(t), v_{\varepsilon i}^+(t)) \right) u_{\varepsilon i}^-(t) dx, \tag{4.2.38}
\]

i.e.,

\[
\frac{1}{2} \frac{d}{dt} \left\| u_{\varepsilon i}^+(t) \right\|_{L^2(\Omega^\varepsilon)}^2 + D \left\| \nabla u_{\varepsilon i}^+(t) \right\|_{L^2(\Omega^\varepsilon)}^2 
\leq \frac{Q_2^2}{2D} \int_{\Omega^\varepsilon} \left| u_{\varepsilon i}^-(t) \right|^2 dx + \frac{D}{2} \int_{\Omega^\varepsilon} \left\| \nabla u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 dx - \int_{\Omega^\varepsilon} S_1 R(u_{\varepsilon i}^+(t), v_{\varepsilon i}^+(t))_{i} u_{\varepsilon i}^+(t) dx,
\]

i.e.,

\[
\frac{1}{2} \frac{d}{dt} \left\| u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 + D \left\| \nabla u_{\varepsilon i}^-(t) \right\|_{L^2(\Omega^\varepsilon)}^2 
\leq \frac{Q_2^2}{2D} \int_{\Omega^\varepsilon} \left| u_{\varepsilon i}^-(t) \right|^2 dx - \int_{\Omega^\varepsilon} S_1 R(u_{\varepsilon i}^+(t), v_{\varepsilon i}^+(t))_{i} u_{\varepsilon i}^+(t) dx. \tag{4.2.39}
\]

\(^{27}\)Here we have used the boundary conditions (4.2.26)-(4.2.28) for type I species. Since \(p > n + 2\), from theorem 3.4.3.4 it follows that \(u_{\varepsilon i} \in F_u\) and \(v_{\varepsilon i} \in F_v\) which implies \(u_{\varepsilon i} \in L^\infty((0, T) \times \Omega^\varepsilon)^I\) and \(v_{\varepsilon i} \in L^\infty((0, T) \times \Omega^\varepsilon)^F\) respectively. This gives \(S_1 R(u_{\varepsilon i}, v_{\varepsilon i}) \in L^p((0, T); L^p(\Omega^\varepsilon)^F)\). But \(L^p(\Omega^\varepsilon)^F \hookrightarrow H^{1,p}(\Omega^\varepsilon)^*\). Thus by the definition (3.1.3) we have

\[
\langle S_1 R(u_{\varepsilon i}, v_{\varepsilon i})_{i}, u_{\varepsilon i}^-(t) \rangle_{H^{1,q}(\Omega^\varepsilon)^* \times L^q(\Omega^\varepsilon)^\prime} = \langle S_1 R(u_{\varepsilon i}, v_{\varepsilon i})_{i}, u_{\varepsilon i}^+(t) \rangle_{L^p(\Omega^\varepsilon) \times L^q(\Omega^\varepsilon)^\prime}
\]
Now we simplify the second term on the r.h.s. of (4.2.38).

\[-(S_1 R(u_{\varepsilon, i}^+, v_{\varepsilon, k}^+))_i\]

\[= - \sum_{j=1}^{J} s_{ij} \left( k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \right) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{-v_{m_j}} - k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} \]

\[= - \sum_{j=1}^{J} \left( s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \right) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{-v_{m_j}} + s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} \]

\[= \text{Term 1} + \text{Term 2}, \quad (4.2.39)\]

where

\[\text{Term 1} = - \sum_{j=1}^{J} \left( s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \right) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{-v_{m_j}} + s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} \]

\[= \sum_{j=1}^{J} \left( s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \right) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{-v_{m_j}} + s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} \]

\[\text{and} \quad \text{Term 2} = \sum_{j=1}^{J} \left( s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \right) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{-v_{m_j}} + s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} \]

\[\leq \sum_{j=1}^{J} \left( s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \right) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{-v_{m_j}} + s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} \]

Moreover \(u_{\varepsilon, i}^+(t) = 0\) in the support of \(u_{\varepsilon, i}^-(t)\) and \(u_{\varepsilon, k}^+(t) = 0\) in the support of \(u_{\varepsilon, k}^-(t)\), therefore

\[\int_{\Omega^e} - S_1 R(u_{\varepsilon, i}^+(t), v_{\varepsilon, k}^+(t)) u_{\varepsilon, k}^-(t) \, dx \]

\[\leq \int_{\Omega^e} \sum_{j=1}^{J} \left( s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \right) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} + s_{ij} k_j \prod_{m=1}^{l_1}(u_{\varepsilon, sm}^+) \prod_{m=1}^{l_2}(v_{\varepsilon, sm}^+)^{v_{m_j}} \]

\[= 0. \quad (4.2.42)\]

From (4.2.38) and (4.2.42) we get

\[\frac{d}{dt} \left\| u_{\varepsilon, i}^-(t) \right\|_{L^2(\Omega^e)}^2 \leq \frac{Q^2}{D} \left\| u_{\varepsilon, i}^-(t) \right\|_{L^2(\Omega^e)}^2. \quad (4.2.43)\]

\[\text{From (4.2.8), } -d_i \geq 0; \text{ from (4.2.6), } -\bar{q}_e \cdot \bar{n} \geq 0; \text{ and by definition (4.1.7), } u_{\varepsilon, k}^- \geq 0.\]
Note that \( u_{\varepsilon t_i}(0) > 0 \), i.e., \( u_{\varepsilon t_i}^{-}(0) = 0 \). A straightforward application of Gronwall’s inequality gives
\[
\left\| u_{\varepsilon t_i}^{-}(t) \right\|_{L^2(\Omega^\varepsilon)}^2 = 0 \quad \text{for a.e. } t \text{ and for all } i = 1, 2, ..., I_1,
\]
\[
\Rightarrow u_{\varepsilon t_i}^{-} = 0 \quad \text{for a.e. in } (0, T) \times \Omega^\varepsilon \text{ and for all } i = 1, 2, ..., I_1,
\]
\[
\Rightarrow u_{\varepsilon t_i}^{-} \geq 0 \quad \text{for a.e. in } (0, T) \times \Omega^\varepsilon \text{ and for all } i = 1, 2, ..., I_1.
\]

(b) **Positivity of the type-II species:** In this case testing the \( k \)-th PDE of (4.2.31) by \(-v_{\varepsilon t_k}^{-}\) and proceeding in the same way as in part (a) yields the proof.

(c) **Positivity of the solution for the immobile species:** The positivity follows by similar arguments as in lemma 3.1 in [Krä08].

### 4.2.1.2 Existence of the Global Solution of the Problem (4.2.36)-(4.2.37)

**Theorem 4.2.1.2.** Let \((\tilde{u}_{\varepsilon x}, \tilde{v}_{\varepsilon x}) \in F_{\varepsilon}^u \times G_{\varepsilon}^v\). Then there exists a positive global solution \( w_{\varepsilon} \in H_{\varepsilon}^v \) of the problem (4.2.36)-(4.2.37).

**Proof.** (i) **Positivity:** This is shown in the lemma 4.2.1.1.2.

(ii) **Existence of the local solution:** Let \( x \in \Omega^\varepsilon \) be fixed but chosen arbitrarily. The function \( \psi_{\varepsilon m}(\cdot) \) is continuous w.r.t. \( w_{\varepsilon \varepsilon m} \) for every fixed \( t \). Since
\[
|\psi_0(w_{\varepsilon}) - \psi_0(\tilde{w}_{\varepsilon})|^2_I = \sum_{m=1}^{I_2} |\psi_0(w_{\varepsilon \varepsilon m}) - \psi_0(\tilde{w}_{\varepsilon \varepsilon m})|^2_I,
\]
the vector function \( \psi_0(\cdot) \) is also continuous w.r.t. \( w_{\varepsilon \varepsilon m} \) for every fixed \( t \). Moreover, \( \psi_0(w_{\varepsilon}) \) is measurable w.r.t. \( t \) and
\[
|\psi_0(w_{\varepsilon})|_I = \left[ \sum_{m=1}^{I_2} |\psi_0(w_{\varepsilon \varepsilon m})|^2_I \right]^{\frac{1}{2}} \leq \left[ \sum_{m=1}^{I_2} 1 \right]^{\frac{1}{2}} = I_2^{\frac{1}{2}} =: m(t),
\]
i.e., \( \psi_0(\cdot) \) is bounded by a measurable function \( m(\cdot) \). Thus the application of the Carathéodory’s theorem yields the existence of an absolutely continuous function \( w_{\varepsilon}(x) \) on \([0, T_1]\) which solves (4.2.36)-(4.2.37) (cf. theorem 2.1.1 in [CL55]), i.e., \( w_{\varepsilon}(x) \in [H^{1,1}(0, T_1)]^{I_2} \), where \( T_1 \leq T \), i.e., the solution is local. Since \( x \) is arbitrary, for a.e. \( x \in \Omega^\varepsilon \), \( w_{\varepsilon}(x) \in [H^{1,1}(0, T_1)]^{I_2} \). For all \( \phi \in [C_{0}^{\infty}((0, T_1))]^{I_2} \), the weak formulation of (4.2.36) is given by
\[
\int_0^{T_1} \left\langle \frac{\partial w_{\varepsilon}(t)}{\partial t}, \phi(t) \right\rangle_{I_2} dt = -k_d \int_0^{T_1} \left\langle \psi_0(w_{\varepsilon}(t)), \phi(t) \right\rangle_{I_2} dt,
\]
i.e.,
\[
\int_0^{T_1} \left\langle w_{\varepsilon}(t), \frac{\partial \phi(t)}{\partial t} \right\rangle_{I_2} dt = k_d \int_0^{T_1} \left\langle \psi_0(w_{\varepsilon}(t)), \phi(t) \right\rangle_{I_2} dt. \tag{4.2.44}
\]
Since \( w_{\varepsilon} \) is a function of both \( x \) and \( t \), we shall show that the weak derivative of \( w_{\varepsilon} \) depends on both \( x \) and \( t \) and belongs to \( L^p((0, T); L^p(\Gamma_\varepsilon))^{I_2} \) which accomplishes the claim that \( w_{\varepsilon} \in H_{\varepsilon}^v \). Let us choose another function \( \zeta \in C_{0}^{\infty}(\Gamma_\varepsilon) \). Multiplying (4.2.44) by \( \zeta \) and integrating over \( \Gamma_\varepsilon \), we obtain
\[
\int_0^{T_1} \int_{\Gamma_\varepsilon} \left\langle w_{\varepsilon}(t, x), \frac{\partial \phi(t)}{\partial t} \zeta(x) \right\rangle_{I_2} ds dt = k_d \int_0^{T_1} \int_{\Gamma_\varepsilon} \left\langle \psi_0(w_{\varepsilon}(t, x)), \phi(t) \zeta(x) \right\rangle_{I_2} ds dt,
\]
\[
\int_0^{T_1} \int_{\Gamma_\varepsilon} \left\langle w_{\varepsilon}(t, x), \frac{\partial \phi(t)}{\partial t} \zeta(x) \right\rangle_{I_2} ds dt = -k_d \int_0^{T_1} \int_{\Gamma_\varepsilon} \left\langle \psi_0(w_{\varepsilon}(t, x)), \phi(t) \zeta(x) \right\rangle_{I_2} ds dt. \tag{4.2.45}
\]
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for all $\phi \in [C_0^\infty((0,T_1))]^{I_2}$ and $\zeta \in C_0^\infty(\Gamma_\varepsilon)$. As $\phi \in [C_0^\infty((0,T_1))]^{I_2}$ and $\zeta \in C_0^\infty(\Gamma_\varepsilon)$, $\phi \zeta \in L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$ such that the weak time derivative is in $L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$, i.e.,

$$\int_0^{T_1} \left< w_{\varepsilon,t}, \frac{\partial \phi (t)}{\partial t} \psi \right>_{L^p(\Gamma_\varepsilon)^{I_2} \times L^q(\Gamma_\varepsilon)^{I_2}} \, dt = k_d \int_0^{T_1} \left< \phi (w_{\varepsilon,t}(t)), \phi(t) \psi \right>_{L^p(\Gamma_\varepsilon)^{I_2} \times L^q(\Gamma_\varepsilon)^{I_2}} \, dt.$$ 

Therefore for any $\eta \in L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$ such that $\frac{\partial \eta}{\partial t} \in L^q((0,T_1);L^q(\Gamma_\varepsilon))^{I_2}$, we have

$$\int_0^{T_1} \left< w_{\varepsilon,t}(t,x), \frac{\partial \eta(t,x)}{\partial t} \right>_{L^p(\Gamma_\varepsilon)^{I_2} \times L^q(\Gamma_\varepsilon)^{I_2}} \, dt = k_d \int_0^{T_1} \left< \phi (w_{\varepsilon,t}(t,x)), \eta(t,x) \right>_{L^p(\Gamma_\varepsilon)^{I_2} \times L^q(\Gamma_\varepsilon)^{I_2}} \, dt.$$ 

This leads to the fact that the weak derivative of $w_{\varepsilon,t}$, $\frac{\partial w_{\varepsilon,t}}{\partial t} \in L^p((0,T_1);L^p(\Gamma_\varepsilon))^{I_2}$, i.e., $w_{\varepsilon,t} \in H^{1,p}((0,T_1);L^p(\Gamma_\varepsilon))^{I_2}$.

(iii) **Extension of the solution:** Clearly,

$$|\psi_\varepsilon(w_{\varepsilon,t})| = \left[ \sum_{k=1}^{I_2} |\psi_\varepsilon(w_{\varepsilon,k})|^2 \right]^{\frac{1}{2}} \leq \left[ \sum_{i=1}^{I_2} 1^2 \right]^{\frac{1}{2}} = I_2 \frac{1}{2},$$

i.e., the r.h.s. of (4.2.36)-(4.2.37) is bounded. Therefore from corollary II.3.4 of [MM02], there exists a global solution of the problem (4.2.36)-(4.2.37) on $[0,T)$ for any $T > 0$.

### 4.2.1.3 Existence of the Global Solution of the Problem (4.2.31)-(4.2.37)

**Lemma 4.2.1.3.1.** Suppose that $p > n + 2$ is fixed and $x \in L^\infty(\partial \Omega_{n})$. If we define the map $Q_{\partial \Omega_{n}} : [H^{1,p}(\Omega_{\varepsilon})]^{I_2} \to [H^{1,q}(\Omega_{\varepsilon})]^{I_2^*}$ by

$$\langle Q_{\partial \Omega_{n}} (\phi), \xi \rangle := \sum_{k=1}^{I_2} \langle Q_{\partial \Omega_{n}} (\phi_k), \xi_k \rangle := \sum_{k=1}^{I_2} \int_{\partial \Omega_{n}} x \phi_k \xi_k \, ds \quad \text{for} \quad \xi \in [H^{1,q}(\Omega_{\varepsilon})]^{I_2},$$

then $Q_{\partial \Omega_{n}}$ is well defined and continuous.

**Proof.** For $\phi \in H^{1,p}(\Omega_{\varepsilon})^{I_2}$, the map is given by

$$\langle Q_{\partial \Omega_{n}} (\phi), \xi \rangle = \sum_{k=1}^{I_2} \int_{\partial \Omega_{n}} x \phi_k \xi_k \, ds$$

$$\leq ||x||_{L^\infty(\partial \Omega_{n})} \sum_{k=1}^{I_2} \int_{\partial \Omega_{n}} |\phi_k| |\xi_k| \, ds$$

$$\leq ||x||_{L^\infty(\partial \Omega_{n})} \sum_{k=1}^{I_2} \int_{\partial \Omega} |\phi_k| |\xi_k| \, ds$$

$$\leq ||x||_{L^\infty(\partial \Omega_{n})} \sum_{k=1}^{I_2} ||\phi_k||_{L^p(\partial \Omega)} ||\xi_k||_{L^q(\partial \Omega)}. \quad (4.2.47)$$

From theorem 3.4.2.2, we know that for $1 \leq p < \infty$ and $\phi_k \in H^{1,p}(\Omega_{\varepsilon})$, there exists an extension $\hat{\phi}_k$ of $\phi_k$ (for the sake of notation we still denote the extension by $\hat{\phi}_k$) such that

$$||\hat{\phi}_k||_{H^{1,p}(\Omega)} \leq C ||\phi_k||_{H^{1,p}(\Omega_{\varepsilon})}, \quad (4.2.48)$$

where $C$ is independent of $\varepsilon$. Also from theorem B.5, for a domain $\Omega$ with sufficiently smooth boundary and for $1 \leq p < \infty$, there exists a bounded linear operator $T : H^{1,p}(\Omega) \to L^p(\partial \Omega)$ such that for $\phi_k \in H^{1,p}(\Omega_{\varepsilon})$, $T \phi_k := \phi_k|_{\partial \Omega}$ and

$$||T \phi_k||_{L^p(\partial \Omega)} \leq C ||\phi_k||_{H^{1,p}(\Omega_{\varepsilon})}, \quad (4.2.49)$$
where $C$ depends on $p$ and $\Omega$ only. Combining (4.2.48) and (4.2.49), we obtain
\[
\|\phi_k\|_{L^p(\partial \Omega)} \leq C \|\phi_k\|_{H^{1,p}(\Omega)} \leq C \|\phi_k\|_{H^{1,q}(\Omega^\varepsilon)}.
\] (4.2.50)
An inequality similar to (4.2.50) holds for $\xi_k$ too, i.e.,
\[
\|\xi_k\|_{L^q(\partial \Omega)} \leq C \|\xi_k\|_{H^{1,q}(\Omega)} \leq C \|\xi_k\|_{H^{1,q}(\Omega^\varepsilon)}.
\] (4.2.51)
Using (4.2.50) and (4.2.51) in (4.2.47), we get

\[
\langle Q_{\partial \Omega^\varepsilon}(\phi), \xi \rangle \leq C \sum_{k=1}^{I_2} \|\phi_k\|_{H^{1,p}(\Omega^\varepsilon)} \|\xi_k\|_{H^{1,q}(\Omega^\varepsilon)}
\leq C \left[ \sum_{k=1}^{I_2} \|\phi_k\|_{H^{1,p}(\Omega^\varepsilon)}^p \right]^{\frac{1}{p}} \left[ \sum_{k=1}^{I_2} \|\xi_k\|_{H^{1,q}(\Omega^\varepsilon)}^q \right]^{\frac{1}{q}}, \text{ by discrete Hölder’s inequality}
\leq C \|\phi\|_{L^p(\partial \Omega^\varepsilon)^{I_2}} \|\xi\|_{L^q(\partial \Omega^\varepsilon)^{I_2}}
\leq C \|\phi\|_{H^{1,p}(\Omega^\varepsilon)} \|\xi\|_{H^{1,q}(\Omega^\varepsilon)}
\leq C \sup_{||\xi||_{H^{1,q}(\Omega^\varepsilon)} = 1} \langle Q_{\partial \Omega^\varepsilon}(\phi), \xi \rangle
\leq C \|\phi\|_{H^{1,p}(\Omega^\varepsilon)} \|\xi\|_{H^{1,q}(\Omega^\varepsilon)}
\leq C. \tag{4.2.52}
\]
This shows that the map $Q_{\partial \Omega^\varepsilon} : [H^{1,p}(\Omega^\varepsilon)]^{I_2} \to [H^{1,q}(\Omega^\varepsilon)]^{I_2}$ is well-defined and bounded, hence continuous.

**Lemma 4.2.1.3.2.** Let $p > n + 2$ be fixed. Then the map $R_{\Gamma^\varepsilon} : [L^p(\Gamma^\varepsilon)]^{I_2} \to [H^{1,q}(\Omega^\varepsilon)]^{I_2}$ given by
\[
\langle R_{\Gamma^\varepsilon}(v), \eta \rangle := \sum_{k=1}^{I_2} \langle R_{\Gamma^\varepsilon}(v_k), \eta_k \rangle := \sum_{k=1}^{I_2} \varepsilon \int_{\Gamma^\varepsilon} v_k(x) \eta_k(x) \, d\sigma_x, \text{ for } \eta \in [H^{1,q}(\Omega^\varepsilon)]^{I_2}, \tag{4.2.52}
\]
is well defined and continuous.

**Proof.** We proceed like previous lemma. Here the map is given as\(^{29}\)
\[
\langle R_{\Gamma^\varepsilon}(v), \eta \rangle = \sum_{k=1}^{I_2} \varepsilon \int_{\Gamma^\varepsilon} v_k(x) \eta_k(x) \, d\sigma_x \quad \eta \in [H^{1,q}(\Omega^\varepsilon)]^{I_2}
\leq \sum_{k=1}^{I_2} \varepsilon \left( \int_{\Gamma^\varepsilon} |v_k(x)|^p \, d\sigma_x \right)^{\frac{1}{p}} \left( \int_{\Gamma^\varepsilon} |\eta_k(x)|^q \, d\sigma_x \right)^{\frac{1}{q}}
\leq \sum_{k=1}^{I_2} \left( \varepsilon \int_{\Gamma^\varepsilon} |v_k(x)|^p \, d\sigma_x \right)^{\frac{1}{p}} \left( \varepsilon \int_{\Gamma^\varepsilon} |\eta_k(x)|^q \, d\sigma_x \right)^{\frac{1}{q}}
\leq C \sum_{k=1}^{I_2} \|v_k\|_{L^p(\Gamma^\varepsilon)} \left[ \int_{\Omega^\varepsilon} |\eta_k(x)|^q + \varepsilon^q |\nabla \eta_k(x)|^q \right]^{\frac{1}{q}}
\leq C \max(1, \varepsilon^q)^{\frac{1}{q}} \sum_{k=1}^{I_2} \|v_k\|_{L^p(\Gamma^\varepsilon)} \|\eta_k\|_{H^{1,q}(\Omega^\varepsilon)}
\]
\(^{29}\)Note that we have used the theorem 3.4.1.3. Also see that $\varepsilon = \varepsilon^{\frac{1}{p} + \frac{q}{2}}$.\]
Let us denote this problem by (4.2.55)-(4.2.61). We denote the problem (4.2.53) has a solution. Let $K$ be a solution of the linear system

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term on both sides of the first PDE in the problem (see propositions 4.1.1.2.1 and 4.1.1.2.2) hold good. For technical reason, we add an extra section 4.1.1.2. We also note that all the properties of conditions. We define the Lyapunov functional in the following way: Let $g = \partial v - \nabla \cdot g \in R_v$ be defined as

$$\begin{align*}
\frac{\partial v}{\partial t} - \nabla \cdot (D \nabla v - \tilde{q} v) = S_2 \tilde{R}(u_{v_{\delta \xi}}, v_{v_{\delta \xi}}) & \text{ in } (0, T) \times \Omega_v^p, \\
-(D \nabla v - \tilde{q} v) \cdot \tilde{n} = 0 & \text{ on } (0, T) \times \partial \Omega_{in}, \\
-D \nabla v \cdot \tilde{n} = 0 & \text{ on } (0, T) \times \partial \Omega_{out}, \\
v_{v_{\delta \xi}}(0, x) = v_0(x) & \text{ in } \Omega_v^p, \\
\frac{\partial w_{v_{\delta \xi}}}{\partial t} = -k_d \psi_d(w_{v_{\delta \xi}}) & \text{ on } (0, T) \times \Gamma_\varepsilon, \\
w_{v_{\delta \xi}}(0, x) = w_0(x) & \text{ on } \Gamma_\varepsilon.
\end{align*}$$

Let us denote this problem by $(P_{v_{\delta \xi}}^+)$. We follow the approach shown in section 4.1.1 but here we pay special attention to the boundary terms due to the presence of inflow-outflow boundary conditions. We define the Lyapunov functional in the following way: Let $\mu^0 \in R_{I_2}$ be a solution of the linear system

$$S^T_2 \mu^0 = -\log K,$$  

(4.2.53)

where $K \in R^{I_2}$ is the vector of equilibrium constants $K_j = k^f_j$. Due to (4.2.4), the system (4.2.53) has a solution. Let $g_k : R_0^+ \rightarrow R$ be defined as

$$g_k(v_{v_{\delta \xi}}) := (\mu^0_k - 1 + \log v_{v_{\delta \xi}}) v_{v_{\delta \xi}} + e^{(1-\mu^0_k)} \text{ for each } k = 1, 2, \ldots, I_2.$$  

(4.2.54)

We define $g : R_0^{I_2} \rightarrow R, f_k : R_0^{I_2} \rightarrow R$ and $F : R_0^{I_2} \rightarrow R$ in a similar way as we did in section 4.1.1.2. We also note that all the properties of $g_k, g, f_k$ and $F_k$ from section 4.1.1.2 (see propositions 4.1.1.2.1 and 4.1.1.2.2) hold good. For technical reason, we add an extra term on both sides of the first PDE in the problem $(P_{v_{\delta \xi}}^+)$, i.e., for any $\kappa > 0$, we have

$$\begin{align*}
\frac{\partial v_{v_{\delta \xi}}}{\partial t} - \nabla \cdot (D \nabla v_{v_{\delta \xi}} - \tilde{q} v_{v_{\delta \xi}}) + \kappa v_{v_{\delta \xi}} = S_2 \tilde{R}(u_{v_{\delta \xi}}, v_{v_{\delta \xi}}) + \kappa v_{v_{\delta \xi}} & \text{ in } (0, T) \times \Omega_v^p, \\
-(D \nabla v_{v_{\delta \xi}} - \tilde{q} v_{v_{\delta \xi}}) \cdot \tilde{n} = 0 & \text{ on } (0, T) \times \partial \Omega_{in}, \\
-D \nabla v_{v_{\delta \xi}} \cdot \tilde{n} = 0 & \text{ on } (0, T) \times \partial \Omega_{out}, \\
v_{v_{\delta \xi}}(0, x) = v_0(x) & \text{ in } \Omega_v^p, \\
\frac{\partial w_{v_{\delta \xi}}}{\partial t} = -k_d \psi_d(w_{v_{\delta \xi}}) & \text{ on } (0, T) \times \Gamma_\varepsilon, \\
w_{v_{\delta \xi}}(0, x) = w_0(x) & \text{ on } \Gamma_\varepsilon.
\end{align*}$$

(4.2.55)

We denote the problem (4.2.55)-(4.2.61) by $(P_{v_{\delta \xi}}^+_{M})$. Since a solution of $(P_{v_{\delta \xi}}^+_{M})$ is also a solution of $(P_{v_{\delta \xi}}^+)$, we prove the global existence of the weak solution of $(P_{v_{\delta \xi}}^+_{M})$. Let us
define the fixed point operator $Z_1 : G^v_ε \rightarrow G^v_ε$ via $v_{εś} := Z_1(\hat{v}_{εś})$, where $v_{εś}$ is the solution of the linear problem given by

$$\frac{∂v_{εś}}{∂t} - \nabla \cdot (Dv_{εś} - \bar{q}_ε v_{εś}) + κv_{εś} = S_2 R(\hat{u}_{εś}, \hat{v}_{εś}) + κ\hat{v}_{εś} \quad \text{in} \quad (0,T) × Ω^p_ε, \tag{4.2.62}$$

$$- (Dv_{εś} - \bar{q}_ε v_{εś}) \cdot \bar{n} = 0 \quad \text{on} \quad (0,T) × ∂Ω_{in}, \tag{4.2.63}$$

$$-Dv_{εś} \cdot \bar{n} = 0 \quad \text{on} \quad (0,T) × ∂Ω_{out}, \tag{4.2.64}$$

$$-Dv_{εś} \cdot \bar{n} = \frac{∂w_{εś}}{∂t} \quad \text{on} \quad (0,T) × Π_ε, \tag{4.2.65}$$

$$v_{εś}(0,x) = v_0(x) \quad \text{in} \quad Ω^p_ε, \tag{4.2.66}$$

$$\frac{∂w_{εś}}{∂t} = -k_d ψ(q(w_{εś})) \quad \text{on} \quad (0,T) × Π_ε, \tag{4.2.67}$$

$$w_{εś}(0,x) = w_0(x) \quad \text{on} \quad Π_ε. \tag{4.2.68}$$

**Remark 4.2.1.3.4.** For fixed $\hat{u}_{εś}$ and $\hat{v}_{εś}$, the subproblem (4.2.67)-(4.2.68) has a unique positive global solution $w_{εś}$ in $H^w_ε$. The reformulation of the problem (4.2.62)-(4.2.66) is given by

$$\frac{∂v_{εś}}{∂t} + A v_{εś} = f_{\text{bound}}(v_{εś}) + f(\hat{u}_{εś}, \hat{v}_{εś}), \tag{AP}$$

$$v_{εś}(0,x) = v_0(x),$$

where $A$ is defined as in the remark 4.1.1.1.1 and satisfies the maximal regularity on $[H^{1,q}(Ω^p_ε)^*]_2$, $f_{\text{bound}}(v_{εś}) := Q_{\text{in}} (v_{εś}) + R_{\hat{v}_{εś}} \left( -\frac{∂w_{εś}}{∂t} \right) - \bar{q}_ε \cdot \nabla v_{εś}$, and $f(\hat{u}_{εś}, \hat{v}_{εś}) := κ\hat{v}_{εś} + S_2 R(\hat{u}_{εś}, \hat{v}_{εś})$, where $κ > 0$. Note that the theorem 3.4.3.4 implies $\hat{u}_{εś} ∈ L^∞((0,T) × Ω^p_ε)^1$. Similar arguments as in remark 4.1.1.1.1 leads to the fact that $f ∈ L^p((0,T); H^{1,q}(Ω^p_ε)^*)_2$. Using lemmas 4.2.1.3.1, 4.2.1.3.2 and the assumption (4.2.7), the boundary term $f_{\text{bound}} ∈ L^p((0,T); H^{1,q}(Ω^p_ε)^*)_2$. The condition $v_0 ∈ \left[ [H^{1,q}(Ω^p_ε)^*], H^{1,p}(Ω^p_ε)_1 - \frac{1}{p}, p \right]_2$ is fulfilled by (4.2.3). Then theorem 3.3.1 assures the existence of a unique solution of the problem (AP). Therefore the operator $Z_1$ is well-defined.

The application of Schaefer’s fixed point theorem resides on the verification of the following two conditions:

(i) The operator $Z_1$ is continuous and compact.

(ii) The set $\{v_{εś} ∈ G^v_ε \mid \exists λ ∈ [0,1] : v_{εś} = λZ_1(v_{εś})\}$ is bounded, i.e., there exists a constant $C > 0$ independent of $v_{εś}$ and $λ$ such that any arbitrary solution $v_{εś} ∈ G^v_ε$ of the equation

$$v_{εś} = λZ_1(v_{εś}) \tag{4.2.69}$$

satisfies

$$\|v_{εś}\|_{G^v_ε} ≤ C. \tag{4.2.70}$$
Equations (4.2.62)-(4.2.68) and (4.1.69) imply
\[
\frac{\partial v_{\varepsilon\delta}}{\partial t} - \nabla \cdot (D\nabla v_{\varepsilon\delta} - \bar{q}_\varepsilon v_{\varepsilon\delta}) + \kappa v_{\varepsilon\delta} = \lambda S_2 \hat{R}(\hat{u}_{\varepsilon\delta}, v_{\varepsilon\delta}) + \lambda \kappa v_{\varepsilon\delta} \quad \text{in} \quad (0, T) \times \Omega^p_{\varepsilon},
\]
(4.2.71)
\[
v_{\varepsilon\delta}(0, x) = \lambda v_0(x) \quad \text{in} \quad \Omega^p_{\varepsilon},
\]
(4.2.72)
\[-(D\nabla v_{\varepsilon\delta} - \bar{q}_\varepsilon v_{\varepsilon\delta}) \cdot \bar{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_n,
\]
(4.2.73)
\[-D\nabla v_{\varepsilon\delta} \cdot \bar{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{out},
\]
(4.2.74)
\[-D\nabla v_{\varepsilon\delta} \cdot \bar{n} = \lambda \varepsilon \frac{\partial w_{\varepsilon\delta}}{\partial t} \quad \text{on} \quad (0, T) \times \Gamma_{\varepsilon},
\]
(4.2.75)
\[-\frac{\partial w_{\varepsilon\delta}}{\partial t} = -k_d \psi_\delta(w_{\varepsilon\delta}) \quad \text{on} \quad (0, T) \times \Gamma_{\varepsilon},
\]
(4.2.76)
\[w_{\varepsilon\delta}(0, x) = w_0(x) \quad \text{on} \quad \Gamma_{\varepsilon}.
\]
(4.2.77)

Note that \(w_{\varepsilon\delta}\) is the solution of the ODE problem (4.2.76)-(4.2.77). Let us call the problem (4.2.71)-(4.2.77) as \(\mathcal{P}_{\varepsilon\lambda}^{2+}\). To show the inequality (4.2.70), we aim to prove a theorem like 4.1.1.2.3 which is the following:

**Theorem 4.2.1.3.5.** Let \(r \in \mathbb{N} \quad (r \geq 2)\), \(0 \leq t \leq T\) and \(0 \leq \lambda \leq 1\). Suppose that \(\hat{u}_{\varepsilon\delta} \in \mathcal{F}_{\varepsilon}^r\).
Further assume that \(v_{\varepsilon\delta} \in \mathcal{G}_{\varepsilon}^r\) is a solution of \((\mathcal{P}_{\varepsilon\lambda}^{2+})\). Then the following inequality holds good:
\[
F_r(v_{\varepsilon\delta}(t)) \leq e^{C_{34}t} F_r(v_{\varepsilon\delta}(0)) \quad \text{for a.e.} \ t,
\]
(4.2.78)
where \(C_{34}\) is independent of \(\varepsilon, \delta, \lambda\) and \(t\).

Our starting point is the following lemma which is similar to the lemma 4.1.1.2.7.

**Lemma 4.2.1.3.6.** Let \(p > n + 2\), \(\hat{u}_{\varepsilon\delta} \in \mathcal{F}_{\varepsilon}^r\) and \(r \in \mathbb{N} \quad (r \geq 2)\). Assume that \(v_{\varepsilon\delta} \in \mathcal{G}_{\varepsilon}^r\) is a solution of \((\mathcal{P}_{\varepsilon\lambda}^{2+})\) and for \(\tau > 0\),
\[
v_{\varepsilon\delta, \tau} := v_{\varepsilon\delta} + \tau.
\]
(4.2.79)

Then the following inequality holds:
\[
\int_0^t \left\langle \frac{\partial v_{\varepsilon\delta, \tau}}{\partial t}, \partial f_r(v_{\varepsilon\delta, \tau}) \right\rangle_{H^{1, 1}(\Omega^p_{\varepsilon})^2, H^{1, 1}(\Omega^p_{\varepsilon})^2} d\theta 
\leq h(t, \tau, v_{\varepsilon\delta, \tau}) + l(t, \tau, v_{\varepsilon\delta, \tau}) + C_{34} \int_0^t F_r(v_{\varepsilon\delta, \tau}) d\theta
\]
(4.2.80)
where \(h(t, \tau, v_{\varepsilon\delta, \tau})\) and \(l(t, \tau, v_{\varepsilon\delta, \tau})\) tend to zero as \(\tau \to 0\) for a.e. \(t\), and \(C_{34}\) is independent of \(\varepsilon, \delta, \lambda\) and \(t\).

**Proof.** Obviously \(v_{\varepsilon\delta, \tau} \in \mathcal{G}_{\varepsilon}^r\). For \(p > n + 2\), \(v_{\varepsilon\delta, \tau} \in L^\infty((0, T) \times \Omega^p_{\varepsilon})^2\) (cf. theorem 3.4.3.4) and \(\partial f_r(v_{\varepsilon\delta, \tau}) \in L^q((0, T); H^{1, 1}(\Omega^p_{\varepsilon}))^2\). Using \(\partial f_r(v_{\varepsilon\delta, \tau})\) in the weak formualtion of the PDE (4.2.71), we get
\[
\int_0^t \left\langle \partial^2 v_{\varepsilon\delta, \tau}, \partial f_r(v_{\varepsilon\delta, \tau}) \right\rangle_{H^1(\Omega^p_{\varepsilon})^2, H^1(\Omega^p_{\varepsilon})^2} d\theta + \kappa \sum_{k=1}^{I_2} \int_0^t \int_{\Omega^p_{\varepsilon}} v_{\varepsilon\delta} \partial f_r(v_{\varepsilon\delta, \tau})_k dx d\theta
\]
\[
- \sum_{k=1}^{I_2} \left[ \int_0^t \int_{\Omega^p_{\varepsilon}} \bar{q}_\varepsilon \cdot \bar{n} v_{\varepsilon\delta} \left( \partial f_r(v_{\varepsilon\delta, \tau}) \right)_k dx d\theta + \lambda \kappa k_d \int_0^t \int_{\Gamma_{\varepsilon}} \psi_\delta(w_{\varepsilon\delta}) \left( \partial f_r(v_{\varepsilon\delta, \tau}) \right)_k dx d\theta \right]
\]
\[
+ \sum_{l=1}^n \int_0^t \int_{\Omega^p_{\varepsilon}} D_{\varepsilon, \delta} \nabla v_{\varepsilon\delta} \partial f_r(v_{\varepsilon\delta, \tau}) dx d\theta + \int_0^t \int_{\Omega^p_{\varepsilon}} \langle \bar{q}_\varepsilon \cdot \nabla v_{\varepsilon\delta}, \partial f_r(v_{\varepsilon\delta, \tau}) \rangle_{I_2} dx d\theta
\]
\[
= \int_0^t \left\langle S_2 \hat{R}(\hat{u}_{\varepsilon\delta}, v_{\varepsilon\delta}), \partial f_r(v_{\varepsilon\delta, \tau}) \right\rangle_{H^1(\Omega^p_{\varepsilon})^2, H^1(\Omega^p_{\varepsilon})^2} d\theta + \lambda \kappa \sum_{k=1}^{I_2} \int_0^t \int_{\Omega^p_{\varepsilon}} v_{\varepsilon\delta} \partial f_r(v_{\varepsilon\delta, \tau})_k dx d\theta,
\]
Now we simplify and estimate the terms $I_{\text{diff}}$, $I_{\text{bound}}$, $I_{\text{adv}}$, $I_{\text{rec}}$ and $I_{E_x}$ one by one. With the help of (4.2.9), the term $I_{\text{rec}}$ can be estimated in the same way as we did in the lemma 4.1.1.2.7 and this will give

$$I_{\text{rec}} \leq \lambda r C \sum_{k=1}^{l_2} \int_{\Omega^\varepsilon_i} \tau \left[ |\mu_k^0| + T |\Omega| |\log \tau| + v_{\varepsilon_k,\tau} \right] d\tau d\theta =: h(t, \tau, \varepsilon, \tau) \quad \text{for a.e. } t,$$

where $C$ is independent of $\lambda$, $\hat{u}_{\varepsilon,\tau}$ and $v_{\varepsilon,\tau}$ and all the other terms of $h(t, \tau, \varepsilon, \tau)$ are bounded and tend to zero as $\tau \to 0$ for a.e. $t$, i.e.,

$$I_{\text{rec}} \leq h(t, \tau, \varepsilon, \tau) \to 0 \text{ as } \tau \to 0 \quad \text{for a.e. } t. \quad (4.2.82)$$

$$I_{E_x} = \kappa(1 - \lambda) \sum_{i=1}^{l_2} \int_{\Omega^\varepsilon_i} v_{\varepsilon_k} \partial f_r(v_{\varepsilon_k}, \tau) d\tau$$

$$= \kappa(1 - \lambda) \sum_{i=1}^{l_2} \int_{\Omega^\varepsilon_i} r(\tau - v_{\varepsilon_k,\tau}) f_{r-1}(v_{\varepsilon_k,\tau})(\mu_k^0 + \log v_{\varepsilon_k,\tau}) d\tau d\theta \quad \text{since } v_{\varepsilon_k,\tau} = v_{\varepsilon_k} + \tau$$

$$= \tau \kappa(1 - \lambda) \sum_{i=1}^{l_2} \int_{\Omega^\varepsilon_i} r(\mu_k^0 + \log v_{\varepsilon_k,\tau}) f_{r-1}(v_{\varepsilon_k,\tau}) d\tau d\theta$$

$$+ r \kappa(1 - \lambda) \sum_{i=1}^{l_2} \int_{\Omega^\varepsilon_i} v_{\varepsilon_k,\tau} (\mu_k^0 + \log v_{\varepsilon_k,\tau}) f_{r-1}(v_{\varepsilon_k,\tau}) d\tau d\theta. \quad (4.2.83)
It can be shown that

\[- v_{\varepsilon \delta k, \tau} (\mu_0 + \log v_{\varepsilon \delta k, \tau}) \leq e^{-(1+\mu_0^2)} \quad \forall i. \tag{4.2.84}\]

We have \(\log v_{\varepsilon \delta k, \tau} \leq g_k(v_{\varepsilon \delta k, \tau})\) and \(g_k(v_{\varepsilon \delta k, \tau}) \geq (e-1)e^{-\mu_0^2}\). Choosing a constant \(C = \max_{1 \leq k \leq I_2} \left(1 + |\mu_0|^2 e^{-\mu_0^2} (e-1)\right)\), we obtain

\[\mu_0^2 + \log v_{\varepsilon \delta k, \tau} \leq \mu_0^2 + g_k(v_{\varepsilon \delta k, \tau}) \leq 0 \leq \frac{1}{2} \frac{1}{\mu_0^2} (e-1) e^{-\mu_0^2} (e-1). \tag{4.2.85}\]

Combining (4.2.83), (4.2.84) and (4.2.85), we get

\[I^{(t)}_E \leq (1-\lambda) \left[ r^2 \kappa \sum_{i=1}^{I_2} \int_0^t \int_{\Omega_E} C g_k(v_{\varepsilon \delta k, \tau}) f_{r-1}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta + \kappa \sum_{i=1}^{I_2} \int_0^t \int_{\Omega_E} e^{-(1+\mu_0^2)} f_{r-1}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \right] \]

\[\leq r^2 \kappa(1-\lambda) \sum_{i=1}^{I_2} \int_0^t \int_{\Omega_E} C g_k(v_{\varepsilon \delta k, \tau}) f_{r-1}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \]

\[+ \kappa(1-\lambda) \sum_{i=1}^{I_2} \int_0^t \int_{\Omega_E} g(v_{\varepsilon \delta, \tau}) f_{r-1}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \]

\[\leq r^2 \kappa(1-\lambda)C \sum_{i=1}^{I_2} \int_0^t \int_{\Omega_E} g(v_{\varepsilon \delta, \tau}) f_{r-1}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \]

\[+ \kappa(1-\lambda) \sum_{i=1}^{I_2} \int_0^t \int_{\Omega_E} g(v_{\varepsilon \delta, \tau}) f_{r-1}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \]

\[\leq I_2 r^2 \tau C \int_0^t \int_{\Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta + I_2 r^2 \kappa(e-1)^{-1} \int_0^t \int_{\Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \quad \text{since } g_k(v_{\varepsilon \delta, \tau}) \leq g(v_{\varepsilon \delta, \tau}) \]

\[\leq I_2 r^2 \tau C \int_0^t \int_{\Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta + I_2 r^2 \kappa(e-1)^{-1} \int_0^t \int_{\Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \quad \text{since } 0 \leq \lambda \leq 1 \]

\[\text{and } f_r = f_{r-1}g \text{ for a.e. } t. \]

As \(\tau \to 0\), \(f_r(v_{\varepsilon \delta, \tau})\) is bounded in \(L^1((0,T) \times \Omega_E^0)\). Therefore for a.e. \(t\) the first term in the r.h.s. of the above inequality tends to zero as \(\tau \to 0\). Denote the first term by \(l(t,\tau,v_{\varepsilon \delta, \tau})\), then

\[I^{(t)}_E \leq l(t,\tau,v_{\varepsilon \delta, \tau}) + I_2 r^2 \kappa(e-1)^{-1} \int_0^t \int_{\Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \, dx \, d\theta \quad \text{for a.e. } t. \tag{4.2.86}\]

Again,

\[I^{(t)}_{\text{advec}} = - \int_0^t \int_{\Omega_E} \langle \tilde{q}_\varepsilon \cdot \nabla v_{\varepsilon \delta}, \partial f_{r}(v_{\varepsilon \delta, \tau}) \rangle I_2 \, dx \, d\theta \]

\[= - \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_E} \tilde{q}_\varepsilon \cdot \nabla v_{\varepsilon \delta} \left( \partial f_{r}(v_{\varepsilon \delta, \tau}) \right)_k \, dx \, d\theta \]

\[= - \sum_{k=1}^{I_2} \sum_{l=1}^{n} \int_0^t \int_{\Omega_E} \frac{\partial v_{\varepsilon \delta}}{\partial x_l} \left( \partial f_{r}(v_{\varepsilon \delta, \tau}) \right)_k \, dx \, d\theta \]

\[= - \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_E} \frac{\partial f_{r}(v_{\varepsilon \delta, \tau})}{\partial x_l} q_l \, dx \, d\theta \]

\[= \int_0^t \int_{\Omega_E} \nabla_x f_{r}(v_{\varepsilon \delta, \tau}) \cdot \tilde{q}_\varepsilon \, dx \, d\theta \]

\[= \int_0^T \int_{\partial \Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \nabla \cdot \tilde{q}_\varepsilon \, ds \, d\theta - \int_0^T \int_{\partial \Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \tilde{q}_\varepsilon \cdot \tilde{n} \, ds \, d\theta \]

\[= - \int_0^t \int_{\partial \Omega_E} f_{r}(v_{\varepsilon \delta, \tau}) \tilde{q}_\varepsilon \cdot \tilde{n} \, ds \, d\theta, \text{ since } \nabla \cdot \tilde{q}_\varepsilon = 0 \text{ in } \Omega_E^0\]
where \( C_{26} := ||\vec{q} \cdot \vec{n}||_{L^\infty((0,T) \times \partial \Omega_m)} \) and \( f_r \geq 0 \). Note that \( C_{26} \) is independent of \( \varepsilon, \delta, \lambda, \tau \) and \( t \). Again,

\[
\begin{align*}
I^{(t)}_{\text{bound}} &= \sum_{k=1}^{I_2} \left[ \int_0^t \int_{\partial \Omega_m} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon,\tau} \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta + \lambda \varepsilon k d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon,k}) \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k d\sigma_x d\theta \right] \\
&= \sum_{k=1}^{I_2} \left[ \int_0^t \int_{\partial \Omega_m} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon,\tau} \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta \\
&\quad + \lambda \varepsilon k d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon,k}) \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k d\sigma_x d\theta \right] \\
&= \sum_{k=1}^{I_2} \left[ \int_0^t \int_{\partial \Omega_m} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon,\tau} \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta - \int_0^t \int_{\partial \Omega_m} \vec{q}_\varepsilon \cdot \vec{n} \tau \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta \right] \\
&\quad + \sum_{k=1}^{I_2} \left[ \lambda \varepsilon k d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon,k}) \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k d\sigma_x d\theta \right] \\
&=: \sum_{k=1}^{I_2} \left[ \text{Boundary}_{1,k} + \text{Boundary}_{2,k} + \text{Boundary}_{3,k} \right].
\end{align*}
\]

where

\[
\text{Boundary}_{1,k} := \int_0^t \int_{\partial \Omega_m} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon,\tau} \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta,
\]

\[
\text{Boundary}_{2,k} := -\int_0^t \int_{\partial \Omega_m} \vec{q}_\varepsilon \cdot \vec{n} \tau \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta
\]

\[
\text{Boundary}_{3,k} := \lambda \varepsilon k d \int_0^t \int_{\Gamma_\varepsilon} \psi_\delta(w_{\varepsilon,k}) \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k d\sigma_x d\theta.
\]

Now,

\[
\text{Boundary}_{1,k} = \int_0^t \int_{\partial \Omega_m} \vec{q}_\varepsilon \cdot \vec{n} v_{\varepsilon,\tau} \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta
\]

\[
= \int_0^t \int_{\partial \Omega_m} -|\vec{q}_\varepsilon \cdot \vec{n}| v_{\varepsilon,\tau} \left( \partial f_r(v_{\varepsilon,\tau}) \right)_k ds d\theta \\
= \int_0^t \int_{\partial \Omega_m} -r f_{r-1}(v_{\varepsilon,\tau}) |\vec{q}_\varepsilon \cdot \vec{n}| v_{\varepsilon,\tau} \left( \mu_k^0 + \log v_{\varepsilon,k,\tau} \right) ds d\theta.
\]
It can be shown that \(-v_{\varepsilon\delta_k,\tau} \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \leq \frac{1}{e(e-1)} g_k(v_{\varepsilon\delta_k,\tau})\). This gives

\[\text{Boundary}_{1,k} \leq \int_0^t \int_{\partial \Omega_{in}} \frac{1}{e(e-1)} g_k(v_{\varepsilon\delta_k,\tau}) |\tilde{q}_e \cdot \tilde{n}| \, ds \, d\theta \]

\[\leq r \left| \tilde{q}_e \cdot \tilde{n} \right|_{L^\infty((0,T) \times \partial \Omega_{in})} \frac{1}{e(e-1)} \int_0^t \int_{\partial \Omega_{in}} f_{r-1}(v_{\varepsilon\delta,\tau}) g_k(v_{\varepsilon\delta,\tau}) \, ds \, d\theta \]

\[\leq r \left| \tilde{q}_e \cdot \tilde{n} \right|_{L^\infty((0,T) \times \partial \Omega_{in})} \frac{1}{e(e-1)} \int_0^t \int_{\partial \Omega_{in}} f_{r-1}(v_{\varepsilon\delta,\tau}) g(v_{\varepsilon\delta,\tau}) \, ds \, d\theta, \text{ since } g_k \leq g \]

\[= C_{27} \int_0^t \int_{\partial \Omega_{in}} f_r(v_{\varepsilon\delta,\tau}(t,x)) \, ds \, d\theta, \quad (4.2.92)\]

where \(C_{27} := r \left| \tilde{q}_e \cdot \tilde{n} \right|_{L^\infty((0,T) \times \partial \Omega_{in})} \frac{1}{e(e-1)} \) is independent of \(\varepsilon, \delta, \lambda, \tau \) and \(t\).

\[\text{Boundary}_{2,k} := \int_0^t \int_{\partial \Omega_{in}^+} \tilde{q}_e \cdot \tilde{n} \tau (\partial f_r(v_{\varepsilon\delta,\tau})) \, ds \, d\theta \]

\[= \int_0^t \int_{\partial \Omega_{in}^+} \tilde{q}_e \cdot \tilde{n} \tau f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, ds \, d\theta. \]

Let \(\partial \Omega_{in}^+ := \{ x \in \partial \Omega_{in} : \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \geq 0 \} \) and \(\partial \Omega_{in}^- := \{ x \in \partial \Omega_{in} : \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \leq 0 \} \). On the boundary \(\partial \Omega_{in}^-\), the integrand is nonpositive and it can be estimated by zero. This gives

\[\text{Boundary}_{2,k} \leq \int_0^t \int_{\partial \Omega_{in}^+} \tilde{q}_e \cdot \tilde{n} \tau f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, ds \, d\theta. \]

From the definition of \(g_k\), \((e-1)e^{-\mu_0^k} \leq g_k(v_{\varepsilon\delta_k,\tau})\) and it can be shown that \(\log v_{\varepsilon\delta_k,\tau} \leq v_{\varepsilon\delta_k,\tau} \leq g_k(v_{\varepsilon\delta_k,\tau})\). Choosing \(C_{28} := \max_{1 \leq k \leq 1_2} \left\{ \left( 1 + \left| \mu_0^k \right| e^{\mu_0^k} (e-1)^{-1} \right) \right\}\), we have

\[\mu_0^0 + \log v_{\varepsilon\delta_k,\tau} \leq C_{28} g_k(v_{\varepsilon\delta_k,\tau}) \leq C_{28} g(v_{\varepsilon\delta_k,\tau}). \]

\[\text{Boundary}_{2,k} \leq \int_0^t \int_{\partial \Omega_{in}^+} \tilde{q}_e \cdot \tilde{n} \tau r f_{r-1}(v_{\varepsilon\delta,\tau}) C_{28} g(v_{\varepsilon\delta,\tau}) \, ds \, d\theta \]

\[\leq C_{29} r \tau \left| \tilde{q}_e \cdot \tilde{n} \right|_{L^\infty((0,T) \times \partial \Omega_{in})} \int_0^t \int_{\partial \Omega_{in}} f_r(v_{\varepsilon\delta,\tau}) \, ds \, d\theta \]

\[= C_{29} \int_0^T \int_{\partial \Omega_{in}} f_r(v_{\varepsilon\delta,\tau}) \, ds \, d\theta, \quad (4.2.93)\]

where \(C_{29} := C_{28} r \tau \left| \tilde{q}_e \cdot \tilde{n} \right|_{L^\infty((0,T) \times \partial \Omega_{in})}\) is independent of \(\varepsilon, \delta, \lambda \) and \(t\).

\[\text{Boundary}_{3,k} = \lambda \varepsilon k_d \int_0^t \int_{\Gamma_{e}} \psi_\delta(w_{\varepsilon\delta_k}) (\partial f_r(v_{\varepsilon\delta,\tau})) \, d\sigma_x \, d\theta \]

\[\leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} (\partial f_r(v_{\varepsilon\delta,\tau})) \, d\sigma_x \, d\theta, \quad (4.2.94)\]

\[= k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]

\[\text{Boundary}_{3,k} \leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]

\[\text{Boundary}_{3,k} \leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]

\[\text{Boundary}_{3,k} \leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]

\[\text{Boundary}_{3,k} \leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]

\[\text{Boundary}_{3,k} \leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]

\[\text{Boundary}_{3,k} \leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]

\[\text{Boundary}_{3,k} \leq k_d \varepsilon \int_0^t \int_{\Gamma_{e}} r f_{r-1}(v_{\varepsilon\delta,\tau}) \left( \mu_0^k + \log v_{\varepsilon\delta_k,\tau} \right) \, d\sigma_x \, d\theta. \]
\[
\leq \varepsilon k_d C_{28} r \int_0^t \int_{\Gamma_\varepsilon} f_{r-1}(v_{\varepsilon,\tau}) g(v_{\varepsilon,\tau}) \, d\sigma_x \, d\theta, \tag{4.2.94}
\]

where \( C_{30} \) \((= k_d C_{28} r)\) is independent of \( \varepsilon, \delta, \lambda \) and \( t \). Substituting the estimates for Boundary\(_{p,k}\) for \( 1 \leq p \leq 3 \) in (4.2.88), we obtain\(^3\)

\[
I_{\text{bound}}^{(t)} = \sum_{k=1}^{I_2} \text{Boundary}_{1,k} + \text{Boundary}_{2,k} + \text{Boundary}_{3,k}
\]

\[
\leq \sum_{k=1}^{I_2} C_{31} \left[ \int_0^t \int_{\partial \Omega_{tn}} f_r(v_{\varepsilon,\tau}) \, ds \, d\theta + \int_0^t \int_{\partial \Omega_{tn}} f_r(v_{\varepsilon,\tau}) \, ds \, d\theta + \varepsilon \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon,\tau}) \, d\sigma_x \, d\theta \right]
\]

\[
= 2C_{31} I_2 \int_0^t \int_{\partial \Omega_{tn}} f_r(v_{\varepsilon,\tau}) \, ds \, d\theta + \varepsilon C_{31} I_2 \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon,\tau}) \, d\sigma_x \, d\theta
\]

\[
\leq C_{32} \left[ \int_0^t \int_{\partial \Omega} f_r(v_{\varepsilon,\tau}) \, ds \, d\theta + \varepsilon \int_0^T \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon,\tau}) \, d\sigma_x \, d\theta \right] \quad \text{for a.e. } t. \tag{4.2.95}
\]

The term \( I_{\text{diff}} \) can be estimated as in lemma 4.1.1.2.7.

\[
I_{\text{diff}}^{(t)} = -D \sum_{k=1}^{I_2} \sum_{l=1}^n \int_0^t \int_{\Omega_r} r(r-1) f_{r-2} \sum_{k=1}^{I_2} \left( \mu^0 + \log v_{\varepsilon,\tau} \right) \left( \mu_k^0 + \log v_{\varepsilon,\tau} \right) \frac{\partial v_{\varepsilon,\tau}}{\partial x_1} \frac{\partial v_{\varepsilon,\tau}}{\partial x_1} \, dx \, d\theta
\]

\[
= -D \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_r} r(r-1) f_{r-2} \left( \sum_{k=1}^{I_2} \left( \mu^0 + \log v_{\varepsilon,\tau} \right) \left( \mu_k^0 + \log v_{\varepsilon,\tau} \right) \right)^2 \, dx \, d\theta \quad \text{for a.e. } t. \tag{4.2.96}
\]

Combining (4.2.81), (4.2.82), (4.2.86), (4.2.87), (4.2.95) and (4.2.96), we get\(^4\)

\[
\int_0^t \left( \partial \theta v_{\varepsilon,\tau}, \partial f_r(v_{\varepsilon,\tau}) \right)_{[H^{1,\alpha}(\Omega^\varepsilon)]^2} \, dx \times [H^{1,\alpha}(\Omega^\varepsilon)]^2 \, d\theta
\]

\[
= I_{\text{diff}}^{(t)} + I_{\text{bound}}^{(t)} + I_{\text{advec}}^{(t)} + I_{\text{reak}}^{(t)} + I_{\text{Ex}}^{(t)}
\]

\[
\leq -D \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_r} r(r-1) f_{r-2} \left( \sum_{k=1}^{I_2} \left( \mu^0 + \log v_{\varepsilon,\tau} \right) \left( \mu_k^0 + \log v_{\varepsilon,\tau} \right) \right)^2 \, dx \, d\theta + C_{32} \int_0^t \int_{\partial \Omega} f_r(v_{\varepsilon,\tau}) \, ds \, d\theta + \varepsilon C_{31} \int_0^t \int_{\Gamma_\varepsilon} f_r(v_{\varepsilon,\tau}) \, d\sigma_x \, d\theta + C_{26} \int_0^t \int_{\partial \Omega} f_r(v_{\varepsilon,\tau}) \, ds \, d\theta + h(t, \tau, v_{\varepsilon,\tau}) + l(t, \tau, v_{\varepsilon,\tau}) + I_{\text{reak}}(e(e-1))^{-1} \int_0^t \int_{\Omega_r} f_r(v_{\varepsilon,\tau}) \, dx \, d\theta
\]

\(^3\)See the above calculation for Boundary\(_{2,k}\).

\(^4\)Where \( C_{31} = \max(C_{27}, C_{29}, C_{30}) \) and \( C_{32} = I_2 \max(2C_{31}, C_{31}) \).

\(^4\)The idea to further estimate the term \( I_{\text{diff}} + I_{\text{bound}} + I_{\text{advec}} + I_{\text{reak}} + I_{\text{Ex}} \) is borrowed from [Krä08]. We also note that \( f_r(u) = [g(u)]^r = [g^2(u)]^2 = f_r^2(u) \).
\[ -D \sum_{l=1}^{n} \int_{Q_{l}} r(t-1)f_{x_{l}}dx d\theta + (C_{32} + C_{36}) \int_{0}^{t} \int_{\Omega} f_{r}(v_{x_{l},\tau}) ds d\theta + C_{32} \varepsilon \int_{0}^{t} \int_{\Gamma_{x}} f_{r}(v_{x_{l},\tau}) d\sigma d\theta + h(t, \tau, v_{x_{l},\tau}) + l(t, \tau, v_{x_{l},\tau}) + I_{2} r k(e(e-1))^{-1} \int_{0}^{t} \int_{Q_{l}} f_{r}(v_{x_{l},\tau}) dx d\theta \]

\[ = -D \sum_{l=1}^{n} \int_{0}^{t} \int_{\Gamma_{x}} f_{r}(v_{x_{l},\tau}) d\sigma d\theta + (C_{32} + C_{36}) \int_{0}^{t} \int_{\Omega} f_{r}(v_{x_{l},\tau}) ds d\theta + C_{32} \varepsilon \int_{0}^{t} \int_{\Gamma_{x}} f_{r}(v_{x_{l},\tau}) d\sigma d\theta + h(t, \tau, v_{x_{l},\tau}) + l(t, \tau, v_{x_{l},\tau}) + I_{2} r k(e(e-1))^{-1} \int_{0}^{t} \int_{Q_{l}} f_{r}(v_{x_{l},\tau}) dx d\theta \]

for a.e. \( t \), where \( C_{26} \) and \( C_{32} \) are independent of \( \varepsilon, \delta, \lambda \) and \( t \). We further simplify the terms in \((4.2.97)\).
Also using theorem 3.4.3.2 we have
\[
\int_0^t \int_{\partial \Omega} f_{\tau}^2(v_{\varepsilon, \tau}(t, x)) ds \, d\theta
\]
\[
= \int_0^t \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\partial \Omega)}^2 d\theta
\]
\[
\leq C_8 \left( \int_0^t \left\| \nabla f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^n \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)} + \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^2 \right) d\theta
\]
\[
\leq C_8 \int_0^t \left( \left\| \nabla f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^n + \Lambda_\varepsilon \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^2 + \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^2 \right) d\theta
\]
due to Young's inequality
\[
= C_8 \int_0^t \left( \left\| \nabla f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^n + \Lambda_\varepsilon \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^2 + \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^2 \right) d\theta, \quad \Lambda_\varepsilon = \hat{\Lambda}_\varepsilon + 1, \quad (4.2.99)
\]
where \( C_8 \) (independent of \( \varepsilon \) and \( \varsigma \)) is a constant in Young’s inequality which will be chosen later. Further note that
\[
\left\| \nabla f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^n = \sum_{l=1}^n \left\| \frac{\partial f_{\tau}^2(v_{\varepsilon, \tau})}{\partial x_l} \right\|_{L^2(\Omega_p^\delta)}^2
\]
\[
= \frac{\varsigma^2}{4} \int_{\Omega_p^\delta} f_{\tau}^2(v_{\varepsilon, \tau}) \sum_{l=1}^n \sum_{k=1}^{l_2} \left( \mu_k^0 + \log v_{\varepsilon, \tau} \right) \frac{\partial v_{\varepsilon, \tau}}{\partial x_l} \right)^2 dx. \quad (4.2.100)
\]
Combining (4.2.97), (4.2.98), (4.2.99) and (4.2.100), we obtain
\[
\int_0^t \left\langle \partial_\theta v_{\varepsilon, \tau}, \partial f_{\tau}(v_{\varepsilon, \tau}) \right\rangle_{[H^{1,\alpha}(\Omega_p^\delta)]^{r_2} \times [H^{1,\alpha}(\Omega_p^\delta)]^{l_2}} \, d\theta
\]
\[
\leq -D \sum_{l=1}^n \int_0^t \int_{\Omega_p^\delta} r(r - 1) f_{\tau}^2 \left( \sum_{l=1}^n \left( \mu_k^0 + \log v_{\varepsilon, \tau} \right) \frac{\partial v_{\varepsilon, \tau}}{\partial x_l} \right)^2 \left( \sum_{l=1}^n \left( \mu_k^0 + \log v_{\varepsilon, \tau} \right) \frac{\partial v_{\varepsilon, \tau}}{\partial x_l} \right)^2 \right) \, dx \, d\theta
\]
\[
+ (C_{32} + C_{26}) C_8 \int_0^t \int_{\Omega_p^\delta} f_{\tau}^2 \left( \sum_{l=1}^n \left( \mu_k^0 + \log v_{\varepsilon, \tau} \right) \frac{\partial v_{\varepsilon, \tau}}{\partial x_l} \right)^2 \, dx \, d\theta
\]
\[
+ (C_{32} + C_{26}) C_8 \int_0^t \int_{\Omega_p^\delta} \Lambda_\varepsilon \left\| f_{\tau}^2(v_{\varepsilon, \tau}(t)) \right\|_{L^2(\Omega_p^\delta)}^2 \, dx \, d\theta + C_{32} \int_0^t \int_{\Omega_p^\delta} f_{\tau}(v_{\varepsilon, \tau}) \, dx \, d\theta
\]
\[
\leq \left[ -Dr(r - 1) + \varsigma C_{33} \frac{r_2^2}{4} \right] \int_0^t \int_{\Omega_p^\delta} f_{\tau}^2 \left( \sum_{l=1}^n \left( \mu_k^0 + \log v_{\varepsilon, \tau} \right) \frac{\partial v_{\varepsilon, \tau}}{\partial x_l} \right)^2 \, dx \, d\theta + h(t, \tau, v_{\varepsilon, \tau})
\]
\[
+ l(t, \tau, v_{\varepsilon, \tau}) + \left( \right) \int_0^t \int_{\Omega_p^\delta} f_{\tau}(v_{\varepsilon, \tau}) \, dx \, d\theta \quad \text{for a.e. } t. \quad (4.2.101)
\]
Choosing \( \varsigma \leq \frac{4D(r-1)}{C_{33} r_2^2} \), this shows that \( \Lambda_\varepsilon + 1 \) is independent of \( \varepsilon, \lambda \) and \( \delta \). This gives
\[
\int_0^t \left\langle \partial_\theta v_{\varepsilon, \tau}, \partial f_{\tau}(v_{\varepsilon, \tau}) \right\rangle_{[H^{1,\alpha}(\Omega_p^\delta)]^{r_2} \times [H^{1,\alpha}(\Omega_p^\delta)]^{l_2}} \, d\theta
\]
\[
\leq h(t, \tau, v_{\varepsilon, \tau}) + l(t, \tau, v_{\varepsilon, \tau}) + \left( \right) \int_0^t \int_{\Omega_p^\delta} f_{\tau}(v_{\varepsilon, \tau}) \, dx \, d\theta
\]
\[35\] Where \( C_{33} = C_8 (C_{26} + C_{32}) \).
Now we use the theorem 4.2.1.3.5 to obtain

\[ \leq h(t, \tau, v_{\xi, \tau}) + l(t, \tau, v_{\xi, \tau}) + C_{34} \int_0^t \int_{\Omega^0} f_r(v_{\xi, \tau}) \, dx \, d\theta \]

\[ \leq h(t, \tau, v_{\xi, \tau}) + l(t, \tau, v_{\xi, \tau}) + C_{34} \int_0^t F_r(v_{\xi, \tau}) \, d\theta \text{ for a.e. } t. \quad (4.2.102) \]

where \( C_{34} := (C_{33} \Lambda_c + C_{32} |\nabla| + I_2 \rho \kappa(c(e-1)^{-1})) \), and \( h(t, \tau, v_{\xi, \tau}) \) and \( l(t, \tau, v_{\xi, \tau}) \) tend to zero as \( \tau \to 0 \) for a.e. \( t \).

**Proof of theorem 4.2.1.3.5.** Let \( v_{\xi, \tau} \) be a solution of the problem \( \tilde{P}^{2+}_{\xi, \lambda, \delta} \). Since we only know the nonnegativity of \( v_{\xi, \tau} \), let \( v_{\xi, \tau} := v_{\xi, \tau} + \tau \) for \( \tau > 0 \). Clearly \( v_{\xi, \tau} \in G^\omega_{\xi} \). Here also we introduce the regularization of \( v_{\xi, \tau} \) and replicating the steps of theorem 4.1.1.2.3, we obtain an inequality similar to (4.1.34), i.e.,

\[ |F_r(v_{\xi, \tau}(t)) - F_r(v_{\xi, \tau}(0))| \leq h(t, \tau, v_{\xi, \tau}) + l(t, \tau, v_{\xi, \tau}) + C_{34} \int_0^t F_r(v_{\xi, \tau}) \, d\theta \]

\[ \Rightarrow F_r(v_{\xi, \tau}(t)) - F_r(v_{\xi, \tau}(0)) \leq h(t, \tau, v_{\xi, \tau}) + l(t, \tau, v_{\xi, \tau}) + C_{34} \int_0^t F_r(v_{\xi, \tau}) \, d\theta \text{ for a.e. } t. \quad (4.2.103) \]

Since \( v_{\xi, \tau} \to v_{\xi} \) as \( \tau \to 0 \), \( h(t, \tau, v_{\xi, \tau}) \to 0 \) and \( l(t, \tau, v_{\xi, \tau}) \to 0 \) for a.e. \( t \). \( F_r(v_{\xi, \tau}) \) is continuous (cf. lemma 4.1.1.2.4), i.e., \( F_r(v_{\xi, \tau}) \to F_r(v_{\xi}) \) as \( \tau \to 0 \). Taking the limit on both sides of (4.2.103) as \( \tau \to 0 \), we get

\[ F_r(v_{\xi}(t)) - F_r(v_{\xi}(0)) \leq C_{34} \int_0^t F_r(v_{\xi}) \, d\theta \text{ for a.e. } t, \]

i.e.,

\[ F_r(v_{\xi}(t)) \leq F_r(v_{\xi}(0)) + C_{34} \int_0^t F_r(v_{\xi}) \, d\theta \text{ for a.e. } t. \]

Gronwall’s inequality gives

\[ F_r(v_{\xi}(t)) \leq e^{C_{34}t} F_r(v_{\xi}(0)) \text{ for a.e. } t. \quad (4.2.104) \]

where \( C_{34} \) is independent of \( \xi, \delta, \lambda, r \) and \( t \). This establishes the inequality (4.2.78).

Now we use the theorem 4.2.1.3.5 to obtain \( L^r \) - and \( L^\infty \) - estimates of the solution \( v_{\xi} \).

Let

\[ C_{36}(r) := C_{36} := \left[ I_2 \sup_{k, \xi, \tau > 0} \sup_{0 \leq t \leq T} \frac{|\Omega|}{C_{13}} \, C_{34} \left( 1 + \left( I_2^{1/2} \left| ||v_0||_{L^\infty(\Omega_t^0)} \right| \right)^{r(1+\alpha)} \right) \right]^{1/2} \quad (4.2.105) \]

and

\[ C_{37} := \sup_{\xi, \delta > 0} \left[ 1 + \left( I_2^{1/2} \left| ||v_0||_{L^\infty(\Omega_t^0)} \right| \right)^{1+\alpha} \right]. \quad (4.2.106) \]

**Corollary 4.2.1.3.7.** Let \( \tilde{u}_{\xi} \in F^\omega_{\xi} \) be fixed. For any arbitrary solution \( v_{\xi} \in G^\omega_{\xi} \) of \( \tilde{P}^{2+}_{\xi, \lambda, \delta} \) the following estimates hold true:

\[ \sup_{\xi, \delta > 0} ||v_{\xi}(t)||_{L^r(\Omega_t^0)} \leq C_{36} < \infty \text{ for all } r \text{ and for a.e. } t, \quad (4.2.107) \]

and

\[ \sup_{\xi, \delta > 0} ||v_{\xi}(t)||_{L^\infty(\Omega_t^0)} \leq C_{37} < \infty \text{ for a.e. } t. \quad (4.2.108) \]
Proof. Given \( r \in \mathbb{N} \) \((r \geq 2)\) and for the problem \((\tilde{P}^{2+}_{\Omega M})\), \(v_{\varepsilon}(0, x) = \lambda v_0(x)\). From inequality (4.2.104), we have
\[
F_r(v_{\varepsilon}(t)) \leq e^{C_{3M}t} F_r(v_{\varepsilon}(0)) \quad \text{for a.e. } t.
\]
A straightforward application of Gronwall’s inequality and arguments similar to the proof of corollary 4.1.1.2.8 yield the desired results.

\[\textbf{Corollary 4.2.1.3.8.} \text{Let the assumptions (4.2.1)-(4.2.10), } \hat{u}_{\varepsilon} \in F^w, \; 0 \leq \lambda \leq 1 \text{ and } r \in \mathbb{N} \text{ be satisfied. Then there exists a constant } C \text{ independent of } \hat{u}_{\varepsilon}, \; v_{\varepsilon}, \; \varepsilon, \; \lambda \text{ and } t \text{ such that any arbitrary solution } v_{\varepsilon} \in G^\nu_{\varepsilon} \text{ of the problem } (\tilde{P}^{2+}_{\Omega M}) \text{ satisfies}
\]
\[|||v_{\varepsilon}|||_{G^\nu_{\varepsilon}} \leq C. \quad (4.2.109)\]

\[\textbf{Proof.} \text{For } p > n+2, \; \hat{u}_{\varepsilon} \in L^\infty((0, T) \times \Omega^p_{\varepsilon})^I. \text{ Note that } v_{\varepsilon} \text{ satisfies the estimates (4.2.107) and (4.2.108). The abstract formulation of the problem (4.2.71)-(4.2.75) is given by}
\]
\[\frac{\partial v_{\varepsilon}}{\partial t} + Av_{\varepsilon} = f\text{bound}(v_{\varepsilon}) + f(v_{\varepsilon}), \quad (4.2.110)\]
\[v_{\varepsilon}(0, x) = v_0(x), \quad (4.2.111)\]

where the operator \(A\) is defined as in remark 4.1.1.1 with maximal regularity on \([H^{1,q}(\Omega^p_{\varepsilon})]^I\), \(f(v_{\varepsilon}) = \lambda S_2 R(\hat{u}_{\varepsilon}, v_{\varepsilon}) + \lambda \kappa v_{\varepsilon}\) and \(f\text{bound}(v_{\varepsilon}) = -\hat{q}_e - \nabla v_{\varepsilon} + Q_{\partial\Omega^p_{\varepsilon}}(v_{\varepsilon}) + R_{\Gamma}(\lambda \frac{\partial v_{\varepsilon}}{\partial n}). \text{ Choosing } r \text{ sufficiently large in (4.2.107) and application of Hölder’s inequality imply that } f \in L^p((0, T); H^{1,q}(\Omega^p_{\varepsilon}))^I. \text{ Since from lemma 4.2.1.3.1 } Q_{\partial\Omega^p_{\varepsilon}}(v_{\varepsilon}) \in L^p((0, T); H^{1,q}(\Omega^p_{\varepsilon}))^I, \text{ by lemma 4.2.1.3.2 } R_{\Gamma}(\lambda \frac{\partial v_{\varepsilon}}{\partial n}) \in L^p((0, T); H^{1,q}(\Omega^p_{\varepsilon}))^I \text{ and } -\hat{q}_e - \nabla v_{\varepsilon} \in L^p((0, T); H^{1,q}(\Omega^p_{\varepsilon}))^I, \text{ the term } f\text{bound} \text{ is in } L^p((0, T); H^{1,q}(\Omega^p_{\varepsilon}))^I. \text{ Moreover from (4.2.3), we have } v_0 \in [H^{1,q}(\Omega^p_{\varepsilon}), H^{1,p}(\Omega^p_{\varepsilon})]_{\frac{1}{p} - 1}^I. \text{ Therefore from theorem 3.3.1 there exists a unique } v_{\varepsilon} \in G^\nu_{\varepsilon} \text{ such that}
\]
\[|||v_{\varepsilon}|||_{G^\nu_{\varepsilon}} \leq C, \quad (4.2.112)\]
\[\text{where } C \text{ is independent of } \lambda \text{ and } v_{\varepsilon}. \]

\[\textbf{Lemma 4.2.1.3.9.} \text{The operator } Z_1 \text{ is compact and continuous.}\]

\[\textbf{Proof.} \text{We will only show the compactness of } Z_1 \text{ as the continuity follows analogously. Let } \hat{u}_{\varepsilon} \in F^w \text{ be fixed. Let } \{v_{\varepsilon n}\}_{n=1}^\infty \text{ be a bounded sequence in } G^\nu_{\varepsilon}. \text{ For } p > n+2, \; G^\nu_{\varepsilon} \hookrightarrow L^\infty((0, T) \times \Omega^p_{\varepsilon})^I. \text{ Then up to a subsequence (still denoted by same symbol) } \{\hat{v}_{\varepsilon n}\}_{n=1}^\infty \text{ is strongly convergent in } L^\infty((0, T) \times \Omega^p_{\varepsilon})^I. \text{ Therefore the r.h.s of the PDE}
\]
\[\frac{\partial v_{\varepsilon n}}{\partial t} - \nabla(D\nabla v_{\varepsilon n} - \hat{q}_e v_{\varepsilon n}) + \kappa v_{\varepsilon n} = S_2 R(\hat{u}_{\varepsilon}, \hat{v}_{\varepsilon n}) + \kappa \hat{v}_{\varepsilon n}\]
\[\text{is strongly convergent in } L^p((0, T); L^p(\Omega^p_{\varepsilon}))^I, \text{ i.e., in } L^p((0, T); H^{1,q}(\Omega^p_{\varepsilon}))^I. \text{ Thus by theorem 3.3.1, the sequence } \{v_{\varepsilon n}\}_{n=1}^\infty \text{ is strongly convergent in } G^\nu_{\varepsilon}. \]

\[\textbf{Proof of theorem 4.2.1.3.3.} \text{The compactness and continuity of the operator } Z_1 \text{ is shown in the lemma 4.2.1.3.9 and the corollary 4.2.1.3.8 gives the estimate (4.2.70). By Schaefer’s fixed point theorem the operator } Z_1 \text{ has a fixed point, i.e., the problem } (\tilde{P}^{2+}_{\Omega M}) \text{ has a solution. This solution is also a solution of } (\tilde{P}^{2+}_{\Omega M}). \]
4.2.1.4 Existence of the Global Solution of the Complete Problem ($P_{l,\gamma}^{\Delta^2}$)

Theorem 4.2.1.4.1. There exists a positive weak solution $(u_{\epsilon}, v_{\epsilon}, w_{\epsilon}) \in F_{\epsilon}^w \times G_{\epsilon}^w \times H_{\epsilon}^w$ of the following problem:

\[
\frac{\partial u_{\epsilon}}{\partial t} - \nabla \cdot (D \nabla u_{\epsilon} - \overline{q}_e u_{\epsilon}) = S_1 \tilde{R}(u_{\epsilon}, v_{\epsilon}) \quad \text{in } (0, T) \times \Omega_{\epsilon}^p, \quad u_{\epsilon}(0, x) = u_0(x) \quad \text{in } \Omega_{\epsilon}^p,
\]

\[
-(D \nabla u_{\epsilon} - \overline{q}_e u_{\epsilon}) \cdot \overline{n} = d \quad \text{on } (0, T) \times \partial \Omega_{in},
\]

\[
-D \nabla u_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_{out},
\]

\[
-D \nabla u_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \Gamma_{\epsilon}.
\]

\[
\frac{\partial v_{\epsilon}}{\partial t} - \nabla \cdot (D \nabla v_{\epsilon} - \overline{q}_e v_{\epsilon}) = S_2 \tilde{R}(u_{\epsilon}, v_{\epsilon}) \quad \text{in } (0, T) \times \Omega_{\epsilon}^p, \quad v_{\epsilon}(0, x) = v_0(x) \quad \text{in } \Omega_{\epsilon}^p,
\]

\[
-(D \nabla v_{\epsilon} - \overline{q}_e v_{\epsilon}) \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_{in},
\]

\[
-D \nabla v_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_{out},
\]

\[
-D \nabla v_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \Gamma_{\epsilon},
\]

\[
\frac{\partial w_{\epsilon}}{\partial t} = -k_\gamma \psi_\delta(w_{\epsilon}) \quad \text{on } (0, T) \times \Gamma_{\epsilon},
\]

\[
w_{\epsilon}(0, x) = w_0(x) \quad \text{on } \Gamma_{\epsilon}.
\]

The positivity of the solution has already been shown in lemma 4.2.1.2. To prove the existence of the global solution of the problem ($P_{l,\gamma}^{\Delta^2}$), we employ the similar techniques which we used to solve ($P_{l,\gamma}^{\Delta^2}$). Here also the basic ingredients are Schaefer’s fixed point theorem, a Lyapunov functional and theorem 3.3.1. The Lyapunov functional is defined in the following way: Let $\mu_0 \in \mathbb{R}^{l_1}$ be the solution of

\[
S_1^T \mu_0 = -\log K, \quad (4.2.113)
\]

where $K \in \mathbb{R}^d$ is the vector of equilibrium constants $K_j = \frac{h_j}{e_{\epsilon}^j}$. Due to (4.2.4), (4.2.113) has a solution. The function $g_l : \mathbb{R}_{l_1}^+ \to \mathbb{R}$ is defined by $g_l(u_{\epsilon}) := \left( \mu_0^l - 1 + \log u_{\epsilon} \right) u_{\epsilon} \varepsilon + \varepsilon (1 - \mu_0^l) u_{\epsilon} \varepsilon + \varepsilon (1 - \mu_0^l)$. We define the functions $g$, $f_r$ and $F_r$ in the similar way as we did in section 4.1.1.2 and their relevant properties hold good (see propositions 4.1.1.2.1 and 4.1.1.2.2). Now for technical reasons, we modify the right hand side of the first PDE in ($P_{l,\gamma}^{\Delta^2}$), i.e., for any $\kappa > 0$, we get

\[
\frac{\partial u_{\epsilon}}{\partial t} - \nabla \cdot (D \nabla u_{\epsilon} - \overline{q}_e u_{\epsilon}) + \kappa u_{\epsilon} = S_1 \tilde{R}(u_{\epsilon}, v_{\epsilon}) + \kappa u_{\epsilon} \quad \text{in } (0, T) \times \Omega_{\epsilon}^p, \quad u_{\epsilon}(0, x) = u_0(x) \quad \text{in } \Omega_{\epsilon}^p,
\]

\[
-(D \nabla u_{\epsilon} - \overline{q}_e u_{\epsilon}) \cdot \overline{n} = d \quad \text{on } (0, T) \times \partial \Omega_{in},
\]

\[
-D \nabla u_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_{out},
\]

\[
-D \nabla u_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \Gamma_{\epsilon}.
\]

\[
\frac{\partial v_{\epsilon}}{\partial t} - \nabla \cdot (D \nabla v_{\epsilon} - \overline{q}_e v_{\epsilon}) = S_2 \tilde{R}(u_{\epsilon}, v_{\epsilon}) \quad \text{in } (0, T) \times \Omega_{\epsilon}^p, \quad v_{\epsilon}(0, x) = v_0(x) \quad \text{in } \Omega_{\epsilon}^p,
\]

\[
-(D \nabla v_{\epsilon} - \overline{q}_e v_{\epsilon}) \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_{in},
\]

\[
-D \nabla v_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_{out},
\]

\[
-D \nabla v_{\epsilon} \cdot \overline{n} = 0 \quad \text{on } (0, T) \times \Gamma_{\epsilon},
\]

\[
\frac{\partial w_{\epsilon}}{\partial t} = -k_\gamma \psi_\delta(w_{\epsilon}) \quad \text{on } (0, T) \times \Gamma_{\epsilon},
\]

\[
w_{\epsilon}(0, x) = w_0(x) \quad \text{on } \Gamma_{\epsilon}.
\]
Chapter 4. Existence of a Unique Global Solution and Homogenization

\[
\frac{\partial w_{\varepsilon\delta}}{\partial t} = -k_d \psi_{\delta}(w_{\varepsilon\delta}) \quad \text{on } (0,T) \times \Gamma_{\varepsilon},
\]  
(4.2.124)

\[
w_{\varepsilon\delta}(0,x) = w_0(x) \quad \text{on } \Gamma_{\varepsilon}.
\]  
(4.2.125)

Let us denote this problem by \((P^{2+}_{\varepsilon\delta M})\). We define the fixed point operator \(Z_2 : F_{\varepsilon}^u \rightarrow F_{\varepsilon}^u\) via \(Z_2(\hat{u}_{\varepsilon\delta}) := u_{\varepsilon\delta}\), where \(u_{\varepsilon\delta}\) is the solution of the following linear problem

\[
\frac{\partial u_{\varepsilon\delta}}{\partial t} - \nabla \cdot (D\nabla u_{\varepsilon\delta} - \vec{q}_{\varepsilon\delta} u_{\varepsilon\delta}) + \kappa u_{\varepsilon\delta} = \bar{S}_1 \hat{R}(\hat{u}_{\varepsilon\delta}, v_{\varepsilon\delta}) + \kappa \hat{u}_{\varepsilon\delta} \quad \text{in } (0,T) \times \Omega_{\varepsilon}^p,
\]  
(4.2.126)

\[
u_{\varepsilon\delta}(0,x) = u_0(x) \quad \text{in } \Omega_{\varepsilon}^p,
\]  
(4.2.127)

\[-(D\nabla u_{\varepsilon\delta} - \vec{q}_{\varepsilon\delta} u_{\varepsilon\delta}) \cdot \vec{n} = d \quad \text{on } (0,T) \times \partial\Omega_{\varepsilon}^{in},
\]  
(4.2.128)

\[-D\nabla u_{\varepsilon\delta} \cdot \vec{n} = 0 \quad \text{on } (0,T) \times \partial\Omega_{\varepsilon}^{out},\]

\[-D\nabla v_{\varepsilon\delta} \cdot \vec{n} = 0 \quad \text{on } (0,T) \times \partial\Omega_{\varepsilon}^{in},\]

\[-D\nabla v_{\varepsilon\delta} \cdot \vec{n} = 0 \quad \text{on } (0,T) \times \Gamma_{\varepsilon},
\]  
(4.2.130)

where \(v_{\varepsilon\delta} \in G_{\varepsilon}^v\) is the solution of the problem

\[
\frac{\partial v_{\varepsilon\delta}}{\partial t} - \nabla \cdot (D\nabla v_{\varepsilon\delta} - \vec{q}_{\varepsilon\delta} v_{\varepsilon\delta}) = \bar{S}_2 \hat{R}(\hat{u}_{\varepsilon\delta}, v_{\varepsilon\delta}) \quad \text{in } (0,T) \times \Omega_{\varepsilon}^p,
\]  
(4.2.131)

\[
v_{\varepsilon\delta}(0,x) = v_0(x) \quad \text{in } \Omega_{\varepsilon}^p,
\]  
(4.2.132)

\[-(D\nabla v_{\varepsilon\delta} - \vec{q}_{\varepsilon\delta} v_{\varepsilon\delta}) \cdot \vec{n} = 0 \quad \text{on } (0,T) \times \partial\Omega_{\varepsilon}^{in},
\]  
(4.2.133)

\[-D\nabla v_{\varepsilon\delta} \cdot \vec{n} = 0 \quad \text{on } (0,T) \times \partial\Omega_{\varepsilon}^{out},\]

\[-D\nabla v_{\varepsilon\delta} \cdot \vec{n} = 0 \quad \text{on } (0,T) \times \Gamma_{\varepsilon},
\]  
(4.2.134)

and \(w_{\varepsilon\delta} \in H_{\varepsilon}^w\) is the solution of the problem

\[
\frac{\partial w_{\varepsilon\delta}}{\partial t} = -k_d \psi_{\delta}(w_{\varepsilon\delta}) \quad \text{on } (0,T) \times \Gamma_{\varepsilon},
\]  
(4.2.136)

\[
w_{\varepsilon\delta}(0,x) = w_0(x) \quad \text{on } \Gamma_{\varepsilon}.
\]  
(4.2.137)

Note that for fixed \(\hat{u}_{\varepsilon\delta}\) the problem \((4.2.131)-(4.2.137)\) has a solution and satisfies the estimates \((4.2.107)-(4.2.108)\). The operator \(Z_2\) is well-defined (can be verified as in remark 4.2.1.3.4). Now in order to apply the Schaefer’s fixed point theorem, we show the following two condition:

(i) The operator \(Z_2\) is continuous and compact.

(ii) The set \(\{u_{\varepsilon\delta} \in F_{\varepsilon}^u | \exists \lambda \in [0,1] : u_{\varepsilon\delta} = \lambda Z_2(u_{\varepsilon\delta})\}\) is bounded, i.e., there exists a constant \(C > 0\) independent of \(u_{\varepsilon\delta}\) and \(\lambda\) such that any arbitrary solution \(u_{\varepsilon\delta} \in F_{\varepsilon}^u\) of the equation

\[
u_{\varepsilon\delta} = \lambda Z_2(u_{\varepsilon\delta})
\]  
(4.2.138)

satisfies

\[\|\|u_{\varepsilon\delta}\|\|_{F_{\varepsilon}^u} \leq C.\]  
(4.2.139)
Combining (4.2.126)-(4.2.137) and (4.2.138), we obtain
\[
\frac{\partial u_{\varepsilon}}{\partial t} - \nabla \cdot (D \nabla u_{\varepsilon} - \overline{q}_\varepsilon u_{\varepsilon}) + \kappa u_{\varepsilon} = \lambda S_1 \overline{R}(u_{\varepsilon}, v_{\varepsilon}) + \lambda \kappa u_{\varepsilon} \quad \text{in} \quad (0, T) \times \Omega^p_{\varepsilon}, \tag{4.2.140}
\]
\[
u_{\varepsilon}(0, x) = \lambda u_0(x) \quad \text{in} \quad \Omega^p_{\varepsilon}, \tag{4.2.141}
\]
\[-(D \nabla u_{\varepsilon} - \overline{q}_\varepsilon u_{\varepsilon}) \cdot \vec{n} = \lambda d \quad \text{on} \quad (0, T) \times \partial \Omega_{\text{in}}, \tag{4.2.142}
\]
\[-D \nabla u_{\varepsilon} \cdot \vec{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{\text{out}}, \tag{4.2.143}
\]
\[-D \nabla u_{\varepsilon} \cdot \vec{n} = 0 \quad \text{on} \quad (0, T) \times \Gamma_{\varepsilon}, \tag{4.2.144}
\]
where \( v_{\varepsilon} \in G^w_{\varepsilon} \) is the solution of the problem
\[
\frac{\partial v_{\varepsilon}}{\partial t} - \nabla \cdot (D \nabla v_{\varepsilon} - \overline{q}_\varepsilon v_{\varepsilon}) = S_2 \overline{R}(u_{\varepsilon}, v_{\varepsilon}) \quad \text{in} \quad (0, T) \times \Omega^p_{\varepsilon}, \tag{4.2.145}
\]
\[
v_{\varepsilon}(0, x) = v_0(x) \quad \text{in} \quad \Omega^p_{\varepsilon}, \tag{4.2.146}
\]
\[-(D \nabla v_{\varepsilon} - \overline{q}_\varepsilon v_{\varepsilon}) \cdot \vec{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{\text{in}}, \tag{4.2.147}
\]
\[-D \nabla v_{\varepsilon} \cdot \vec{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{\text{out}}, \tag{4.2.148}
\]
\[-D \nabla v_{\varepsilon} \cdot \vec{n} = \varepsilon \frac{\partial w_{\varepsilon}}{\partial t} \quad \text{on} \quad (0, T) \times \Gamma_{\varepsilon}, \tag{4.2.149}
\]
and \( w_{\varepsilon} \in H^w_{\varepsilon} \) is the solution of the problem
\[
\frac{\partial w_{\varepsilon}}{\partial t} = -k_\varepsilon \psi_\varepsilon(w_{\varepsilon}) \quad \text{on} \quad (0, T) \times \Gamma_{\varepsilon}, \tag{4.2.150}
\]
\[
w_{\varepsilon}(0, x) = w_0(x) \quad \text{on} \quad \Gamma_{\varepsilon}. \tag{4.2.151}
\]
Let us call the problem (4.2.140)-(4.2.151) as \((P^2_{\varepsilon, \lambda, M})\). The inequality (4.2.139) is the consequence of the following three results:

**Lemma 4.2.1.4.2.** Let \( p > n + 2, 0 \leq \lambda \leq 1 \) and \( r \in \mathbb{N} \; (r \geq 2) \). Assume that \( u_{\varepsilon} \in F^u_{\varepsilon} \) is a solution of \((P^2_{\varepsilon, \lambda, M})\) and for \( \tau > 0 \),
\[
u_{\varepsilon, \tau} := u_{\varepsilon} + \tau.
\]

Then the following inequality holds:
\[
\int_0^t \left\langle \frac{\partial u_{\varepsilon, \tau}}{\partial \theta}, \partial f_r(u_{\varepsilon, \tau}) \right\rangle_{[H^{1,q}(\Omega^p_{\varepsilon})]^1 \times H^{1,q}(\Omega^p_{\varepsilon})} d\theta \leq h(t, \tau, u_{\varepsilon, \tau}) + l(t, \tau, u_{\varepsilon, \tau}) + C_{38} \int_0^t F_r(u_{\varepsilon, \tau}) d\theta \quad \text{for a.e. } t,
\]
where \( h(t, \tau, u_{\varepsilon, \tau}) \) and \( l(t, \tau, u_{\varepsilon, \tau}) \) tend to zero as \( \tau \to 0 \) for a.e. \( t \), and \( C_{38} \) is independent of \( \varepsilon, \delta, \lambda \) and \( t \).

**Proof.** Note that for \( u_{\varepsilon} \in F^u_{\varepsilon} \), the problem (4.2.145)-(4.2.151) has a solution \((v_{\varepsilon}, w_{\varepsilon}) \in G^w_{\varepsilon} \times H^w_{\varepsilon}\) with estimates (4.2.107)-(4.2.108). We use \( \partial f_r(u_{\varepsilon}) \in L^q((0, T); H^{1,q}(\Omega^p_{\varepsilon})) \) as the test function in the weak formulation of (4.2.140). Replicating the steps of lemma 4.2.1.3.6 and use of (4.2.10) will finish the proof.

**Theorem 4.2.1.4.3.** Let \( r \in \mathbb{N} \; (r \geq 2) \), \( 0 \leq t \leq T \) and \( 0 \leq \lambda \leq 1 \). Suppose that \( u_{\varepsilon} \in F^u_{\varepsilon} \) is a solution of \((P^2_{\varepsilon, \lambda, M})\). Then the following inequality holds good:
\[
F_r(u_{\varepsilon}(t)) \leq e^{C_{38} t} F_r(u_{\varepsilon}(0)) \quad \text{for a.e. } t, \tag{4.2.152}
\]
where \( C_{38} \) is independent of \( \varepsilon, \delta, \lambda \) and \( t \).
**Proof.** The proof follows from lemma 4.2.1.4.2 by using arguments similar to the proof of theorem 4.2.1.3.5.

**Corollary 4.2.1.4.4.** For any arbitrary solution $u_{\varepsilon} \in \mathcal{F}_{\varepsilon}^u$ of $(P_{\varepsilon, \Delta M}^{2+})$ the following estimates hold true:

$$||u_{\varepsilon}(t)||_{L^r(\Omega^p_{\varepsilon})} \leq C_{39} < \infty \quad \text{for all } r \text{ and for a.e. } t,$$

and

$$||u_{\varepsilon}(t)||_{L^\infty(\Omega^p_{\varepsilon})} \leq C_{40} < \infty \quad \text{for a.e. } t,$$

where $C_{39}$ and $C_{40}$ are independent of $\varepsilon$, $\delta$, $\lambda$ and $t$.

**Proof.** By using arguments from the proof of corollary 4.2.1.3.7 in (4.2.152) yield the proof.

**Corollary 4.2.1.4.5.** Let the assumptions (4.2.1)-(4.2.10), $0 \leq \lambda \leq 1$ and $r \in \mathbb{N}$ be satisfied. Then there exists a constant $C$ independent of $u_{\varepsilon}$, $\varepsilon$, $\delta$, $\lambda$ and $t$ such that any arbitrary solution $u_{\varepsilon} \in \mathcal{F}_{\varepsilon}^u$ of the problem $(P_{\varepsilon, \Delta M}^{2+})$ satisfies

$$||u_{\varepsilon}||_{\mathcal{F}_{\varepsilon}^u} \leq C.$$

**Proof.** The proof is analogous to the proof of corollary 4.2.1.3.8.

**Lemma 4.2.1.4.5.** The operator $Z_2$ is compact and continuous.

**Proof.** Here we shall prove only the compactness of $Z_2$. Let $\{u_{\varepsilon,n}\}_{n=1}^\infty$ be a bounded sequence in $\mathcal{F}_{\varepsilon}^u$. The proof will be done if we can show that up to a subsequence the r.h.s of the PDE

$$\frac{\partial u_{\varepsilon,n}}{\partial t} - \nabla \cdot (\nabla u_{\varepsilon,n} - \overline{q}_{\varepsilon} u_{\varepsilon,n}) + \kappa u_{\varepsilon,n} = S_1 \bar{R}(\bar{u}_{\varepsilon,n}, v_{\varepsilon,n}) + \kappa \bar{u}_{\varepsilon,n}$$

is strongly convergent in $L^p((0,T);H^{1,\alpha}(\Omega^p_{\varepsilon}))$, where $v_{\varepsilon,n} \in \mathcal{G}_{\varepsilon}^p$ is the solution of the problem

$$\frac{\partial v_{\varepsilon,n}}{\partial t} - \nabla \cdot (D\nabla v_{\varepsilon,n} - \overline{q}_{\varepsilon} v_{\varepsilon,n}) = S_2 \bar{R}(\bar{u}_{\varepsilon,n}, v_{\varepsilon,n}) \quad \text{in } (0,T) \times \Omega^p_{\varepsilon},$$

$$v_{\varepsilon,n}(0,x) = v_0(x) \quad \text{in } \Omega^p_{\varepsilon},$$

$$-(D\nabla v_{\varepsilon,n} - \overline{q}_{\varepsilon} v_{\varepsilon,n}) \cdot \bar{n} = 0 \quad \text{on } (0,T) \times \partial \Omega_{in},$$

$$-D\nabla v_{\varepsilon,n} \cdot \bar{n} = 0 \quad \text{on } (0,T) \times \partial \Omega_{out},$$

$$-D\nabla v_{\varepsilon,n} \cdot \bar{n} = \varepsilon \frac{\partial w_{\varepsilon,n}}{\partial t} \quad \text{on } (0,T) \times \Gamma_{\varepsilon},$$

with estimate (4.2.109) and $w_{\varepsilon,n} \in \mathcal{H}_{\varepsilon}^w$ is the solution of the problem

$$\frac{\partial w_{\varepsilon,n}}{\partial t} = -k_d \psi(g(w_{\varepsilon,n})) \quad \text{on } (0,T) \times \Gamma_{\varepsilon},$$

$$w_{\varepsilon,n}(0,x) = w_0(x) \quad \text{on } \Gamma_{\varepsilon}.$$

Thus the sequence $\{v_{\varepsilon,n}\}_{n=1}^\infty$ is bounded in $\mathcal{G}_{\varepsilon}^p$. Since $\mathcal{F}_{\varepsilon}^p, \mathcal{G}_{\varepsilon}^p \hookrightarrow L^\infty((0,T) \times \Omega^p_{\varepsilon})$, up to a subsequence (still denoted by same symbol), $\{u_{\varepsilon,n}\}_{n=1}^\infty$ and $\{v_{\varepsilon,n}\}_{n=1}^\infty$ are strongly convergent in $L^\infty((0,T) \times \Omega^p_{\varepsilon})$ and this yields the strong convergence of the r.h.s of the PDE (4.2.156) in $L^p((0,T);H^{1,\alpha}(\Omega^p_{\varepsilon}))$.  

$\blacksquare$
4.2. Model M2

Proof of theorem 4.2.1.4.1. The corollary 4.2.1.4.5 and lemma 4.2.1.4.5 show that the conditions of Schaefer’s fixed point theorem are satisfied. Hence there exists at least one fixed point of \(Z_2\), i.e., the problem \((P^2_{\varepsilon_M})\) has a solution \((u_{\varepsilon^1}, v_{\varepsilon^1}, w_{\varepsilon^1}) \in \mathcal{F}_{\varepsilon} \times \mathcal{G}_{\varepsilon} \times \mathcal{H}_{\varepsilon}^w\). The solution of \((P^2_{\varepsilon_M})\) is also a solution of \((P^2_{\varepsilon})\).

Proof of theorem 4.2.1.1.1. Since in lemma 4.2.1.1.2 we have shown that the solution of \((P^2_{\varepsilon})\) is nonnegative, the solution also solves the problem \((P^2_{\varepsilon})\). In the next section, we prove the uniqueness of the solution of \((P^2_{\varepsilon})\).

4.2.1.5 Uniqueness of the Solution of the Problem \((P^2_{\varepsilon})\)

Theorem 4.2.1.5.1. There exists a unique positive global solution \((u_{\varepsilon^1}, v_{\varepsilon^1}, w_{\varepsilon^1}) \in \mathcal{F}_{\varepsilon} \times \mathcal{G}_{\varepsilon} \times \mathcal{H}_{\varepsilon}^w\) of the problem \((P^2_{\varepsilon})\).

Proof. On the contrary, let us assume that \((u_{\varepsilon^1,1}, v_{\varepsilon^1,1}, w_{\varepsilon^1,1})\) and \((u_{\varepsilon^1,2}, v_{\varepsilon^1,2}, w_{\varepsilon^1,2})\) be the solutions of the problem \((P^2_{\varepsilon})\). Set \(\bar{u}_{\varepsilon} := u_{\varepsilon,1} - u_{\varepsilon,2}\), \(\bar{v}_{\varepsilon} := v_{\varepsilon,1} - v_{\varepsilon,2}\) and \(\bar{w}_{\varepsilon} := w_{\varepsilon,1} - w_{\varepsilon,2}\). Let the systems satisfied by \((u_{\varepsilon,1}, v_{\varepsilon,1}, w_{\varepsilon,1})\) and \((u_{\varepsilon,2}, v_{\varepsilon,2}, w_{\varepsilon,2})\) be denoted by \((P^2_{\varepsilon^1})\) and \((P^2_{\varepsilon^2})\) respectively. Subtracting the systems of equations of \((P^2_{\varepsilon^1})\) and \((P^2_{\varepsilon^2})\), we get

\[
\begin{align*}
\frac{\partial \bar{u}_{\varepsilon}}{\partial t} - \nabla \cdot (D \nabla \bar{u}_{\varepsilon} - \bar{q}_e \bar{u}_{\varepsilon}) = & \left(S_1 R(u_{\varepsilon,1}, v_{\varepsilon,1}, w_{\varepsilon,1}) - S_1 R(u_{\varepsilon,2}, v_{\varepsilon,2}, 2)\right) \text{ in } (0, T) \times \Omega_{\varepsilon}^p, \quad (4.2.164) \\
\bar{u}_{\varepsilon}(0) = & 0 \quad \text{in } \Omega_{\varepsilon}^p, \quad (4.2.165) \\
- (D \nabla \bar{u}_{\varepsilon} - \bar{q}_e \bar{u}_{\varepsilon}) \cdot \tilde{n} = & 0 \quad \text{on } (0, T) \times \partial \Omega_{\varepsilon}, \quad (4.2.166) \\
- D \nabla \bar{u}_{\varepsilon} \cdot \tilde{n} = & 0 \quad \text{on } (0, T) \times \partial \Omega_{\varepsilon}^o, \quad (4.2.167) \\
- D \nabla \bar{u}_{\varepsilon} \cdot \tilde{n} = & 0 \quad \text{on } (0, T) \times \Gamma_{\varepsilon}, \quad (4.2.168)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial \bar{v}_{\varepsilon}}{\partial t} - \nabla \cdot (D \nabla \bar{v}_{\varepsilon} - \bar{q}_e \bar{v}_{\varepsilon}) = & \left(S_2 R(u_{\varepsilon,1}, v_{\varepsilon,1}, w_{\varepsilon,1}) - S_2 R(u_{\varepsilon,2}, v_{\varepsilon,2}, 2)\right) \text{ in } (0, T) \times \Omega_{\varepsilon}^p, \quad (4.2.169) \\
\bar{v}_{\varepsilon}(0) = & 0 \quad \text{in } \Omega_{\varepsilon}^p, \quad (4.2.170) \\
- (D \nabla \bar{v}_{\varepsilon} - \bar{q}_e \bar{v}_{\varepsilon}) \cdot \tilde{n} = & 0 \quad \text{on } (0, T) \times \partial \Omega_{\varepsilon}, \quad (4.2.171) \\
- D \nabla \bar{v}_{\varepsilon} \cdot \tilde{n} = & 0 \quad \text{on } (0, T) \times \partial \Omega_{\varepsilon}^o, \quad (4.2.172) \\
- D \nabla \bar{v}_{\varepsilon} \cdot \tilde{n} = & \bar{w}_{\varepsilon} \frac{d \bar{w}_{\varepsilon}}{d t} \quad \text{on } (0, T) \times \Gamma_{\varepsilon}, \quad (4.2.173) \\
\frac{\partial \bar{w}_{\varepsilon}}{\partial t} = & - k_d \psi_{\beta}(w_{\varepsilon,1}) - \psi_{\beta}(w_{\varepsilon,2}) \quad \text{on } (0, T) \times \Gamma_{\varepsilon}, \quad (4.2.174) \\
\bar{w}_{\varepsilon}(0) = & 0 \quad \text{on } \Gamma_{\varepsilon}. \quad (4.2.175)
\end{align*}
\]

(i) Uniqueness of the ODE (4.2.42)-(4.2.43): Here \(\bar{w}_{\varepsilon} := w_{\varepsilon,1} - w_{\varepsilon,2} \in H^{1,p}(0, T); L^p(\Gamma_{\varepsilon})\). We multiply the ODE (4.2.174) with \(\bar{w}_{\varepsilon}\) and integrate over \((0, t) \times \Gamma_{\varepsilon}\). Employing the Lipschitz continuity of \(\psi_{\beta}\) and a straightforward application of Gronwall’s inequality yield the desired result.

(ii) Uniqueness of the PDE (4.2.32)-(4.2.36) and (4.2.37)-(4.2.41): Testing the equation
(4.2.169) with $\bar{v}_{\varepsilon k}$, we obtain

$$
\frac{1}{2} \sum_{k=1}^{I_2} \int_{0}^{t} \frac{d}{d\theta} \left| \bar{v}_{\varepsilon k} (t) \right|_{L^2(\Omega^p)}^2 d\theta + D \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \left| \nabla \bar{v}_{\varepsilon k} \right|^2 dx d\theta
$$

$$
+ \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \bar{q}_e \nabla \bar{v}_{\varepsilon k} \bar{v}_{\varepsilon k} dx d\theta
$$

$$
- \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\partial \Omega m} \bar{q}_e \bar{v}_{\varepsilon k}^2 ds d\theta + \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Gamma_x} \varepsilon \frac{\partial \bar{v}_{\varepsilon k}}{\partial t} \bar{v}_{\varepsilon k} d\sigma_x d\theta
$$

$$
= \sum_{k=1}^{I_2} \int_{0}^{t} \left( S_2 R (u_{\varepsilon,1}, v_{\varepsilon,1})_k - S_2 R (u_{\varepsilon,2}, v_{\varepsilon,2})_k, v_{\varepsilon,1} - v_{\varepsilon,2} \right) d\theta,
$$

i.e.,

$$
\frac{1}{2} \sum_{k=1}^{I_2} \int_{0}^{t} \frac{d}{d\theta} \left| \bar{v}_{\varepsilon k} (t) \right|_{L^2(\Omega^p)}^2 d\theta + D \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \left| \nabla \bar{v}_{\varepsilon k} \right|^2 dx d\theta
$$

$$
= \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\partial \Omega m} \bar{q}_e \bar{v}_{\varepsilon k}^2 ds d\theta - \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Gamma_x} \varepsilon \frac{\partial \bar{v}_{\varepsilon k}}{\partial t} \bar{v}_{\varepsilon k} d\sigma_x d\theta
$$

$$
- \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \bar{q}_e \nabla \bar{v}_{\varepsilon k} \bar{v}_{\varepsilon k} dx d\theta
$$

$$
=: I_{advect}
$$

$$
+ \sum_{k=1}^{I_2} \int_{0}^{t} \left( S_2 R (u_{\varepsilon,1}, v_{\varepsilon,1})_k - S_2 R (u_{\varepsilon,2}, v_{\varepsilon,2})_k, v_{\varepsilon,1} - v_{\varepsilon,2} \right) d\theta.
$$

(4.2.176)

We simplify the boundary, advective and reaction terms separately. We start with the advective term.

$$
I_{advect} = - \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \bar{q}_e \nabla \bar{v}_{\varepsilon k} \bar{v}_{\varepsilon k} dx d\theta
$$

$$
\leq \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \bar{q}_e \left| \nabla \bar{v}_{\varepsilon k} \right| \bar{v}_{\varepsilon k} \left| dx d\theta \right|
$$

$$
\leq Q \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \left| \nabla \bar{v}_{\varepsilon k} \right| \bar{v}_{\varepsilon k} \left| \left| dx d\theta \right|, \text{where } Q = \left| \bar{q}_e \right|_{L^\infty((0,T) \times \Omega^p)}
$$

$$
\leq \frac{2Q^2}{D} \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \bar{v}_{\varepsilon k}^2 dx d\theta + \frac{D}{8} \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \left| \nabla \bar{v}_{\varepsilon k} \right|^2 dx d\theta
$$

$$
\leq C \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \bar{v}_{\varepsilon k}^2 dx d\theta + \frac{D}{8} \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Omega^p} \left| \nabla \bar{v}_{\varepsilon k} \right|^2 dx d\theta.
$$

(4.2.177)

Next we simplify the boundary term.

$$
I_{bound} = \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\partial \Omega m} \bar{q}_e \cdot \bar{n} \left| \bar{v}_{\varepsilon k} \right|^2 ds d\theta - \sum_{k=1}^{I_2} \int_{0}^{t} \int_{\Gamma_x} \varepsilon \frac{\partial \bar{v}_{\varepsilon k}}{\partial t} \bar{v}_{\varepsilon k} d\sigma_x d\theta.
$$
By part (i), \( w_{\varepsilon,1}(t,x) = w_{\varepsilon,2}(t,x) \) for a.e. \( t \) and \( x \). This implies that the boundary term on \( \Gamma \) vanishes. On \( \partial \Omega_m, \vec{q}_\varepsilon \cdot \vec{n} \leq 0 \). Thus the integrand on \( \partial \Omega_m \) is nonpositive. Therefore

\[
I_{\text{bound}} \leq 0. \tag{4.2.178}
\]

Finally, we simplify the reaction term.

\[
I_{\text{reac}} = \sum_{k=1}^{I_2} \int_0^t \left( S_2 R(u_{\varepsilon,1}, \varepsilon_{\varepsilon,1})_k - S_2 R(u_{\varepsilon,2}, \varepsilon_{\varepsilon,2})_k, \varepsilon_{\varepsilon,1} - \varepsilon_{\varepsilon,2} \right) \, d\theta
\]

\[
\leq \frac{1}{2} \sum_{k=1}^{I_2} \int_0^t \left[ \left| S_2 R(u_{\varepsilon,1}, \varepsilon_{\varepsilon,1})_k - S_2 R(u_{\varepsilon,2}, \varepsilon_{\varepsilon,2})_k \right|^2_{L^2(\Omega_p^e)} + \left| \varepsilon_{\varepsilon,1} - \varepsilon_{\varepsilon,2} \right|^2_{L^2(\Omega_p^e)} \right] \, d\theta.
\tag{4.2.179}
\]

Note that

\[
\left| S_2 R(u_{\varepsilon,1}, \varepsilon_{\varepsilon,1}) - S_2 R(u_{\varepsilon,2}, \varepsilon_{\varepsilon,2}) \right|^2_{L^2(\Omega_p^e)} \leq \int_{\Omega_p^e} \left( \sum_{j=1}^J |v_{kj}| \left| R_j(u_{\varepsilon,1}, \varepsilon_{\varepsilon,1}) - R_j(u_{\varepsilon,2}, \varepsilon_{\varepsilon,2}) \right| \right)^2 \, dx.
\]

Expanding the term \( R_j(u_{\varepsilon,1}, \varepsilon_{\varepsilon,1}) - R_j(u_{\varepsilon,2}, \varepsilon_{\varepsilon,2}) \), we will obtain two terms in which each term contains a factor of the type \( u_{\varepsilon,1} - u_{\varepsilon,2} \) and \( \varepsilon_{\varepsilon,1} - \varepsilon_{\varepsilon,2} \) whereas all the other factors are in \( L^\infty((0,T) \times \Omega_p^e) \). Therefore we obtain

\[
\left| S_2 R(u_{\varepsilon,1}, \varepsilon_{\varepsilon,1}) - S_2 R(u_{\varepsilon,2}, \varepsilon_{\varepsilon,2}) \right|^2_{L^2(\Omega_p^e)} \leq \tilde{C} \left[ \sum_{i=1}^{I_1} \int_{\Omega_e^p} \left| u_{\varepsilon,1} - u_{\varepsilon,2} \right|^2 \, dx + \sum_{k=1}^{I_2} \int_{\Omega_p^e} \left| \varepsilon_{\varepsilon,1} - \varepsilon_{\varepsilon,2} \right|^2 \, dx \right]. \tag{4.2.180}
\]

Combining (4.2.176), (4.2.177), (4.2.178), (4.2.179) and (4.2.180), we obtain

\[
\frac{1}{2} \sum_{k=1}^{I_2} \int_0^t \frac{d}{d\theta} \left| \tilde{\varepsilon}_{\varepsilon,k} (t) \right|^2_{L^2(\Omega_p^e)} \, d\theta + D \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_p^e} \left| \nabla \tilde{\varepsilon}_{\varepsilon,k} \right|^2 \, dx \, d\theta
\]

\[
= I_{\text{bound}} + I_{\text{advec}} + I_{\text{reac}}
\]

\[
\leq 0 + \tilde{C} \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_p^e} \left| \tilde{\varepsilon}_{\varepsilon,k} \right|^2 \, dx \, d\theta + \frac{D}{2} \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_p^e} \left| \nabla \tilde{\varepsilon}_{\varepsilon,k} \right|^2 \, dx \, d\theta
\]

\[
+ C \left[ \int_0^t \int_{\Omega_e^p} \sum_{i=1}^{I_1} \left| \tilde{u}_{\varepsilon,i} \right|^2 \, dx \, d\theta + \int_0^t \int_{\Omega_p^e} \sum_{k=1}^{I_2} \left| \tilde{\varepsilon}_{\varepsilon,k} \right|^2 \, dx \, d\theta \right] \tag{4.2.181}
\]

\[
\Rightarrow \frac{1}{2} \sum_{k=1}^{I_2} \int_0^t \frac{d}{d\theta} \left| \tilde{\varepsilon}_{\varepsilon,k} (t) \right|^2_{L^2(\Omega_p^e)} \, d\theta \leq \tilde{C} \left( \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_e^p} \left| \tilde{\varepsilon}_{\varepsilon,k} \right|^2 \, dx \, d\theta + \sum_{i=1}^{I_1} \int_0^t \int_{\Omega_p^e} \left| \tilde{u}_{\varepsilon,i} \right|^2 \, dx \, d\theta \right). \tag{4.2.182}
\]

Now we test the equation (4.2.164) by \( \tilde{u}_{\varepsilon,i} \) and proceed in the similar fashion as above, we obtain an inequality like (4.2.181) as

\[
\frac{1}{2} \sum_{i=1}^{I_1} \int_0^t \frac{d}{d\theta} \left| \tilde{u}_{\varepsilon,i} (t) \right|^2_{L^2(\Omega_e^p)} \, d\theta \leq \tilde{C} \left( \sum_{k=1}^{I_2} \int_0^t \int_{\Omega_e^p} \left| \tilde{\varepsilon}_{\varepsilon,k} \right|^2 \, dx \, d\theta + \sum_{i=1}^{I_1} \int_0^t \int_{\Omega_p^e} \left| \tilde{u}_{\varepsilon,i} \right|^2 \, dx \, d\theta \right). \tag{4.2.182}
\]
Adding (4.2.181) and (4.2.182), we get
\[
\frac{1}{2} \int_0^t \frac{d}{d\theta} \left( |\tilde{u}_{j\delta}(t)|^2_{L^2(\Omega^e)} + |\tilde{v}_{j\delta}(t)|^2_{L^2(\Omega^e)} \right) d\theta \leq \bar{C}_3 \int_0^t \left( |\tilde{u}_{j\delta}(t)|^2_{L^2(\Omega^e)} + |\tilde{v}_{j\delta}(t)|^2_{L^2(\Omega^e)} \right) d\theta,
\]
i.e.,
\[
|\tilde{u}_{j\delta}(t)|^2_{L^2(\Omega^e)} + |\tilde{v}_{j\delta}(t)|^2_{L^2(\Omega^e)} \leq 2\bar{C}_3 \int_0^t \left( |\tilde{u}_{j\delta}(\theta)|^2_{L^2(\Omega^e)} + |\tilde{v}_{j\delta}(\theta)|^2_{L^2(\Omega^e)} \right) d\theta.
\]
Since \(u_{j\delta,1}(0) = u_{j\delta,2}(0)\) and \(v_{j\delta,1}(0) = v_{j\delta,2}(0)\) for all \(i\) and \(k\), therefore Gronwall’s inequality gives
\[
|\tilde{u}_{j\delta}(t)|^2_{L^2(\Omega^e)} + |\tilde{v}_{j\delta}(t)|^2_{L^2(\Omega^e)} = 0 \quad \text{for a.e. } t
\]
\[
\implies u_{j\delta,1} = u_{j\delta,2} \text{ and } v_{j\delta,1} = v_{j\delta,2},
\]
Hence the problem \((P^2_{\varepsilon\delta})\) has a unique positive global weak solution in \(F^u \times G^\nu \times H^w\).

**4.2.2 Homogenization of the Problem \((P^2_{\varepsilon\delta})\)**

Now keeping \(\delta\) fixed, we upscale model M2 from the micro to the macro scale. The basic ingredients are the *a-priori estimates* of the solutions, *two-scale convergence* and periodic *unfolding*. We begin with the equations for inmobile species.

**4.2.2.1 A-priori Estimates of the Solution of the Problem (4.2.12)-(4.2.23)**

**4.2.2.1.1 A-priori Estimates of the Solution of the ODE (4.2.22)-(4.2.23)**

**Theorem 4.2.2.1.1.1.** Let \(w_{j\delta}\) be the solution of the problem \((4.2.22)-(4.2.23)\), then it satisfies the following estimates:

\[
\varepsilon \sum_{m=1}^{l_2} \int_0^T \int_{\Gamma_e} \frac{1}{2} |w_{j\delta m}(t,x)|^2 d\sigma_x dt + \varepsilon \sum_{m=1}^{l_2} \int_0^T \int_{\Gamma_e} \left| \frac{\partial w_{j\delta m}(t,x)}{\partial t} \right|^2 d\sigma_x dt + \varepsilon \sum_{m=1}^{l_2} \int_0^T \int_{\Gamma_e} \left| \frac{\partial w_{j\delta m}(t,x)}{\partial \theta} \right|^p d\sigma_x dt \leq C_{A1} \varepsilon \sum_{m=1}^{l_2} \int_0^T \int_{\Gamma_e} \left| \psi_\delta \left( w_{j\delta m}(\theta,x), w_{j\delta m}(\theta,x) \right) \right| d\sigma_x d\theta,
\]

where \(C_{A1}\) is independent of \(\varepsilon, \delta, m\) and \(t\).

**Proof.** The proof consists of several steps.

(a) Multiplying both sides of (4.2.22) by \(w_{j\delta}\) and integrating over \((0,t) \times \Gamma_e\), we obtain

\[
\int_0^t \int_{\Gamma_e} \frac{\partial w_{j\delta}(t,x)}{\partial \theta} w_{j\delta}(t,x) d\sigma_x d\theta = -k_d \int_0^t \int_{\Gamma_e} \psi_\delta \left( w_{j\delta}(\theta,x), w_{j\delta}(\theta,x) \right) d\sigma_x d\theta,
\]
i.e.,

\[
\sum_{m=1}^{l_2} \int_0^t \int_{\Gamma_e} \frac{\partial w_{j\delta m}(t,x)}{\partial \theta} w_{j\delta m}(t,x) d\sigma_x d\theta = -k_d \sum_{m=1}^{l_2} \int_0^t \int_{\Gamma_e} \psi_\delta \left( w_{j\delta m}(\theta,x), w_{j\delta m}(\theta,x) \right) d\sigma_x d\theta,
\]
i.e.,

\[
\frac{1}{2} \sum_{m=1}^{l_2} \int_0^t \frac{\partial}{\partial \theta} \left( w_{j\delta m}(t,x) \right)^2 d\sigma_x d\theta \leq k_d \sum_{m=1}^{l_2} \int_0^t \int_{\Gamma_e} \psi_\delta \left( w_{j\delta m}(\theta,x) \right) \left| w_{j\delta m}(\theta,x) \right| d\sigma_x d\theta,
\]
i.e.,
\[
\frac{1}{2} \sum_{m=1}^{l_2} \int_0^t \int_{\Gamma_t} \langle \dot{\theta} \rangle \int_{\Gamma_t} |w_{\varepsilon \delta m}(\theta, x)|^2 \, d\sigma_x \, d\theta
\]
\[
\leq \frac{1}{2} \sum_{m=1}^{l_2} \left[ \int_0^t \int_{\Gamma_t} k_2^2 \, d\sigma_x \, d\theta + \int_0^t \int_{\Gamma_t} |w_{\varepsilon \delta m}(\theta, x)|^2 \, d\sigma_x \, d\theta \right]
\]
\[
\leq \frac{1}{2} \sum_{m=1}^{l_2} \left[ \int_{\Gamma_t} |w_{\varepsilon \delta m}(0, x)|^2 \, d\sigma_x + k_2^2 \int_0^t \int_{\Gamma_t} |w_{\varepsilon \delta m}(\theta, x)|^2 \, d\sigma_x \, d\theta \right]
\]
i.e.,
\[
\varepsilon \sum_{m=1}^{l_2} \int_{\Gamma_t} |w_{\varepsilon \delta m}(t, x)|^2 \, d\sigma_x
\]
\[
\leq \varepsilon \sum_{m=1}^{l_2} \int_{\Gamma_t} |w_{\varepsilon \delta m}(0, x)|^2 \, d\sigma_x + k_2^2 \int_0^t \int_{\Gamma_t} |w_{\varepsilon \delta m}(\theta, x)|^2 \, d\sigma_x \, d\theta
\]
i.e.,
\[
\varepsilon \sum_{m=1}^{l_2} \int_{\Gamma_t} |w_{\varepsilon \delta m}(t, x)|^2 \, d\sigma_x \leq \tilde{C} + \varepsilon \int_0^t \left( \sum_{m=1}^{l_2} \int_{\Gamma_t} |w_{\varepsilon \delta m}(\theta, x)|^2 \, d\sigma_x \right) \, d\theta,
\]
where \( \tilde{C} := |\Gamma| |\Omega|^{1-\frac{2}{p}} \sum_{m=1}^{l_2} \sup_{\varepsilon > 0} |w_{0 \varepsilon \delta m}|^2_{L^p(\Omega)} + k_2^2 |\Gamma| |\Omega| \sum_{m=1}^{l_2} \int_0^t \int_{\Gamma_t} |w_{\varepsilon \delta m}(\theta, x)|^2 \, d\sigma_x \, d\theta \) is a constant independent of \( \varepsilon \). Application of Gronwall’s inequality gives
\[
\varepsilon \sum_{m=1}^{l_2} \int_{\Gamma_t} |w_{\varepsilon \delta m}(t, x)|^2 \, d\sigma_x \leq \tilde{C}(1 + te^t),
\]
i.e.,
\[
\varepsilon \sum_{m=1}^{l_2} \int_0^T \int_{\Gamma_t} |w_{\varepsilon \delta m}(t, x)|^2 \, d\sigma_x \, dt \leq \tilde{C} \int_0^T (1 + te^t) \, dt,
\]
\[^{36}\text{Note that } \sum_{m=1}^{l_2} \sup_{\varepsilon > 0} |w_{0 \varepsilon \delta m}|^2_{L^p(\Omega)} < \infty \text{ by (4.2.5).} \]
(b) Now multiplying the equation (4.2.22) by $\frac{\partial \delta w}{\partial t}$ and integrating over $(0,T) \times \Gamma$, we get

$$\int_0^T \int_{\Gamma} \left( \frac{\partial \delta w}{\partial t}, \frac{\partial \delta w}{\partial t} \right) d\sigma_x dt = -k_d \int_0^T \int_{\Gamma} \left( \psi \delta (w_{\varepsilon m}(t,x)), \frac{\partial \delta w}{\partial t} \right) d\sigma_x dt,$ngh

i.e.,

$$\sum_{m=1}^{I_2} \int_0^T \int_{\Gamma} \frac{|\partial \delta w_{\varepsilon m}(t,x)|^2}{\partial t} d\sigma_x dt \leq \frac{1}{2} \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma} \left[ k_d^2 |\psi \delta (w_{\varepsilon m}(t,x))|^2 + \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^2 \right] d\sigma_x dt,$ngh

i.e.,

$$\frac{1}{2} \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma} \frac{|\partial \delta w_{\varepsilon m}(t,x)|^2}{\partial t} d\sigma_x dt \leq \frac{1}{2} \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma} \left[ k_d^2 |\psi \delta (w_{\varepsilon m}(t,x))|^2 + \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^2 \right] d\sigma_x dt,$ngh

i.e.,

$$\varepsilon \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^2 d\sigma_x dt \leq k_d^2 I_2 |\Omega| \frac{\Omega}{|Y|} T =: C_{43}, \quad (4.2.185)$$

where $C_{43}$ is independent of $\varepsilon$, $\delta$, $m$ and $t$.

(c) Again multiplying the $m$-th ODE of (4.2.22) by $\frac{\partial \delta w_{\varepsilon m}}{\partial t}$ and integrating over $(0,T) \times \Gamma$, we get

$$\int_0^T \int_{\Gamma} \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^p d\sigma_x dt$$

$$= - \int_0^T \int_{\Gamma} k_d \psi \delta (w_{\varepsilon m}(t,x)) \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^p d\sigma_x dt,$ngh

i.e.,

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^p d\sigma_x dt \leq \int_0^T \int_{\Gamma} k_d \left| \psi \delta (w_{\varepsilon m}(t,x)) \right| \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^{p-1} d\sigma_x dt,$ngh

i.e.,

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^p d\sigma_x dt \leq \int_0^T \int_{\Gamma} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^{p-1} k_d d\sigma_x dt,$ngh

i.e.,

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^p d\sigma_x dt \leq \int_0^T \int_{\Gamma} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^{p-1} k_d d\sigma_x dt,$ngh

i.e.,

$$\int_0^T \int_{\Gamma} \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^p d\sigma_x dt \leq \int_0^T \int_{\Gamma} \left[ \left| \frac{\partial \delta w_{\varepsilon m}(t,x)}{\partial t} \right|^p + \frac{1}{p'} k_d^p \right] d\sigma_x dt,$ngh

where $k_d$ is a constant.
i.e.,
\[
\frac{1}{p} \int_0^T \int_{\Gamma_e} \left| \frac{\partial w_{\varepsilon m}(t,x)}{\partial t} \right|^p \, d\sigma_x \, dt \leq \frac{k_d^p}{p} \int_0^T \int_{\Gamma_e} \, d\sigma_x \, dt,
\]
i.e.,
\[
\int_0^T \int_{\Gamma_e} \left| \frac{\partial w_{\varepsilon m}(t,x)}{\partial t} \right|^p \, d\sigma_x \, dt \leq k_d^p \frac{\Omega}{|\Gamma|},
\]
i.e.,
\[
\varepsilon \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^p \, d\sigma_x \, dt \leq C_{44},
\]
where \( C_{44} \) is independent of \( \varepsilon, \delta, m \) and \( t \).

(d) Multiplying both sides of the \( m \)-th ODE of (4.2.22) by \( w_{\varepsilon m} \left| w_{\varepsilon m} \right|^{p-2} \) and integrating
\[
\int_0^T \int_{\Gamma_e} w_{\varepsilon m}(t,x) \left| w_{\varepsilon m}(t,x) \right|^{p-2} \frac{\partial w_{\varepsilon m}(t,x)}{\partial t} \, d\sigma_x \, dt = -k_d \int_0^T \int_{\Gamma_e} w_{\varepsilon m}(t,x) \left| w_{\varepsilon m}(t,x) \right|^{p-2} \psi_1(t,x) \, d\sigma_x \, dt,
\]
i.e.,
\[
\int_0^T \int_{\Gamma_e} \left| \frac{\partial w_{\varepsilon m}(t,x)}{\partial t} \right|^p \, d\sigma_x \, dt \leq \int_0^T \int_{\Gamma_e} \left[ \frac{p-1}{p} \left| w_{\varepsilon m}(t,x) \right|^p + k_d^p \right] \, d\sigma_x \, dt,
\]
i.e.,
\[
\int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^p \, d\sigma_x \leq \int_{\Gamma_e} \left| w_{\varepsilon m}(0,x) \right|^p \, d\sigma_x + \int_0^T \int_{\Gamma_e} \left[ \frac{p-1}{p} \left| w_{\varepsilon m}(t,x) \right|^p + k_d^p \right] \, d\sigma_x \, dt,
\]
i.e.,
\[
\varepsilon \sum_{m=1}^{I_2} \int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^p \, d\sigma_x \leq \varepsilon \sum_{m=1}^{I_2} \int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^p \, d\sigma_x \, dt + \frac{k_d^p}{p} \int_0^T \int_{\Gamma_e} \, d\sigma_x \, dt + (p-1) \, k_d^p \int_0^T \int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^p \, d\sigma_x \, dt.
\]
A straightforward application of Gronwall’s inequality and steps similar to part (a) will imply
\[
\varepsilon \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^p \, d\sigma_x \, dt \leq C_{45},
\]
where \( C_{45} \) is independent of \( \varepsilon, \delta, m \) and \( t \).

Therefore adding (4.2.184), (4.2.185), (4.2.186) and (4.2.187), we obtain
\[
\varepsilon \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^2 \, d\sigma_x \, dt + \varepsilon \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma_e} \left| \frac{\partial w_{\varepsilon m}(t,x)}{\partial t} \right|^2 \, d\sigma_x \, dt
\]
\[
+ \varepsilon \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma_e} \left| w_{\varepsilon m}(t,x) \right|^p \, d\sigma_x \, dt + \varepsilon \sum_{m=1}^{I_2} \int_0^T \int_{\Gamma_e} \left| \frac{\partial w_{\varepsilon m}(t,x)}{\partial t} \right|^p \, d\sigma_x \, dt
\]
\[
\leq C_{42} + C_{43} + C_{45} + C_{44} = C_{41},
\]
where \( C_{41} := C_{42} + C_{43} + C_{44} + C_{45} \) is independent of \( \varepsilon, \delta, m \) and \( t \).
4.2.2.1.2 Extension of the Solution of the PDE \((4.2.17)\)-(4.2.21)

**Theorem 4.2.2.1.2.1.** There exists an extension of the solution \(v_{\varepsilon k}\) of the problem \((4.2.17)-(4.2.21)\) to all of \((0,T) \times \Omega\) such that

\[
|||v_{\varepsilon k}|||_{L^r((0,T);L^r(\Omega))} + |||v_{\varepsilon k}|||_{L^\infty((0,T);L^\infty(\Omega))} + |||\nabla v_{\varepsilon k}|||_{L^2((0,T);L^2(\Omega))} \leq C_{46},
\]

where \(C_{46}\) is independent of \(\varepsilon, \delta, k\) and \(t\) but depends on \(r\).

The proof of the above theorem resides on the following lemma:

**Lemma 4.2.2.1.2.2.** Let \(p > n+2\) and \(r \in \mathbb{N}\). Suppose that \(v_{\varepsilon k}\) is the solution of the problem \((4.2.17)-(4.2.21)\), then it satisfies the following estimate

\[
|||v_{\varepsilon k}|||_{L^r((0,T);L^r(\Omega^p_r))} + |||v_{\varepsilon k}|||_{L^\infty((0,T);L^\infty(\Omega^p_r))} + |||\nabla v_{\varepsilon k}|||_{L^2((0,T);L^2(\Omega^p_r))} \leq C_{47},
\]

where \(C_{47}\) is independent of \(\varepsilon, \delta, k\) and \(t\) but depends on \(r\).

**Proof.** The proof of this lemma consists of several intermediate steps.

(a) Following the arguments of lemma 4.1.2.1, we obtain

\[
|||v_{\varepsilon k}|||_{L^r((0,T);L^r(\Omega^p_r))} \leq C_{48}
\]

and

\[
|||v_{\varepsilon k}|||_{L^\infty((0,T);L^\infty(\Omega^p_r))} \leq C_{49},
\]

where \(C_{48}\) and \(C_{49}\) are independent of \(\varepsilon, \delta, k\) and \(t\).

(b) Testing the \(k\)-th PDE of the system of equation \((4.2.17)\) by \(v_{\varepsilon k}\) and integrating over \((0,T)\), we get

\[
\int_0^T \left< \frac{\partial v_{\varepsilon k}}{\partial t}, v_{\varepsilon k} \right> dt - \int_0^T \left< \nabla D \nabla v_{\varepsilon k}, v_{\varepsilon k} \right> dt + \int_0^T \int_{\Omega^p_r} \bar{q}_e \cdot \nabla v_{\varepsilon k} v_{\varepsilon k} dx dt = \int_0^T \left< (S_2 R(u_{\varepsilon k}, v_{\varepsilon k}))_{k}, v_{\varepsilon k} \right> dt,
\]

i.e.,

\[
\frac{1}{2} \int_0^T \frac{d}{dt} |||v_{\varepsilon k}|||_{L^2(\Omega^p_r)}^2 dt + D \int_0^T \int_{\Omega^p_r} \left| \nabla v_{\varepsilon k} \right|^2 dx dt
\]

\[
= \int_0^T \int_{\partial \Omega^p_m} \bar{q}_e \cdot \bar{n} v_{\varepsilon k} v_{\varepsilon k} ds dt - \varepsilon \int_0^T \int_{\Gamma^p \varepsilon} \frac{\partial w_{\varepsilon m}}{\partial t} v_{\varepsilon k} ds \sigma_x dt
\]

\[
- \int_0^T \int_{\Omega^p_r} \bar{q}_e \cdot \nabla v_{\varepsilon k} v_{\varepsilon k} dx dt + \int_0^T \left< (S_2 R(u_{\varepsilon k}, v_{\varepsilon k}))_{k}, v_{\varepsilon k} \right> dt
\]

\[
=: I_{\text{bound}} + I_{\text{advec}} + I_{\text{reac}},
\]

where

\[
I_{\text{bound}} := \int_0^T \int_{\partial \Omega^p_m} \bar{q}_e \cdot \bar{n} v_{\varepsilon k} v_{\varepsilon k} ds dt - \varepsilon \int_0^T \int_{\Gamma^p \varepsilon} \frac{\partial w_{\varepsilon m}}{\partial t} v_{\varepsilon k} ds \sigma_x dt,
\]

\[
I_{\text{advec}} := - \int_0^T \int_{\Omega^p_r} \bar{q}_e \cdot \nabla v_{\varepsilon k} v_{\varepsilon k} ds dt,
\]

\[
I_{\text{reac}} := \int_0^T \left< (S_2 R(u_{\varepsilon k}, v_{\varepsilon k}))_{k}, v_{\varepsilon k} \right> dt.
\]
We simplify the terms \( I_{\text{bound}}, I_{\text{advec}} \) and \( I_{\text{reac}} \) one by one.

\[
I_{\text{bound}} = \int_0^T \int_{\partial \Omega^n} \left[ \hat{q}_e \cdot \hat{n} v_{\varepsilon,k} \right] v_{\varepsilon,k} \, ds \, dt - \varepsilon \int_0^T \int_{\Gamma_v} \frac{\partial v_{\varepsilon,m}}{\partial t} v_{\varepsilon,k} \, d\sigma_x \, dt \\
\leq \int_0^T \int_{\partial \Omega^n} |\hat{q}_e \cdot \hat{n}|^2_{L^\infty((0,T) \times \partial \Omega^n)} v_{\varepsilon,k}^2 \, ds \, dt + \varepsilon k_d \int_0^T \int_{\Gamma_v} |v_{\varepsilon,k}|^2 \, d\sigma_x \, dt + \varepsilon \frac{1}{\varepsilon} |\psi_\delta(w_{\varepsilon,m})| \leq 1 \\
\leq \int_0^T \int_{\partial \Omega^n} |\hat{q}_e \cdot \hat{n}|_{L^\infty((0,T) \times \partial \Omega^n)}^2 v_{\varepsilon,k}^2 \, ds \, dt + \varepsilon \int_0^T \int_{\Gamma_v} \left[ \tau_1 |v_{\varepsilon,k}|^2 + c(\tau_1) k_d^2 \right] \, d\sigma_x \, dt. \tag{4.2.196}
\]

Note that
\[
\int_0^T \int_{\partial \Omega^n} v_{\varepsilon,k}^2 \, ds \, dt = c \left( \left| \nabla v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)} \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 + \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 \right), \text{ by theorem 3.4.3.2} \\
\leq c \left( \tau_2 \left| \nabla v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)} + c(\tau_2) \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 + \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 \right) \\
= c \left( \tau_2 \left| \nabla v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 + \hat{c}(\tau_2) \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 \right), \text{ where } \hat{c}(\tau_2) = c(\tau_2) + 1 \quad (4.2.197)
\]
and from theorem 3.4.1.3,
\[
\varepsilon \int_0^T \int_{\Gamma_v} \left| v_{\varepsilon,k} \right|^2 \, d\sigma_x \, dt \leq c \left( \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 + \varepsilon \left| \nabla v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 \right) \\
\leq c \left( \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 + \left| \nabla v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 \right), \tag{4.2.198}
\]

since \( 0 < \varepsilon \ll 1 \). Combining (4.2.196), (4.2.197) and (4.2.198), we get
\[
I_{\text{bound}} \leq ||\hat{q}_e \cdot \hat{n}||_{L^\infty((0,T) \times \partial \Omega^n)} c \left( \tau_2 \left| \nabla v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 + \hat{c}(\tau_2) \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 \right) \\
+ c\tau_1 \left( \left| v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 + \left| \nabla v_{\varepsilon,k} \right|_{L^2((0,T) \times \Omega^p)}^2 \right) + \varepsilon c(\tau_1) k_d^2 \int_0^T \int_{\Gamma_v} \, d\sigma_x \, dt. \tag{4.2.199}
\]

Next,
\[
I_{\text{advec}} \leq \int_0^T \int_{\Omega^p} \left[ q_e \cdot \nabla v_{\varepsilon,k} \right] \left| v_{\varepsilon,k} \right| \, dx \, dt \\
\leq \tau_3 \int_0^T \int_{\Omega^p} \left| \nabla v_{\varepsilon,k} \right|^2 \, dx \, dt + Q^2 c(\tau_3) \int_0^T \int_{\Omega^p} \left| v_{\varepsilon,k} \right|^2 \, dx \, dt, \tag{4.2.200}
\]
where \( Q = ||\hat{q}_e||_{L^\infty((0,T) \times \Omega^p)} \):
\[
I_{\text{reac}} \leq \int_0^T \left| \left( \left( S_2 R(u_{\varepsilon_k}, v_{\varepsilon_k}) \right)_k, v_{\varepsilon,k} \right) \right| \, dt \leq \frac{1}{2} \int_0^T \int_{\Omega^p} \left[ \left( S_2 R(u_{\varepsilon_k}, v_{\varepsilon_k}) \right)_k \right]^2 + \left| v_{\varepsilon,k} \right|^2 \, dx \, dt. \tag{4.2.201}
\]
Combining (4.2.192), (4.2.199), (4.2.200) and (4.2.201), we get
\[
\frac{1}{2} \int_0^T \frac{d}{dt} \left| v_{\varepsilon,k} \right|_{L^2(\Omega^p)}^2 \, dt + D \int_{S \times \Omega^p} \left| \nabla v_{\varepsilon,k} \right|^2 \, dx \, dt = I_{\text{bound}} + I_{\text{advec}} + I_{\text{reac}}
\]
\[ \leq \|q_\varepsilon \cdot n\| \|v_{\varepsilon k}\|_{L^2((0,T) \times \partial \Omega^\varepsilon)} \quad (\tau_2 \|\nabla v_{\varepsilon k}\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + \tilde{c}(\tau_2) \|v_{\varepsilon k}\|_{L^2((0,T) \times \Omega^\varepsilon)}^2) \\
+ c\tau_1 \left( \left\|v_{\varepsilon k}\right\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + \left\|\nabla v_{\varepsilon k}\right\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 \right) + c\varepsilon(\tau_1)k_2^2 \int_0^T \int_{\Gamma_\varepsilon} \sigma_x \, dt \\
+ \tau_3 \int_0^T \int_{\Omega^\varepsilon} \left|\nabla v_{\varepsilon k}\right|^2 \, dx \, dt + Q^2 c(\tau_3) \int_0^T \int_{\Omega^\varepsilon} \left|v_{\varepsilon k}\right|^2 \, dx \, dt, \\
+ \frac{1}{2} \int_0^T \int_{\Omega^\varepsilon} \left[|S_2 R(u_{\varepsilon, z}, v_{\varepsilon, z})|k_2^2 + \|v_{\varepsilon k}\|^2\right] \, dx \, dt. \tag{4.2.202} \]

Choosing \( \tau_1 = \frac{D}{8\varepsilon}, \tau_2 = \frac{D}{8\|q_\varepsilon \cdot n\|_{L^\infty((0,T) \times \partial \Omega^\varepsilon)}} \) and \( \tau_3 = \frac{D}{8} \), then (4.2.202) reduces to

\[ \frac{1}{2} \int_0^T \frac{d}{dt} \|v_{\varepsilon k}\|^2_{L^2(\Omega^\varepsilon)} \, dt + \frac{5D}{8} \int_0^T \int_{\Omega^\varepsilon} \left|\nabla v_{\varepsilon k}\right|^2 \, dx \, dt \]

\[ \leq \|q_\varepsilon \cdot n\|_{L^\infty((0,T) \times \partial \Omega^\varepsilon)} c\tilde{c}(\tau_2) \|v_{\varepsilon k}\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + c\tau_1 \|v_{\varepsilon k}\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + c\varepsilon(\tau_1)k_2^2 \int_0^T \int_{\Gamma_\varepsilon} \sigma_x \, dt \\
+ \frac{1}{2} \int_0^T \int_{\Omega^\varepsilon} \left|\nabla v_{\varepsilon k}\right|^2 \, dx \, dt + \frac{1}{2} \int_0^T \int_{\Omega^\varepsilon} \left[|S_2 R(u_{\varepsilon, z}, v_{\varepsilon, z})|k_2^2 + \|v_{\varepsilon k}\|^2\right] \, dx \, dt, \tag{4.2.203} \]

i.e.,

\[ \frac{1}{2} \|v_{\varepsilon k}(T)\|^2_{L^2(\Omega^\varepsilon)} = \frac{1}{2} \int_0^T \frac{d}{dt} \|v_{\varepsilon k}\|^2_{L^2(\Omega^\varepsilon)} \, dt + \frac{5D}{8} \int_0^T \int_{\Omega^\varepsilon} \left|\nabla v_{\varepsilon k}\right|^2 \, dx \, dt \]

\[ \leq \frac{1}{2} \|v(0)\|^2_{L^2(\Omega^\varepsilon)} + \|q_\varepsilon \cdot n\|_{L^\infty((0,T) \times \partial \Omega^\varepsilon)} c\tilde{c}(\tau_2) \|v_{\varepsilon k}\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 + c\tau_1 \|v_{\varepsilon k}\|_{L^2((0,T) \times \Omega^\varepsilon)}^2 \\
+ c(\tau_1)k_2^2 T \left|\frac{|\Omega|}{|Y|}\right| + Q^2 c(\tau_3) \|v_{\varepsilon k}\|^2_{L^2((0,T) \times \Omega^\varepsilon)} \\
+ \frac{1}{2} \left[|S_2 R(u_{\varepsilon, z}, v_{\varepsilon, z})|k_2^2 + \|v_{\varepsilon k}\|^2\right]. \tag{4.2.203} \]

From the assumptions (4.2.3) and (4.2.5) it follows that \( \sup_{\varepsilon > 0} \|v(0)\|_{L^2(\Omega^\varepsilon)} < \infty \). Choosing \( r \) sufficiently large in the inequalities (4.2.107) and (4.2.153) and employment of the Hölder’s inequality give \( \sup_{\varepsilon > 0} \|\tilde{S}_2 R(u_{\varepsilon, z}, v_{\varepsilon, z})\|_{L^2((0,T) \times \Omega^\varepsilon)} \leq C_{50} < \infty \). Thus the whole r.h.s. of (4.2.203) is bounded by a constant independent of \( \varepsilon, \delta, k \) and \( t \), i.e.,

\[ \|\nabla v_{\varepsilon k}\|^2_{L^2((0,T) \times \Omega^\varepsilon)} \leq C_{51} \implies \sum_{k=1}^{I_0} \|\nabla v_{\varepsilon k}\|^2_{L^2((0,T) \times \Omega^\varepsilon)} \leq C_{51} I_0 \]

\[ \implies \sup_{\varepsilon > 0} \|\nabla v_{\varepsilon}\|_{L^2((0,T) \times \Omega^\varepsilon)} \leq C_{52}, \tag{4.2.204} \]

where \( C_{52} := (C_{51} I_0)^{\frac{1}{2}} \) is independent of \( \varepsilon, \delta, k, \) and \( t \) but depends on \( r \). Therefore adding (4.2.190), (4.2.191) and (4.2.204), we get

\[ \|v\|_{L^4((0,T);L^4(\Omega^\varepsilon))}^2 \leq C_{48} + C_{49} + C_{52} = C_{47}, \tag{4.2.205} \]

where \( C_{47} := C_{48} + C_{49} + C_{52} \) is independent of \( \varepsilon, \delta, k \) and \( t \) but depends on \( r \).

Proof of theorem 4.2.1.2.1: The estimate (4.2.188) from lemma 4.2.1.2.2 and theorem 3.4.2.3 finish off the proof.
4.2.2.1.3 Extension of the Solution of the PDE (4.2.12)-(4.2.16)

We can extend the solution $u_{\varepsilon} \in \mathcal{F}_\varepsilon$ of the problem (4.2.12)-(4.2.16) to all of $(0,T) \times \Omega$ as we did for $v_{\varepsilon}$ in section 4.2.2.1.2. For the extension we use the following two results:

**Theorem 4.2.2.1.3.1.** There exists an extension of the solution $u_{\varepsilon}$ of the problem (4.2.12)-(4.2.16) to all of $(0,T) \times \Omega$ such that

$$|||u_{\varepsilon}|||_{L^r((0,T);L^r(\Omega))}^2 + |||u_{\varepsilon}|||_{L^\infty((0,T);L^\infty(\Omega))}^2 + |||\nabla u_{\varepsilon}|||_{L^2((0,T);L^2(\Omega))}^2 \leq C_{55},$$

where $C_{55}$ is independent of $\varepsilon$, $\delta$, $i$ and $t$ but depends on $r$.

**Lemma 4.2.2.1.3.2.** Let $p > n+1$ and $r \in \mathbb{N}$. Suppose that $u_{\varepsilon}$ is the solution of the problem (4.2.12)-(4.2.16), then it satisfies the following estimates

$$|||u_{\varepsilon}|||_{L^r((0,T);L^r(\Omega^r))}^2 + |||u_{\varepsilon}|||_{L^\infty((0,T);L^\infty(\Omega^r))}^2 + |||\nabla u_{\varepsilon}|||_{L^2((0,T);L^2(\Omega^r))}^2 \leq C_{54},$$

where $C_{54}$ is independent of $\varepsilon$, $\delta$, $i$ and $t$ but depends on $r$.

### 4.2.2 Convergence of the Micro Solution

#### 4.2.2.1 Convergence of the Micro Solution of the Problem (4.2.22)-(4.2.26)

**Theorem 4.2.2.1.1.** The solution, $v_{\varepsilon}$, of the problem (P)$_{\varepsilon}$ satisfies the following estimate:

$$|||v_{\varepsilon}|||_{L^\infty((0,T);L^2(\Omega))}^2 + |||v_{\varepsilon}|||_{L^2((0,T);H^{1,2}(\Omega))}^2 + |||\chi^r \frac{\partial v_{\varepsilon}}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^r)} \leq C_{55},$$

where $C_{55}$ is independent of $\varepsilon$, $\delta$, $k$ and $t$ but depends on $r$.

**Proof.** (a) The arguments similar to part (a) and (b) of theorem 4.1.2.1 yields

$$|||v_{\varepsilon}|||_{L^\infty((0,T);L^2(\Omega))}^2 \leq C_{56},$$

and

$$|||v_{\varepsilon}|||_{L^2((0,T);H^{1,2}(\Omega))}^2 \leq C_{57},$$

where $C_{56}$ and $C_{57}$ are independent of $\varepsilon$, $\delta$, $k$ and $t$.

(b) Now let $\phi \in H^{1,2}_0(0,T)$ and $\psi \in H^{1,2}(\Omega)$. Then the weak formulation of the $k$-th PDE of the problem (4.2.17)-(4.2.21) is given by

$$\int_0^T \left< \chi \left( \frac{x}{\varepsilon} \right) \frac{\partial v_{\varepsilon_k}(t)}{\partial t}, \phi(t) \psi \right>_{H^{1,2}(\Omega)^r \times H^{1,2}(\Omega)} dt + \int_0^T \int_{\Omega} D\phi(t) \chi \left( \frac{x}{\varepsilon} \right) \nabla v_{\varepsilon_k}(t,x) \nabla \psi(x) dx dt$$

$$- \int_0^T \int_{\partial \Omega_n} \frac{\partial \tilde{q}}{\partial t} \cdot \nabla v_{\varepsilon_k}(t,x) \phi(t) \psi(x) ds dt + \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon_k}(t,x)}{\partial t} \phi(t) \psi(x) d\sigma x dt$$

$$+ \int_0^T \int_{\Omega} \chi \left( \frac{x}{\varepsilon} \right) \tilde{q} \cdot \nabla v_{\varepsilon_k}(t,x) \phi(t) \psi(x) dx dt$$

$$= \int_0^T \chi \left( \frac{x}{\varepsilon} \right) S_2 R(u_{\varepsilon}(t),v_{\varepsilon}(t))_k,\phi(t) \psi \right>_{H^{1,2}(\Omega)^r \times H^{1,2}(\Omega)} dt,$$
i.e.,

\[
\int_0^T \left\langle \chi \left( \frac{x}{\varepsilon} \right) \frac{\partial \psi_{\varepsilon_k}(t)}{\partial t}, \frac{\partial (t) \psi}{} \right\rangle_{H^{1,2}(\Omega) \times H^{1,2}(\Omega)} dt
\]

\[
\leq \int_0^T \int_\Omega D \chi \left( \frac{x}{\varepsilon} \right) |\nabla \psi_{\varepsilon_k}(t, x)| |\nabla \psi(x)| |\phi(t)| dx dt
\]

\[
+ ||\bar{q}||_{L^\infty((0,T) \times \Omega)} \int_0^T \int_\Omega |\nabla \psi_{\varepsilon_k}(t, x)| |\phi(t)| |\psi(x)| dx dt
\]

\[
+ ||\bar{q} \cdot \underline{m}||_{L^\infty((0,T) \times \partial\Omega)} \int_0^T \int_\Omega |\psi(x)| |\phi(t)| ds \, dt + \varepsilon
\]

\[
\int_0^T \int_{\Gamma_\varepsilon} \left| \frac{\partial \psi_{\varepsilon_k}(t, x)}{\partial t} \right| |\phi(t)| |\psi(x)| \, d\sigma \, dt + \int_0^T \left\langle \chi \left( \frac{x}{\varepsilon} \right) S_R(u_{\varepsilon_k}(t, x), v_{\varepsilon_k}(t), \phi(t) \psi) \right\rangle_{L^2(\Omega) \times L^2(\Omega)} dt.
\]

We estimate each term on the r.h.s. of (4.2.211) one by one. The first term can be estimated as

\[
\int_0^T \int_\Omega D \chi \left( \frac{x}{\varepsilon} \right) |\nabla \psi_{\varepsilon_k}(t, x)| |\nabla \psi(x)| |\phi(t)| dx dt
\]

\[
+ \int_0^T \int_\Omega |\psi_{\varepsilon_k}(t, x)|^2 dx \, dt + \int_0^T \int_\Omega |\phi(t)|^2 |\nabla \psi(x)|^2 dx \, dt
\]

\[
+ \frac{D}{2} \int_0^T \int_\Omega |\nabla \psi_{\varepsilon_k}(t, x)|^2 dx \, dt
\]

\[
\leq \frac{D}{2} \int_0^T \int_\Omega \left[ |\nabla \psi_{\varepsilon_k}(t, x)|^2 + |\phi(t)|^2 |\psi(x)|^2 \right] \, dx \, dt
\]

\[
\leq \frac{D}{2} \left[ ||\nabla \psi_{\varepsilon_k}||_{L^2((0,T) \times \Omega)}^2 + ||\phi||_{L^2((0,T) \times \Omega)}^2 ||\psi||_{L^2(\Omega)}^2 \right].
\]

(4.2.212)

Again,\(^{37}\)

\[
\int_0^T \int_{\partial\Omega} |\psi_{\varepsilon_k}(t, x)| |\phi(t)| \, d\sigma \, dt
\]

\[
\leq \frac{1}{2} \int_0^T \int_{\partial\Omega} \left[ |\psi_{\varepsilon_k}(t, x)|^2 + |\phi(t)|^2 \right] \, d\sigma \, dt
\]

\[
\leq \frac{C}{2} \left[ \int_0^T \int_\Omega \left( |\psi_{\varepsilon_k}(t, x)|^2 + |\nabla \psi_{\varepsilon_k}(t, x)|^2 \right) dx \, dt + \int_0^T \int_\Omega \left( |\psi|^2 + |\nabla \psi|^2 \right) dx \, dt \right]
\]

\[
= \frac{C}{2} \left[ \int_0^T \int_\Omega |\nabla \psi_{\varepsilon_k}(t, x)|^2 dx \, dt + \left( \int_0^T |\phi|^2 dx \right) \left( \int_\Omega |\psi|^2 + |\nabla \psi|^2 \right) \right].
\]

(4.2.213)

For the third term,\(^{38}\)

\[
\varepsilon \int_0^T \int_{\Gamma_\varepsilon} \left| \frac{\partial \psi_{\varepsilon_k}(t, x)}{\partial t} \right| |\phi(t)| |\psi(x)| \, d\sigma \, dt
\]

\[
\leq \frac{\varepsilon}{2} \int_0^T \int_{\Gamma_\varepsilon} \left[ \left| \frac{\partial \psi_{\varepsilon_k}(t, x)}{\partial t} \right|^2 + |\phi(t)|^2 |\psi(x)|^2 \right] \, d\sigma \, dt
\]

\[
= \frac{1}{2} \left[ \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \left| \frac{\partial \psi_{\varepsilon_k}(t, x)}{\partial t} \right|^2 \, d\sigma \, dt + \int_0^T |\phi|^2 \, dt \varepsilon \int_{\Gamma_\varepsilon} |\psi|^2 \, d\sigma \right].
\]

\(^{37}\)We have used the boundary inequality (3.4.22).

\(^{38}\)In this case, we used the inequality (3.4.5).
\[
\begin{align*}
&\leq \frac{1}{2} \left[ \varepsilon \int_0^T \int_\Omega \left| \frac{\partial v_{\varepsilon k}}{\partial t} \right|^2 \, d\sigma_x \, dt + \int_0^T |\phi(t)|^2 \, dt \, C \left( \int_{\Omega^p} |\psi(x)|^2 + \varepsilon^2 \int_{\Omega^p} |\nabla \psi(x)|^2 \right) \, dx \right] \\
&\leq \frac{1}{2} \left[ \varepsilon \int_0^T \int_\Omega \left| \frac{\partial v_{\varepsilon k}}{\partial t} \right|^2 \, d\sigma_x \, dt + \int_0^T |\phi(t)|^2 \, dt \, C \left( \int_{\Omega^p} |\psi(x)|^2 + \int_{\Omega^p} |\nabla \psi(x)|^2 \right) \, dx \right], \quad (4.2.214)
\end{align*}
\]

since \( 0 < \varepsilon << 1 \). Finally, the fourth term can be estimated as

\[
\int_0^T \left( \chi \left( \frac{x}{\varepsilon} \right) S_2 R(u_{\varepsilon k}, v_{\varepsilon k})_k, \phi \psi \right)_{L^2(\Omega) \times L^2(\Omega)} \, dt \\
\leq \frac{1}{2} \int_0^T \int_\Omega \left[ \chi \left( \frac{x}{\varepsilon} \right) S_2 R(u_{\varepsilon k}, v_{\varepsilon k})_k \right]^2 + |\phi(t)|^2 |\psi(x)|^2 \, dx \, dt. \quad (4.2.215)
\]

The inequalities (4.2.212), (4.2.213), (4.2.214) and (4.2.215) can be further estimated by (4.2.183) and (4.2.188). Following the similar steps as shown in the proof of theorem 4.1.2.2.1, we obtain

\[
\left\| \chi^2 \frac{\partial v_{\varepsilon k}}{\partial t} \right\|_{L^2((0,T); H^{1,2}(\Omega)^*)} \leq C_{58} \\
\implies \sum_{k=1}^{I_2} \left\| \chi^2 \frac{\partial v_{\varepsilon k}}{\partial t} \right\|^2_{L^2((0,T); H^{1,2}(\Omega)^*)} \leq C_{58}^2 I_2 \\
\implies \left\| \chi^2 \frac{\partial v_{\varepsilon k}}{\partial t} \right\|_{L^2((0,T); H^{1,2}(\Omega)^*)} \leq C_{59} \quad (4.2.216)
\]

where \( C_{59} := (C_{58}^2 I_2) \frac{1}{2} \) is independent of \( \varepsilon, \delta, k \) and \( t \) but depends on \( r \). Adding (4.2.209), (4.2.210) and (4.2.216), we get

\[
|||v_{\varepsilon k}|||_{L^\infty((0,T); L^2(\Omega))^I_2} + |||v_{\varepsilon k}|||_{L^2((0,T); H^{1,2}(\Omega))^I_2} + \left\| \chi^2 \frac{\partial v_{\varepsilon k}}{\partial t} \right\|_{L^2((0,T); H^{1,2}(\Omega)^*)}^2 \\
\leq C_{56} + C_{57} + C_{59} =: C_{55},
\]

where \( C_{55} \) is independent of \( \varepsilon, \delta, k \) and \( t \) but depends on \( r \).

\begin{theorem}
4.2.2.1.2. Let \((v_{\varepsilon k})_{\varepsilon>0}\) satisfies the estimates (4.2.188) and (4.2.208). Then there exists a function \( v_{\delta} \in L^2((0,T); H^{1,2}(\Omega))^I_2 \) and a function \( v_{\delta h}^1 \in L^2((0,T) \times \Omega; H^{1,2}(Y)/\mathbb{R})^I_2 \) such that up to a subsequence, still denoted by same subscript, the following convergence results hold:

(i) \((v_{\varepsilon k})_{\varepsilon>0}\) is weakly convergent to \( v_{\delta} \) in \( L^2((0,T); H^{1,2}(\Omega))^I_2 \). \hspace{1cm} (4.2.217)

(ii) \((v_{\varepsilon k})_{\varepsilon>0}\) is strongly convergent to \( v_{\delta} \) in \( L^2((0,T); L^2(\Omega))^I_2 \). \hspace{1cm} (4.2.218)

(iii) \((v_{\varepsilon k})_{\varepsilon>0}\) and \((\nabla_x v_{\varepsilon k})_{\varepsilon>0}\) are two-scale convergent to \( v_{\delta} \) and \( \nabla_x v_{\delta} + \nabla_y v_{\delta h}^1 \) in the sense of (3.5.3). \hspace{1cm} (4.2.219)
\end{theorem}

\begin{proof}
Given the a-priori estimates (4.2.188) and (4.2.208). With the help of theorem 3.5.13 and lemma 4.1.2.2.2, the proof follows like the proof of theorem 4.1.2.2.3.
\end{proof}

\begin{corollary}
4.2.2.1.3. The limit function \( v_{\delta} \) belongs to \( L^\infty((0,T) \times \Omega \times Y)^I_2 \). \hspace{1cm} (3.5.3)
\end{corollary}

\[39\]The function \( v_{\delta} \) is independent of \( y \).
Proof. Since \((v_{\varepsilon k})_{\varepsilon > 0}\) is strongly convergent in \(L^2((0,T); L^2(\Omega))\), there exists a subsequence \((v'_{\varepsilon})_{\varepsilon > 0}\) which is pointwise convergent to \(v_\delta\) almost everywhere in \((0,T)\times\Omega\), i.e.,

\[
\lim_{\varepsilon' \to 0} v'_{\varepsilon}(t,x) = v_\delta(t,x) \quad \text{a.e.} \quad (t,x) \in (0,T) \times \Omega.
\]

By theorem \ref{thm:existence_global_solution}, we have \(||v_{\varepsilon k}||_{L^\infty((0,T); L^\infty(\Omega))} \leq C_{46}\) for all \(k\), where \(C_{46}\) is independent of \(\varepsilon\) and \(\delta\), therefore

\[
|v_{\delta k}(t,x)|^2 \leq |v_\delta(t,x)|^2 = \sum_{k=1}^{I_2} |v'_{\varepsilon_k}(t,x)|^2 = \lim_{\varepsilon' \to 0} \sum_{k=1}^{I_2} |v'_{\varepsilon_k}(t,x)|^2
\]

\[
\leq \sum_{k=1}^{I_2} \lim_{\varepsilon' \to 0} \text{ess sup}_{t \in (0,T)} \text{ess sup}_{x \in \Omega} |v'_{\varepsilon_k}(t,x)|^2
\]

\[
\leq \sum_{k=1}^{I_2} C_{46}^2
\]

\[
= C_{46}^2 I_2 \quad \text{for a.e. } t \text{ and } x
\]

\[
\implies \text{ess sup}_{t \in (0,T)} \text{ess sup}_{x \in \Omega} |v_{\delta_k}(t,x)|^2 \leq C_{46}^2 I_2
\]

\[
\implies \sup_{\delta > 0} ||v_{\delta_k}||_{L^\infty((0,T); L^\infty(\Omega))} \leq \left( C_{46}^2 I_2 \right)^{\frac{1}{2}} < \infty \quad \text{for all } k,
\]

where \((C_{46}^2 I_2)^{\frac{1}{2}}\) is independent of \(\varepsilon\), \(\delta\) and \(k\). This gives

\[
||v_\delta||_{L^\infty((0,T) \times \Omega \times Y)} I_2 = \max_{1 \leq k \leq I_2} ||v_{\delta_k}||_{L^\infty((0,T) \times \Omega \times Y)}
\]

\[
= \max_{1 \leq k \leq I_2} \text{ess sup}_{(t,x,y) \in (0,T) \times \Omega \times Y} |v_{\delta_k}(t,x)|
\]

\[
\leq \max_{1 \leq k \leq I_2} \text{ess sup}_{y \in Y} \text{ess sup}_{(t,x) \in (0,T) \times \Omega} |v_{\delta_k}(t,x)|
\]

\[
\leq \max_{1 \leq k \leq I_2} \text{ess sup}_{y \in Y} \text{ess sup}_{t \in (0,T)} \text{ess sup}_{x \in \Omega} |v_{\delta_k}(t,x)|
\]

\[
\leq \text{ess sup}_{y \in Y} \left( C_{46}^2 I_2 \right)^{\frac{1}{2}} < \infty
\]

\[
\implies \sup_{\delta > 0} ||v_\delta||_{L^\infty((0,T) \times \Omega \times Y)} I_2 \leq \left( C_{46}^2 I_2 \right)^{\frac{1}{2}} < \infty
\]

This completes the proof.

4.2.2.2 Convergence of the Micro Solution of the Problem (4.2.22)-(4.2.16)

Next we state theorems concerning weak, strong and two-scale convergences of the sequence of functions (solution of (4.2.12)-(4.2.16)) \(u_{\varepsilon k}\). These theorems can be proved in an analogous way like the theorems 4.2.2.1.1 and 4.2.2.1.2.

**Theorem 4.2.2.2.1.** The solution, \(u_{\varepsilon k}\), of the problem \((P_{\varepsilon k})\) satisfies the following estimate:

\[
||u_{\varepsilon k}||_{L^\infty((0,T); L^2(\Omega))} + ||u_{\varepsilon k}||_{L^2((0,T); H^{1,2}(\Omega))} + \left| |\varepsilon \frac{\partial u_{\varepsilon k}}{\partial t}||_{L^2((0,T); H^{1,2}(\Omega)^*)} \right| \leq C_{60},
\]

(4.2.220)

where \(C_{60}\) is independent of \(\varepsilon\), \(\delta\), \(i\) and \(t\) but depends on \(r\).
Let $(u_{\varepsilon})_{\varepsilon>0}$ satisfies the estimates (4.2.206) and (4.2.220). Then there exists a function $u_\delta \in L^2((0,T);H^{1,2}(\Omega)^{I_1})$ and a function $u_\delta^l \in L^2((0,T) \times \Omega;H^{1,2}_per(Y)/\mathbb{R})^{I_1}$ such that up to a subsequence, still denoted by same subscript, the following convergence results hold:

(i) $(u_{\varepsilon})_{\varepsilon>0}$ is weakly convergent to $u_\delta$ in $L^2((0,T);H^{1,2}(\Omega)^{I_1})$. \hspace{1cm} (4.2.221)

(ii) $(u_{\varepsilon})_{\varepsilon>0}$ is strongly convergent to $u_\delta$ in $L^2((0,T);L^2(\Omega)^{I_1})$. \hspace{1cm} (4.2.222)

(iii) $(u_{\varepsilon})_{\varepsilon>0}$ and $(\nabla_x u_{\varepsilon})_{\varepsilon>0}$ are two-scale convergent to $u_\delta$ and $\nabla_x u_\delta + \nabla_y u_\delta^l$ in the sense of (3.5.3). \hspace{1cm} (4.2.223)

**Corollary 4.2.2.2.3.** The limit function $u_\delta$ belongs to $L^\infty((0,T) \times \Omega \times Y)^{I_1}$.\hspace{1cm} 40

**Proof.** The proof is analogous to the proof of the theorem 4.2.2.1.3. \hspace{1cm} ♦

**Theorem 4.2.2.2.4.** The sequences $(S_1 R(u_{\varepsilon},v_{\varepsilon}))_{\varepsilon>0}$ and $(S_2 R(u_{\varepsilon},v_{\varepsilon}))_{\varepsilon>0}$ are strongly convergent to $S_1 R(u_\delta,v_{\delta})$ in $L^2((0,T);L^2(\Omega)^{I_1})$ and to $S_2 R(u_\delta,v_{\delta})$ in $L^2((0,T);L^2(\Omega)^{I_2})$ respectively.

**Proof.** The proof is analogous to the proof of the theorem 4.1.2.6. The $L^\infty$ - estimates in theorems 4.2.2.1.2, 4.2.2.1.3, 4.2.2.2.2 and 4.2.2.2.3, and the strong convergences in theorems 4.2.2.2.1.2 and 4.2.2.2.2 complete the proof. \hspace{1cm} ♦

### 4.2.2.3 Passage to the Limit as $\varepsilon \to 0$

**4.2.2.3.1 Homogenization of the ODE (4.2.22)-(4.2.23)**

Using theorems 3.5.16 and 4.2.2.1.1, we can pass to the two-scale limit on the l.h.s. of (4.2.22) but due to the presence of nonlinear function on the r.h.s. of (4.2.22) one needs to pay special attention while passing the limit as $\varepsilon \to 0$. Here we take the help of periodic unfolding introduced in section 3.6 to pass to the limit in the nonlinear function $\psi_\delta(w_{\varepsilon_m})$.

Let $T_\delta^b : L^2((0,T) \times \Gamma_x) \to L^2((0,T) \times \Omega \times \Gamma)$ be the boundary unfolding operator defined as

$$T_\delta^b w_{\varepsilon_m}(t,x,y) := w_{\varepsilon_m}(t,\varepsilon \left[ x \varepsilon \right] + \varepsilon y), \quad \text{for every} \quad (t,x,y) \in (0,T) \times \Omega \times \Gamma. \hspace{1cm} (4.2.224)$$

Using the unfolding operator $T_\delta^b$, we unfold the $m$-th ODE of (4.2.22)-(4.2.23). See that

$$T_\delta^b \left( \frac{\partial w_{\varepsilon_m}}{\partial t} \right)(t,x,y) = \frac{\partial}{\partial t} \left( w_{\varepsilon_m}(t,\varepsilon \left[ x \varepsilon \right] + \varepsilon y) \right) = \frac{\partial}{\partial t} T_\delta^b \left( w_{\varepsilon_m}(t,x,y) \right),$$

$$T_\delta^b \psi_\delta(w_{\varepsilon_m})(t,x,y) = \psi_\delta(w_{\varepsilon_m})(t,\varepsilon \left[ x \varepsilon \right] + \varepsilon y) = \psi_\delta \left( w_{\varepsilon_m}(t,\varepsilon \left[ x \varepsilon \right] + \varepsilon y) \right) = \psi_\delta(T_\delta^b w_{\varepsilon_m}(t,x,y)).$$

Therefore the unfolded ODE is

$$\frac{\partial}{\partial t} T_\delta^b w_{\varepsilon_m}(t,x,y) = - k_d \psi_\delta(T_\delta^b w_{\varepsilon_m}(t,x,y)) \quad \text{in} \quad (0,T) \times \Omega \times \Gamma, \hspace{1cm} (4.2.225)$$

$$T_\delta^b (w_{\varepsilon_m})(0,x,y) = T_\delta^b w_{\varepsilon_m}(x,y) \quad \text{on} \quad \Omega \times \Gamma. \hspace{1cm} (4.2.226)$$

40 The function $u_\delta$ is independent of $y$. 
Lemma 4.2.2.3.1.1. The sequence \( T^\varepsilon_b(w_{\varepsilon_{\mu \kappa m}}) \) \( \varepsilon > 0 \) is strongly convergent in \( L^2((0, T) \times \Omega \times \Gamma) \).

Proof. For \( \nu, \mu \in \mathbb{N} \), let us consider the sequences \( (T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}))_{\mu = 1}^\infty \) and \( (T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}))_{\nu = 1}^\infty \) which satisfies (4.2.225)-(4.2.226) such that

\[
\frac{\partial}{\partial t} \left( T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}) \right) = -k_0 \left[ \psi_\delta \left( T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}) \right) \right] \quad \text{in} \quad (0, T) \times \Gamma_\varepsilon, \quad (4.2.227)
\]

\[
T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(0, x, y)) = T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(x, y)) \quad \text{on} \quad \Gamma_\varepsilon, \quad (4.2.228)
\]

\[
\frac{\partial}{\partial t} \left( T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}) \right) = -k_0 \left[ \psi_\delta \left( T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}) \right) \right] \quad \text{in} \quad (0, T) \times \Gamma_\varepsilon, \quad (4.2.229)
\]

\[
T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(0, x, y)) = T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(x, y)) \quad \text{on} \quad \Gamma_\varepsilon. \quad (4.2.230)
\]

Subtracting (4.2.227)-(4.2.229), we get

\[
\frac{\partial}{\partial t} \left( T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}) \right)
= -k_0 \left[ \psi_\delta \left( T^\varepsilon_{\mu b}(w_{\varepsilon_{\nu \kappa m}}) \right) - \psi_\delta \left( T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}) \right) \right] \quad \text{in} \quad (0, T) \times \Gamma_\varepsilon. \quad (4.2.231)
\]

Multiplying both sides by \( T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}) \) and integrating over \( (0, T) \times \Omega \times \Gamma \),

we obtain \(^{41}\)

\[
\frac{1}{2} \int_0^T \int_\Omega \int_\Gamma \left| T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(\theta, x, y)) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(\theta, x, y)) \right|^2 \, dx \, dy \, d\theta
= -k_0 \int_0^T \int_\Omega \int_\Gamma \left[ \psi_\delta \left( T^\varepsilon_{\mu b}(w_{\varepsilon_{\nu \kappa m}}(\theta, x, y)) \right) - \psi_\delta \left( T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(\theta, x, y)) \right) \right] \, dx \, dy \, d\theta
\]

\[
\leq k_0 K_{\text{Lip}} \int_0^T \int_\Omega \int_\Gamma \left| T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(\theta, x, y)) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(\theta, x, y)) \right|^2 \, dx \, dy \, d\theta,
\]

i.e.,

\[
\left| T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(t)) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(t)) \right|^2_{L^2(\Omega \times \Gamma)} \leq \left| T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(0)) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(0)) \right|^2_{L^2(\Omega \times \Gamma)}
+ K_1 \int_0^t \left| T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(\theta)) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(\theta)) \right|^2_{L^2(\Omega \times \Gamma)} \, d\theta. \quad (4.2.232)
\]

The application of Gronwall’s inequality yields \(^{42}\)

\[
\int_{\Omega \times \Gamma} \left| T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(t, x, y)) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(t, x, y)) \right|^2 \, dx \, dy
\]

\[
\leq \left( 1 + K_1 t e^{K_1 t} \right) \left[ \int_{\Omega \times \Gamma} \left| T^\varepsilon_{\mu b}(w_{\varepsilon_{\mu \kappa m}}(0, x, y)) - T^\varepsilon_{\nu b}(w_{\varepsilon_{\nu \kappa m}}(0, x, y)) \right|^2 \, dx \, dy \right] \forall t
\]

\(^{41}\)We have used the Lipschitz continuity of \( \psi_\delta \).

\(^{42}\)Confer part (v) of the theorem 3.6.4.
\[
\int_0^T \int_{\Omega \times \Gamma} \left| T_{b}^{\varepsilon} \left( w_{\varepsilon \mu \delta m}(t,x,y) \right) - T_{b}^{\varepsilon} \left( w_{\varepsilon \mu \delta m}(t,x,y) \right) \right|^2 \, dx \, dy \, dt \\
\leq \int_{\Omega \times \Gamma} \left| T_{b}^{\varepsilon} \left( w_{\varepsilon \mu \delta m}(0,x,y) \right) - T_{b}^{\varepsilon} \left( w_{\varepsilon \mu \delta m}(0,x,y) \right) \right|^2 \, dx \, dy \\
\leq C_{G1} \left[ \int_{\Omega \times \Gamma} \left| T_{b}^{\varepsilon} \left( w_{\varepsilon \mu \delta m}(0,x,y) \right) \right|^2 \, dx \, dy \right]
\]

This shows that \( \left( T_{b}^{\varepsilon} w_{\varepsilon \mu \delta m}(\cdot) \right)_{\varepsilon > 0} \) is a Cauchy sequence in \( L^2((0,T) \times \Omega \times \Gamma) \). It is strongly convergent to a limit \( \xi \) in \( L^2((0,T) \times \Omega \times \Gamma) \).

The above lemma shows that \( \left( T_{b}^{\varepsilon} w_{\varepsilon \mu \delta m}(\cdot) \right)_{\varepsilon > 0} \) is weakly convergent to \( \xi \) in \( L^2((0,T) \times \Omega \times \Gamma) \). Since the weak limit of an unfolded sequence is equal to the two-scale limit of the sequence, \( \xi = w_{\delta m} \) (cf. part (vii) of the theorem 3.6.4). Furthermore the continuity of \( \psi(\xi) \) implies its strong convergence to \( \psi(w_{\delta m}) \) in \( L^2((0,T) \times \Omega \times \Gamma) \). Under similar arguments \( \psi(\xi) \) is two-scale convergent to \( \psi(w_{\delta m}) \) in \( L^2((0,T) \times \Omega \times \Gamma) \).

Therefore for all \( m = 1,2,\ldots,I_2 \), the sequences \( \left( w_{\varepsilon \mu \delta m}(\cdot) \right)_{\varepsilon > 0} \) and \( \left( \psi(\xi) \right)_{\varepsilon > 0} \) are two-scale convergent to the limits \( w_{\delta m} \) and \( \psi(\delta m) \) respectively. (4.2.234)

Let us choose a test function \( \phi(t,x,y) \in \left[ C_0^\infty((0,T) \times \Omega; C_0^\infty(Y)) \right]_{l_2} \), then from (4.2.234) and from theorems 3.5.16 and 4.2.2.1.1.1, we have

\[
\sum_{m=1}^{I_2} \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon \mu \delta m}(t,x)}{\partial t} \phi_m(t,x,\frac{x}{\varepsilon}) \, d\sigma_x \, dt \\
= -k_d \sum_{m=1}^{I_2} \lim_{\varepsilon \to 0} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \psi(\xi)(w_{\varepsilon \mu \delta m}(t,x)) \phi_m(t,x,\frac{x}{\varepsilon}) \, d\sigma_x \, dt,
\]

i.e.,

\[
\sum_{m=1}^{I_2} \int_0^T \int_{\Omega \times \Gamma} \frac{\partial w_{\varepsilon \mu \delta m}(t,x,y)}{\partial t} \phi_m(t,x,y) \, dx \, dy \, dt = -k_d \sum_{m=1}^{I_2} \int_0^T \int_{\Omega \times \Gamma} \psi(\xi)(w_{\varepsilon \mu \delta m}(t,x,y)) \phi_m(t,x,y) \, dx \, dy \, dt,
\]
Let us choose the functions \( \phi \) the procedure to obtain the finally arrive to the equations similar to (4.1.100) and (4.1.102). Thus we have \( \phi \) i.e.,

\[
\int_0^T \int_{\Omega \times \Gamma} \left( \frac{\partial w_\delta(t, x, y)}{\partial t}, \phi(t, x, y) \right) dx \, dy \, dt = -k_d \int_0^T \int_{\Omega \times \Gamma} (\psi_\delta(w_\delta(t, x, y)), \phi(t, x, y)) \, dx \, dy \, dt, 
\]

\[
\Longrightarrow \quad \frac{\partial w_\delta}{\partial t} = -k_d \psi_\delta(w_\delta) \quad \text{for a.e. in } (0, T) \times \Omega \times \Gamma; \quad (4.2.235)
\]

\[
w_\delta(0, x, y) = w_0(x, y) \quad \text{for a.e. in } \Omega \times \Gamma. \quad (4.2.236)
\]

### 4.2.2.3.2 Homogenization of the PDE (4.2.17)-(4.2.21)

Let us choose the functions \( \phi_0 \in C_0^\infty((0, T) \times \Omega)^I \) and \( \phi_1 \in C_0^\infty((0, T) \times \Omega; C^\infty_{\text{per}}(Y))^I \). Set \( \phi(t, x, \frac{x}{\varepsilon}) = \phi_0(t, x) + \varepsilon \phi_1(t, x, \frac{x}{\varepsilon}) \in C_0^\infty((0, T) \times \Omega; C^\infty_{\text{per}}(Y))^I \). Using \( \phi \) as test function in the weak formulation of (4.2.17)-(4.2.21), we get

\[
\sum_{k=1}^{I_2} \int_0^T \int_{\Omega_\varepsilon \times \Gamma} \left( \frac{\partial v_{\varepsilon, k}}{\partial t}, \phi_k \right) \, dx \, dt + \sum_{k=1}^{I_2} \int_0^T \int_{\Omega_\varepsilon} \left( D \nabla v_{\varepsilon, k} - \bar{q}_\varepsilon v_{\varepsilon, k} \right) \nabla \phi_k \, dx \, dt 
+ \sum_{k=1}^{I_2} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon, k}}{\partial t} \phi_k \, d\sigma \, dt 
= \sum_{k=1}^{I_2} \int_0^T \left( S_2 R(u_{\varepsilon, k}, v_{\varepsilon, k}), \phi_k \right) \, dt,
\]

i.e.,

\[
I_{\text{time}} + I_{\text{diff}} + I_{\text{bound}} = I_{\text{reac}}, \quad (4.2.237)
\]

where

\[
I_{\text{time}} = \sum_{k=1}^{I_2} \int_0^T \left( \frac{\partial v_{\varepsilon, k}}{\partial t}, \phi_k \right) \, dt, \quad (4.2.238)
\]

\[
I_{\text{diff}} = \sum_{k=1}^{I_2} \int_0^T \int_{\Omega_\varepsilon} \left( D \nabla v_{\varepsilon, k} - \bar{q}_\varepsilon v_{\varepsilon, k} \right) \nabla \phi_k \, dx \, dt, \quad (4.2.239)
\]

\[
I_{\text{bound}} = \sum_{k=1}^{I_2} \varepsilon \int_0^T \int_{\Gamma_\varepsilon} \frac{\partial w_{\varepsilon, k}}{\partial t} \phi_k \, d\sigma \, dt, \quad (4.2.240)
\]

\[
I_{\text{reac}} = \sum_{k=1}^{I_2} \int_0^T \left( S_2 R(u_{\varepsilon, k}, v_{\varepsilon, k}), \phi_k \right) \, dt. \quad (4.2.241)
\]

Now we pass to the two-scale limit in each term separately. Note that for (4.2.238) and (4.2.241) the procedure to obtain the two-scale limit follows like the section 4.1.2.3 and we finally arrive to the equations similar to (4.1.100) and (4.1.102). Thus we have

\[
\lim_{\varepsilon \to 0} I_{\text{time}} = \lim_{\varepsilon \to 0} \sum_{k=1}^{I_2} \int_0^T \left( \frac{\partial v_{\varepsilon, k}}{\partial t}, \phi_k \right) \, dt = |Y|^p \sum_{k=1}^{I_2} \int_0^T \left( \frac{\partial v_{\varepsilon, k}}{\partial t}, \phi_0_k \right)_{H^{1,2}(\Omega)^* \times H^{1,2}(\Omega)} \, dt \quad (4.2.242)
\]

\[
||w_{0,m}||_{L^p(\Omega \times \Gamma)}^p = \int_\Gamma \int_{\Omega} |T_{\varepsilon}^m w_{0,m}(x, y)|^p \, dx \, dy = |\Gamma| \int_{\Omega} |w_{0,m}(x)|^p \, dx < \infty \text{ by (4.2.5).}
\]
and

\[
\lim_{\varepsilon \to 0} I_{\text{react}} = |Y^p| \int_0^T (S_2 R(u_{\delta}, v_{\delta}), \phi_0)_{[H^{1,2}(\Omega)^*] \times [H^{1,2}(\Omega)]} dt. \tag{4.2.243}
\]

Next,

\[
\lim_{\varepsilon \to 0} I_{\text{diff}} = \lim_{\varepsilon \to 0} \sum_{k=1}^{I^2} \int_0^T \int_{\Omega} \left( D \left( D v_{\varepsilon{k}h} - \overline{q}_k v_{\varepsilon{k}h} \right) (\nabla \phi_{0k} + \nabla y \phi_{1k}) \right) dx \, dt.
\]

Again,

\[
\lim_{\varepsilon \to 0} I_{\text{bound}} = \lim_{\varepsilon \to 0} \sum_{k=1}^{I^2} \int_0^T \int_{\Gamma_x} \frac{\partial w_{\varepsilon{k}h}}{\partial t} \phi_{0k} \, d\sigma_x \, dt
\]

Combining the equations (4.2.242), (4.2.243), (4.2.244) and (4.2.245), we obtain

\[
|Y^p| \int_0^T \left( \frac{\partial u_{\delta}}{\partial t}, \phi_0 \right)_{[H^{1,2}(\Omega)^*] \times [H^{1,2}(\Omega)]} dt
+ \sum_{k=1}^{I^2} \int_0^T \int_{\Omega} \int_{\Gamma_y} \left( D \left( \nabla v_{\delta{k}} + \nabla y \frac{v_{1}}{\delta{k}} \right) - v_{\delta{k}} \overline{q}_1 \right) (\nabla \phi_{0k} + \nabla y \phi_{1k}) \, dx \, dy \, dt
+ \int_0^T \int_{\Omega} \int_{\Gamma} \left( \frac{\partial w_{\delta}}{\partial t}, \phi_0 \right) dx \, dy \, dt = |Y^p| \int_0^T (S_2 R(u_{\delta}, v_{\delta}), \phi_0)_{[H^{1,2}(\Omega)^*] \times [H^{1,2}(\Omega)]} dt. \tag{4.2.246}
\]

In an analogy to section 4.1.2.3, here also we decouple the equation (4.2.246) to achieve the homogenized equation and the Cell-Problem. Setting \( \phi_0 \equiv 0 \), the equation (4.2.246) reduces to

\[
\sum_{k=1}^{I^2} \int_0^T \int_{\Omega} \int_{\Gamma_y} \left( D \left( \nabla v_{\delta{k}} + \nabla y \frac{v_{1}}{\delta{k}} \right) - v_{\delta{k}} \overline{q}_1 \right) \cdot \nabla y \phi_{1k} \, dx \, dy \, dt = 0, \tag{4.2.247}
\]

We state the following lemma from [MZ11]:
**Lemma 4.2.2.3.2.1.** Let \( a_j(y) \) for \( j = 1, 2, ..., n \) be the \( Y \)-periodic solution of the integral identity

\[
\int_0^T \int_{\Omega} \int_{Y^p} (e_j + \nabla_y a_j(y)) \cdot \nabla_y \phi_{1_k} \, dx \, dy \, dt = 0, \tag{4.2.248}
\]

and \( a_0(t, x, y) \) be the solution to the integral identity

\[
\int_0^T \int_{\Omega} \int_{Y^p} (\bar{q}_1 + \nabla_y a_0) \cdot \nabla_y \phi_{1_k} \, dx \, dy \, dt = 0, \tag{4.2.249}
\]

for any \( Y \)-periodic smooth function \( \phi_1 \). Then the function

\[
v_{\delta_k}(x, y, t) = \sum_{j=1}^n \frac{\partial v_{\delta_k}(t, x)}{\partial x_j} a_j(y) + a_0(x, y, t)v_{\delta_k}(t, x)
\]

satisfies the integral identity (4.2.247).

**Proof.** The proof is straightforward and hence omitted. \( \triangleleft \)

We set \( \bar{q}_0 = D \int_{Y^p} \nabla_y a_0 \, dy \). Now setting \( \phi_1 \equiv 0 \), then (4.2.246) reduces to

\[
\int_{Y^p} \int_0^T \left( \frac{\partial v_5}{\partial t}, \phi_0 \right) dt + \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_{Y^p} D \left( \nabla v_{\delta_k} + \nabla_y v_{\delta_k}^1 \right) \nabla \phi_{0_k} \, dx \, dy \, dt
\]

\[
= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle dt - \frac{1}{|Y^p|} \int_0^T \int_{\Omega} \int_\Gamma \left( \frac{\partial w_5}{\partial t}, \phi_0 \right) \, dx \, dy \, dt, \tag{4.2.250}
\]

i.e.,

\[
\int_0^T \left( \frac{\partial v_5}{\partial t}, \phi_0 \right) dt + \frac{1}{|Y^p|} \sum_{k=1}^{I_2} \int_0^T \int_{Y^p} D \left( \nabla v_{\delta_k} + \nabla_y v_{\delta_k}^1 \right) \nabla \phi_{0_k} \, dx \, dy \, dt
\]

\[
= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle dt - \frac{1}{|Y^p|} \int_0^T \int_{\Omega} \int_\Gamma \left( \frac{\partial w_5}{\partial t}, \phi_0 \right) \, dx \, dy \, dt. \tag{4.2.250}
\]

Substituting \( v_{\delta_k}^1(x, y, t) = \sum_{j=1}^n \frac{\partial v_{\delta_k}(t, x)}{\partial x_j} a_j(y) + a_0(x, y, t)v_{\delta_k}(t, x) \), for \( k = 1, 2, ..., I_2 \), in (4.2.250) leaves

\[
\int_0^T \left( \frac{\partial v_5}{\partial t}, \phi_0 \right) dt + \int_0^T \int_{Y^p} D \left( \nabla v_{\delta_k} + \nabla_y v_{\delta_k}^1 \right) \nabla \phi_{0_k} \, dx \, dy \, dt
\]

\[
= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle dt - \frac{1}{|Y^p|} \int_0^T \int_{\Omega} \int_\Gamma \left( \frac{\partial w_5}{\partial t}, \phi_0 \right) \, dx \, dy \, dt,
\]
i.e.,

\[
\int_0^T \left( \frac{\partial v_\delta}{\partial t}, \phi_0 \right) dt + \sum_{k=1}^{l_2} \int_0^T \int_\Omega \sum_{i,j=1}^n \left( \frac{D}{|Y_p|} \left( \delta_{jl} + \frac{\partial a_j}{\partial y_l} \right) dy \right) \frac{\partial v_\delta_k}{\partial x_j} \frac{\partial \phi_0_k}{\partial x_l} dx dt
\]

\[
- \frac{1}{|Y_p|} \sum_{k=1}^{l_2} \int_0^T \int_\Omega (\tilde{q}_l - D \nabla_y a_0) v_\delta_k \nabla \phi_0_k dx dy dt
\]

\[
= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle dt - \frac{1}{|Y_p|} \sum_{k=1}^{l_2} \int_0^T \int_\Gamma \left( \frac{\partial w_\delta}{\partial t}, \phi_0 \right) dx dy dt,
\]

i.e.,

\[
\int_0^T \left( \frac{\partial v_\delta}{\partial t}, \phi_0 \right) dt + \sum_{k=1}^{l_2} \int_0^T \int_\Omega P \nabla v_\delta \cdot \nabla \phi_0 dx dt - \frac{1}{|Y_p|} \sum_{k=1}^{l_2} \int_0^T \int_\Omega (\tilde{q}_l - \tilde{q}_0) v_\delta_k \nabla \phi_0_k dx dt
\]

\[
= \int_0^T \langle S_2 R(u_\delta, v_\delta), \phi_0 \rangle dt - \frac{1}{|Y_p|} \int_0^T \int_\Gamma \left( \frac{\partial w_\delta}{\partial t}, \phi_0 \right) dx dy dt,
\]

(4.2.251)

where \( P = (p_{jl})_{1 \leq j \leq n} \) is a positive definite second order symmetric tensor whose components are given by

\[
p_{jl} = \int_{Y_p} \frac{D}{|Y_p|} \left( \delta_{jl} + \frac{\partial a_j}{\partial y_l} \right) dy \quad \text{for} \quad j, l = 1, 2, ..., n.
\]

(4.2.252)

Therefore the strong form of the homogenized equation (4.2.251) is

\[
\frac{\partial v_\delta}{\partial t} - \nabla \left( P \nabla v_\delta - \frac{1}{|Y_p|} (\tilde{q} - \tilde{q}_0) v_\delta \right) = S_2 R(u_\delta, v_\delta)
\]

\[
- \frac{1}{|Y_p|} \int_\Gamma \left( \frac{\partial w_\delta}{\partial t} \right) dy \quad \text{in} \quad (0, T) \times \Omega,
\]

(4.2.253)

\[
- \left( P \nabla v_\delta - \frac{1}{|Y_p|} (\tilde{q} - \tilde{q}_0) v_\delta \right) \cdot \tilde{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{in},
\]

(4.2.254)

\[
- \left( P \nabla v_\delta + \frac{1}{|Y_p|} \tilde{q}_0 v_\delta \right) \cdot \tilde{n} = 0 \quad \text{on} \quad (0, T) \times \partial \Omega_{out},
\]

(4.2.255)

\[
v_\delta(0, x) = v_0(x) \quad \text{in} \quad \Omega.
\]

(4.2.256)

### 4.2.2.3.3 Homogenization of the PDE (4.2.12)-(4.2.16)

The arguments and the procedure to homogenize the PDE (4.2.12)-(4.2.16) are similar to the approach shown in the previous subsection 4.2.2.3.2. Choosing a function \( \phi(t, x, \frac{z}{\varepsilon}) = \phi_0(t, x) + \varepsilon \phi_1(t, x, \frac{z}{\varepsilon}) \), where \( \phi_0 \in C_0^\infty((0, T) \times \Omega) \) and \( \phi_1 \in C_0^\infty((0, T) \times \Omega; C_{per}(Y)) \). Using \( \phi \) as test function in the weak formulation of the PDE (4.2.12)-(4.2.16), we get

\[
\sum_{i=1}^{l_1} \int_0^T \left( \frac{\partial u_{\varepsilon, i}}{\partial t}, \phi_i \right) dt + \sum_{i=1}^{l_1} \int_0^T \int_{Y_p} (D \nabla u_{\varepsilon, i} - \tilde{q}_0 v_{\varepsilon, i}) \nabla \phi_i dx dt = \sum_{i=1}^{l_1} \int_0^T (S_1 R(u_{\varepsilon, i}, v_{\varepsilon, i}, 1), \phi_i) dt,
\]

i.e.,

\[
I_{time} + I_{diff} = I_{reac},
\]

(4.2.257)

where

\[
I_{time} = \sum_{i=1}^{l_1} \int_0^T \left( \frac{\partial u_{\varepsilon, i}}{\partial t}, \phi_i \right) dt,
\]

(4.2.258)
\[ I_{\text{diff}} = \sum_{i=1}^{l_1} \int_0^T \int_{\Omega^e} \left( D \nabla v_{\varepsilon, i} - \bar{q}_i v_{\varepsilon, i} \right) \nabla \phi_i \, dx \, dt, \quad (4.2.259) \]
\[ I_{\text{reac}} = \sum_{i=1}^{l_1} \int_0^T \langle S_1 R(u_{\varepsilon, i}, v_{\varepsilon, i}), \phi_i \rangle \, dt. \quad (4.2.260) \]

Letting \( \varepsilon \to 0 \) in two-scale sense in the terms \( I_{\text{time}}, I_{\text{diff}} \) and \( I_{\text{reac}} \) and proceeding in a similar fashion like the previous subsection, we obtain weak form of the homogenized equation as
\[
\int_0^T \left\langle \frac{\partial u_{\delta}}{\partial t}, \phi_0 \right\rangle \, dt + \sum_{i=1}^{l_1} \int_0^T \int_\Omega P \nabla u_{\delta} \cdot \nabla \phi_0 \, dx \, dt
- \frac{1}{|Y_p|} \sum_{i=1}^{l_1} \int_0^T \int_\Omega (\bar{q} - \bar{q}_0) u_{\delta} \nabla \phi_0 \, dx \, dt = \int_0^T \langle S_1 R(u_{\delta}, v_{\delta}), \phi_0 \rangle \, dt, \quad (4.2.261) \]
where \( P = (p_{jl})_{1 \leq j \leq n}^{1 \leq i \leq n} \) is a positive definite second order symmetric tensor whose components are given by
\[
p_{jl} = \int_{Y_p} \frac{D}{|Y_p|} \left( \delta_{jl} + \frac{\partial a_{ji}}{\partial y_l} \right) \, dy \quad \text{for } j, l = 1, 2, \ldots, n. \quad (4.2.262)\]
The strong form of the homogenized problem (4.2.261) is
\[
\frac{\partial u_{\delta}}{\partial t} - \nabla \left( P \nabla u_{\delta} - \frac{1}{|Y_p|} (\bar{q} - \bar{q}_0) u_{\delta} \right) = S_1 R(u_{\delta}, v_{\delta}) \quad \text{in } (0, T) \times \Omega, \quad (4.2.263)\]
\[
- \left( P \nabla u_{\delta} - \frac{1}{|Y_p|} (\bar{q} - \bar{q}_0) u_{\delta} \right) \cdot \vec{n} = d \quad \text{on } (0, T) \times \partial \Omega, \quad (4.2.264)\]
\[
- \left( P \nabla u_{\delta} + \frac{1}{|Y_p|} \bar{q}_0 u_{\delta} \right) \cdot \vec{n} = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (4.2.265)\]
\[
u_{\delta}(0, x) = u_0(x) \quad \text{in } \Omega. \quad (4.2.266)\]
Therefore the complete homogenized problem is
\[
\frac{\partial u_{\delta}}{\partial t} - \nabla \left( P \nabla u_{\delta} - \frac{1}{|Y_p|} (\bar{q} - \bar{q}_0) u_{\delta} \right) = S_1 R(u_{\delta}, v_{\delta}) \quad \text{in } (0, T) \times \Omega, \quad (4.2.267)\]
\[
- \left( P \nabla u_{\delta} - \frac{1}{|Y_p|} (\bar{q} - \bar{q}_0) u_{\delta} \right) \cdot \vec{n} = d \quad \text{on } (0, T) \times \partial \Omega, \quad (4.2.268)\]
\[
- \left( P \nabla u_{\delta} + \frac{1}{|Y_p|} \bar{q}_0 u_{\delta} \right) \cdot \vec{n} = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (4.2.269)\]
\[
u_{\delta}(0, x) = u_0(x) \quad \text{in } \Omega. \quad (4.2.270)\]
\[
\frac{\partial v_{\delta}}{\partial t} - \nabla \left( P \nabla v_{\delta} - \frac{1}{|Y_p|} (\bar{q} - \bar{q}_0) v_{\delta} \right) = S_2 R(u_{\delta}, v_{\delta}) \quad \text{in } (0, T) \times \Omega, \quad (4.2.271)\]
\[
- \left( P \nabla v_{\delta} - \frac{1}{|Y_p|} (\bar{q} - \bar{q}_0) v_{\delta} \right) \cdot \vec{n} = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (4.2.272)\]
\[
- \left( P \nabla v_{\delta} + \frac{1}{|Y_p|} \bar{q}_0 v_{\delta} \right) \cdot \vec{n} = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad (4.2.273)\]
\[
v_{\delta}(0, x) = v_0(x) \quad \text{in } \Omega, \quad (4.2.274)\]
\[
\frac{\partial w_{\delta}}{\partial t} = -k_d \psi_{\delta}(w_{\delta}) \quad \text{on } (0, T) \times \Omega \times \Gamma, \quad (4.2.275)\]
\[
w_{\delta}(0, x, y) = w_0(x, y) \quad \text{on } \Omega \times \Gamma. \quad (4.2.276)\]
where the solution
\[ u_\delta \in \mathcal{F}_2 \cap L^\infty((0,T);L^\infty(\Omega))^{I_1}, \quad v_\delta \in \mathcal{G}_2 \cap L^\infty((0,T);L^\infty(\Omega))^{I_2} \quad \text{and} \quad w_\delta \in \mathcal{H}_2. \quad (4.2.277) \]

The velocity vector satisfies
\[ \nabla_y \cdot \vec{q}_1 = 0 \quad \text{in} \quad (0,T) \times \Omega \times Y^P \quad \text{and} \quad \nabla \cdot \int_{Y^P} \vec{q}_1 \, dy = 0 \quad \text{in} \quad (0,T) \times \Omega, \quad (4.2.278) \]
\[ \vec{q}_1 = 0 \quad \text{in} \quad (0,T) \times \Omega \times Y^s. \quad (4.2.279) \]

**4.2.2.3.4 Uniqueness of the Solution of (4.2.267)-(4.2.276)**

**Theorem 4.2.3.4.1.** There exists a unique solution of the homogenized problem (4.2.267)-(4.2.276).

**Proof.** Following the steps of theorem 4.2.1.5.1 yields the proof. Note that \( P \) is a second order positive definite symmetric tensor.

**4.2.3 Passage to the Limit as \( \delta \to 0 \) in the Problem \( (P_\delta^2) \)**

**Theorem 4.2.3.1.** For any \( \delta > 0 \), the solution \((u_\delta, v_\delta, w_\delta)\) of the problem (4.2.267)-(4.2.276) satisfies the following estimate:

\[
\begin{align*}
&|||u_\delta|||_{L^2((0,T);L^2(\Omega))} + |||u_\delta|||_{L^\infty((0,T);L^\infty(\Omega))} + |||\nabla u_\delta|||_{L^2((0,T);L^2(\Omega))} + |||v_\delta|||_{L^2((0,T);L^2(\Omega))} + |||v_\delta|||_{L^\infty((0,T);L^\infty(\Omega))} \\
&+ |||\nabla v_\delta|||_{L^2((0,T);H^{1,2}(\Omega)^*)} + |||w_\delta|||_{L^2((0,T);H^{1,2}(\Omega)^*)} + |||w_\delta|||_{L^p((0,T)\times\Omega)} \\
&+ |||w_\delta|||_{L^2((0,T)\times\Omega)_{\Gamma}} + \left\| \int_{\Gamma} \frac{\partial w_\delta(y)}{\partial t} \, dy \right\|_{L^2((0,T)\times\Omega)^{I_2}} \leq C_{\delta_2} < \infty,
\end{align*}
\]

where \( C_{\delta_2} \) is independent of \( \delta \).

**Proof.** The proof consists of several steps.

(i) Multiplying both sides of (4.2.275) by \( \frac{\partial w_\delta}{\partial t} \) and integrating, we obtain

\[
\sum_{m=1}^{l_2} \int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_{\delta m}}{\partial t} \right|^2 \, dx \, d\sigma_y \, dt = -k_d \sum_{m=1}^{l_2} \int_0^T \int_{\Omega} \int_{\Gamma} \psi_\delta(w_{\delta m}) \frac{\partial w_{\delta m}}{\partial t} \, dx \, d\sigma_y \, dt,
\]

i.e.,

\[
\sum_{m=1}^{l_2} \int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_{\delta m}}{\partial t} \right|^2 \, dx \, d\sigma_y \, dt \leq \frac{1}{2} \sum_{m=1}^{l_2} \int_0^T \int_{\Omega} \int_{\Gamma} \left( k_d \psi_\delta(w_{\delta m}) \right)^2 + \left| \frac{\partial w_{\delta m}}{\partial t} \right|^2 \right\| \, dx \, d\sigma_y \, dt,
\]

i.e.,

\[
\frac{1}{2} \sum_{m=1}^{l_2} \int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_{\delta m}}{\partial t} \right|^2 \, dx \, d\sigma_y \, dt \leq \frac{k_d^2}{2} \sum_{m=1}^{l_2} \int_0^T \int_{\Omega} \int_{\Gamma} \, dx \, d\sigma_y \, dt \quad \text{since} \quad \left| \psi_\delta(w_{\delta m}) \right| \leq 1,
\]
Multiplying both sides of (4.2.275) by i.e.,

\[ \frac{\partial w_\delta}{\partial t} \| w_\delta \|_{L^2((0,T) \times \Omega \times \Gamma)}^2 \leq k_d^2 T \| \Omega \| \| \Gamma \| I_2, \]

i.e.,

\[ \frac{\partial w_\delta}{\partial t} \| w_\delta \|_{L^2((0,T) \times \Omega \times \Gamma)}^2 \leq C_{63}, \tag{4.2.281} \]

where \( C_{63} := (k_d^2 T \| \Omega \| \| \Gamma \| I_2)^{\frac{1}{2}} \) is independent of \( \delta \).

(ii) Multiplying both sides of (4.2.275) by \( w_\delta \) and integrating, we get

\[ \sum_{m=1}^{I_2} \int_0^t \int_0^\Omega \int_\Gamma \frac{\partial w_{\delta m}}{\partial \theta} w_{\delta m} \, dx \, d\sigma_y \, d\theta = -k_d \sum_{m=1}^{I_2} \int_0^t \int_0^\Omega \int_\Gamma \psi_\delta(w_{\delta m}) w_{\delta m} \, dx \, d\sigma_y \, d\theta, \]

i.e.,

\[ \frac{1}{2} \sum_{m=1}^{I_2} \int_0^t \int_0^\Omega \int_\Gamma \frac{\partial}{\partial \theta} \| w_{\delta m}(\theta) \|_{L^2(\Omega \times \Gamma)}^2 \, d\theta \leq \frac{1}{2} \sum_{m=1}^{I_2} \int_0^t \int_0^\Omega \int_\Gamma \left[ |k_d \psi_\delta(w_{\delta m})|^2 + |w_{\delta m}|^2 \right] \, dx \, d\sigma_y \, d\theta, \]

i.e.,

\[ \sum_{m=1}^{I_2} \int_0^t \int_0^\Omega \int_\Gamma \left[ |w_{\delta m}(t)|_{L^2(\Omega \times \Gamma)}^2 - |w_m(0)|_{L^2(\Omega \times \Gamma)}^2 \right] \leq \sum_{m=1}^{I_2} \int_0^t \int_0^\Omega \int_\Gamma \left[ k_d^2 + |w_{\delta m}|^2 \right] \, dx \, d\sigma_y \, d\theta. \]

This gives

\[ \sum_{m=1}^{I_2} \| w_{\delta m}(t) \|_{L^2(\Omega \times \Gamma)}^2 \leq \sum_{m=1}^{I_2} \| w_{0m} \|_{L^2(\Omega \times \Gamma)}^2 + I_2 k_d^2 T \| \Omega \| \| \Gamma \| + \int_0^t \sum_{m=1}^{I_2} \| w_{\delta m}(\theta) \|_{L^2(\Omega \times \Gamma)}^2 \, d\theta, \]

i.e.,

\[ \sum_{m=1}^{I_2} \| w_{\delta m}(t) \|_{L^2(\Omega \times \Gamma)}^2 \leq \| w_0 \|_{L^2(\Omega \times \Gamma)}^2 + I_2 k_d^2 T \| \Omega \| \| \Gamma \| + \int_0^t \sum_{m=1}^{I_2} \| w_{\delta m}(\theta) \|_{L^2(\Omega \times \Gamma)}^2 \, d\theta, \]

i.e.,

\[ \sum_{m=1}^{I_2} \| w_{\delta m}(t) \|_{L^2(\Omega \times \Gamma)}^2 \leq C_{64} + \int_0^t \sum_{m=1}^{I_2} \| w_{\delta m}(\theta) \|_{L^2(\Omega \times \Gamma)}^2 \, d\theta, \]

where \( C_{64} := \| w_0 \|_{L^2(\Omega \times \Gamma)}^2 + I_2 k_d^2 T \| \Omega \| \| \Gamma \| \) is independent of \( \delta \) by footnote 43. Gronwall’s inequality yields

\[ \sum_{m=1}^{I_2} \| w_{\delta m}(t) \|_{L^2(\Omega \times \Gamma)}^2 \leq C_{64}(1 + t e^t), \]

i.e.,

\[ \sum_{m=1}^{I_2} \int_0^T \| w_{\delta m}(t) \|_{L^2(\Omega \times \Gamma)}^2 \, dt \leq \int_0^T C_{64}(1 + t e^t) \, dt =: C_{65}, \]

i.e.,

\[ \| w_\delta \|_{L^2((0,T) \times \Omega \times \Gamma)} \leq C_{65}, \tag{4.2.282} \]
where $C_{65}$ is independent of $\delta$.

(iii) Multiplying both sides of the $m$-th ODE of (4.2.275) by $w_{\delta_m}|w_{\delta_m}|^{p-2}$ and integrating

$$
\int_0^t \int_{\Omega} \int_{\Gamma} w_{\delta_m}(\theta, x) |w_{\delta_m}(\theta, x)|^{p-2} \frac{\partial w_{\delta_m}(\theta, x)}{\partial \theta} \, dx \, d\sigma_y \, d\theta
$$

i.e.,

$$
\int_0^t \int_{\Omega} \int_{\Gamma} \frac{1}{p} \frac{\partial}{\partial \theta} w_{\delta_m}(\theta, x)^p \, dx \, d\sigma_y \, d\theta \leq \int_0^t \int_{\Omega} \int_{\Gamma} \left[ \frac{p-1}{p} |w_{\delta_m}(\theta, x)|^p + \frac{k_d^p}{p} \right] \, dx \, d\sigma_y \, d\theta,
$$

i.e.,

$$
\int_0^t \int_{\Omega} \int_{\Gamma} |w_{\delta_m}(t, x)|^p \, dx \, d\sigma_y \leq \int_0^t \int_{\Omega} \int_{\Gamma} |w_{\delta_m}(0, x)|^p \, dx \, d\sigma_y + \int_0^t \int_{\Omega} \int_{\Gamma} ((p-1)|w_{\delta_m}(\theta, x)|^p + k_d^p) \, dx \, d\sigma_y \, d\theta,
$$

i.e.,

$$
\sum_{m=1}^{I_2} \int_{\Omega} \int_{\Gamma} |w_{\delta_m}(t, x)|^p \, dx \, d\sigma_y \leq \sum_{m=1}^{I_2} \int_{\Omega} \int_{\Gamma} |w_{\delta_m}(0, x)|^p \, dx \, d\sigma_y + T \, I_2 \, |\Gamma| \, |\Omega| \, k_d^p + (p-1) \sum_{m=1}^{I_2} \int_0^t \int_{\Omega} \int_{\Gamma} |w_{\delta_m}(\theta, x)|^p \, dx \, d\sigma_y \, d\theta.
$$

A straightforward application of Gronwall’s inequality and steps similar to part (ii) will imply

$$
\|w_\delta\|_{L^p((0, T) \times \Omega \times \Gamma)^2} \leq \left[ \sum_{m=1}^{I_2} \int_0^T \int_{\Omega} \int_{\Gamma} |w_{\delta_m}(t, x)|^p \, dx \, d\sigma_y \, dt \right]^{\frac{1}{p}} \leq C_{66},
$$

(4.2.283)

where $C_{66}$ is independent of $\delta$.

(iv) Integrating both sides of the $m$-th ODE of (4.2.275) and squaring leaves

$$
\left[ \int_{\Gamma} \frac{\partial w_\delta_m}{\partial t} \, d\sigma_y \right]^2 = k_d^2 \left[ \int_{\Gamma} \psi_\delta(w_\delta_m) \, d\sigma_y \right]^2,
$$

i.e.,

$$
\left[ \int_{\Gamma} \frac{\partial w_\delta_m}{\partial t} \, d\sigma_y \right]^2 \leq k_d^2 \left| \Gamma \right|^2,
$$

i.e.,

$$
\left\| \int_{\Gamma} \frac{\partial w_\delta_m}{\partial t} \, d\sigma_y \right\|_{L^2((0, T) \times \Omega)}^2 \leq \left| \Omega \right| \left| \Gamma \right|^2 \, T \, k_d^2,
$$

i.e.,

$$
\sum_{m=1}^{I_2} \left\| \int_{\Gamma} \frac{\partial w_\delta_m}{\partial t} \, d\sigma_y \right\|_{L^2((0, T) \times \Omega)}^2 \leq I_2 \left| \Omega \right| \left| \Gamma \right|^2 \, T \, k_d^2,
$$
Chapter 4. Existence of a Unique Global Solution and Homogenization

i.e.,

\[ \left\| \int_{\Gamma} \frac{\partial w_\delta}{\partial t} \, d\sigma \right\|_{L^2((0,T) \times \Omega)} \leq C_{67}, \]  

(4.2.284)

where \( C_{67} \) is independent of \( \delta \).

(v) From corollary 4.2.2.1.3, we have

\[ \left\| v_\delta \right\|_{L^\infty((0,T) \times \Omega)} \leq C_{68}, \]  

(4.2.285)

where \( C_{68} \) is independent of \( \delta \).

(vi) Note that

\[ \left\| v_\delta \right\|_{L^2((0,T) \times \Omega)} = \sum_{k=1}^{I_2} \left\| v_{\delta_k} \right\|_{L^2((0,T) \times \Omega)} \leq \sum_{k=1}^{I_2} \int_0^T \int_{\Omega} \left| v_{\delta_k}(t,x) \right|^2 \, dx \, dt \]

\[ \leq \sum_{k=1}^{I_2} \left\| v_{\delta_k} \right\|_{L^\infty((0,T) \times \Omega)} T |\Omega| \]

\[ \leq \left\| v_\delta \right\|_{L^\infty((0,T) \times \Omega)} I_2 T |\Omega| \]

\[ \leq C_{68}^2 I_2 T |\Omega|, \quad \text{by (4.2.285)}, \]

i.e.,

\[ \left\| v_\delta \right\|_{L^2((0,T) \times \Omega)} \leq C_{69}, \]  

(4.2.286)

where \( C_{69} := (C_{68}^2 I_2 T |\Omega|)^{\frac{1}{2}} \) is independent of \( \delta \).

(vii) Using corollary 4.2.2.2.2.3, we get

\[ \left\| u_\delta \right\|_{L^\infty((0,T) \times \Omega)} \leq C_{70}, \]  

(4.2.287)

where \( C_{70} \) is independent of \( \delta \).

(viii)

\[ \left\| u_\delta \right\|_{L^2((0,T) \times \Omega)} \leq C_{71}, \]  

(4.2.288)

where \( C_{71} \) is independent of \( \delta \).

(ix) Testing (4.2.271) by \( v_\delta \) leaves

\[ \frac{1}{2} \sum_{k=1}^{I_2} \int_0^T \frac{d}{dt} \left\| v_{\delta_k}(t) \right\|_{L^2(\Omega)}^2 \, dt + \sum_{k=1}^{I_2} \int_0^T \int_{\Omega} P \nabla v_{\delta_k} \cdot \nabla v_{\delta_k} \, dx \, dt \]

\[ = \sum_{k=1}^{I_2} \left[ \frac{1}{|\Omega|} \int_0^T \int_{\partial\Omega_n} \bar{q} \cdot \bar{n} \left| v_{\delta_k}(t) \right|^2 \, ds \, dt - \frac{1}{|\Omega|} \int_0^T \int_{\Omega} \bar{q} \cdot \nabla v_{\delta_k} v_{\delta_k} \, dx \, dt \right. \]

\[ + \left. \frac{1}{|\Omega|} \int_0^T \int_{\Omega} \left( \int_{\Gamma} \frac{\partial v_{\delta_k}}{\partial t} \, d\sigma \right) v_{\delta_k} \, dx \, dt \right]. \]  

(4.2.289)

Recall theorem 3.1.3 which implies \( \left\| v(0) \right\|_{L^\infty(\Omega)} < \infty \) and investing the knowledge of

\[ Notice that \( v_\delta \) is independent of \( y \).
positive definiteness of the symmetric tensor $P$, we obtain

\[
\frac{1}{2} \sum_{k=1}^{l_2} \|v_{\delta_k}(T)\|_{L^2(\Omega)}^2 - \frac{1}{2} \sum_{k=1}^{l_2} \|v_k(0)\|_{L^2(\Omega)}^2 + \sum_{k=1}^{l_2} \int_0^T \int_\Omega |\nabla v_{\delta_k}|^2 \, dx \, dt \leq \sum_{k=1}^{l_2} \left[ \|q - \bar{n}\|_{L^\infty((0,T) \times \partial \Omega_n)} \int_0^T \int_{\partial \Omega} |v_{\delta_k}|^2 \, ds \, dt + \frac{1}{\|Y\|} \int_0^T \int_\Omega |q| |\nabla v_{\delta_k}| |v_{\delta_k}| \, dx \, dt \right] \\
+ \frac{1}{2} \sum_{k=1}^{l_2} \left[ \int_0^T \|S_2 R(u_\delta(t), v_\delta(t))\|_{L^2(\Omega)}^2 \, dt + \int_0^T \|v_{\delta_k}(t)\|_{L^2(\Omega)}^2 \, dt \right] \\
+ \frac{1}{\|Y\|} \int_0^T \int_\Omega |\nabla v_{\delta_k}| |v_{\delta_k}| \, dx \, dt + \frac{1}{2} \sum_{k=1}^{l_2} \left[ \int_0^T \int_\Omega \left| \frac{\partial w_{\delta_k}}{\partial t} \right|^2 \, dx \, dt + \int_0^T \|v_{\delta_k}(t)\|_{L^2(\Omega)}^2 \, dt \right].
\]

(4.2.290)

By Young’s inequality, 45

\[
\frac{1}{\|Y\|} \sum_{k=1}^{l_2} \int_0^T \int_\Omega |q| |\nabla v_{\delta_k}| |v_{\delta_k}| \, dx \, dt + \frac{1}{\|Y\|} \sum_{k=1}^{l_2} \int_0^T \int_\Omega |\bar{q}| |\nabla v_{\delta_k}| |v_{\delta_k}| \, dx \, dt \leq \frac{1}{\|Y\|} \sum_{k=1}^{l_2} \left( C_{46} I_2 \frac{1}{2} \|q\|_{L^2((0,T) \times \Omega)} \|\nabla v_{\delta_k}\|_{L^2((0,T) \times \Omega)} \right) \\
+ \frac{1}{\|Y\|} \sum_{k=1}^{l_2} \left( C_{46} I_2 \frac{1}{2} \|\bar{q}\|_{L^2((0,T) \times \Omega)} \|\nabla v_{\delta_k}\|_{L^2((0,T) \times \Omega)} \right) \text{ by corollary 4.2.2.1.3}
\]

\[
\leq \sum_{k=1}^{l_2} \left[ \frac{\beta}{4} \|\nabla v_{\delta_k}\|_{L^2((0,T) \times \Omega)}^2 + \frac{C_{46} I_2}{\beta \|Y\|} \|q\|_{L^2((0,T) \times \Omega)}^2 \right] \\
+ \sum_{k=1}^{l_2} \left[ \frac{\beta}{4} \|\nabla v_{\delta_k}\|_{L^2((0,T) \times \Omega)}^2 + \frac{C_{46} I_2}{\beta \|Y\|} \|\bar{q}\|_{L^2((0,T) \times \Omega)}^2 \right] \\
\leq \sum_{k=1}^{l_2} \left[ \frac{\beta}{2} \|\nabla v_{\delta_k}\|_{L^2((0,T) \times \Omega)}^2 + \frac{C_{46} I_2}{\beta \|Y\|} (\|q\|_{L^2((0,T) \times \Omega)}^2 + \|\bar{q}\|_{L^2((0,T) \times \Omega)}^2) \right].
\]

(4.2.291)

From boundary inequality (cf. theorem B.7) and Young’s inequality, we have

\[
\frac{1}{\|Y\|} \sum_{k=1}^{l_2} \int_0^T \int_{\partial \Omega} |v_{\delta_k}|^2 \, ds \, dt \leq C_{72} \|q^\cdot \bar{n}\|_{L^\infty(\partial \Omega_n)} \sum_{k=1}^{l_2} \left[ \|\nabla v_{\delta_k}\|_{L^2((0,T);L^2(\Omega))} \|v_{\delta_k}\|_{L^2((0,T);L^2(\Omega))} + \|v_{\delta_k}\|_{L^2((0,T);L^2(\Omega))} \right]
\]

45We note that $\|q\|_{L^2((0,T) \times \Omega)} < \infty$ and $\|\bar{q}\|_{L^2((0,T) \times \Omega)} < \infty$. 
\[
\frac{\beta}{4} \sum_{k=1}^{l_2} \int_0^T \int_\Omega |\nabla v_{\delta_k}|^2 \, dx \, dt \\
+ \left( \frac{(C_{T_2} \| \tilde{q} \cdot \tilde{n} \|_{L^\infty(\partial \Omega_{in})})^2}{\beta |Y|^2} + \frac{C_{T_2} \| \tilde{q} \cdot \tilde{n} \|_{L^\infty(\partial \Omega_{in})}}{|Y|^2} \right) \sum_{k=1}^{l_2} \int_0^T \int_\Omega |v_{\delta_k}|^2 \, dx \, dt
\]
\]
\[
\leq \frac{\beta}{4} \sum_{k=1}^{l_2} \int_0^T \int_\Omega |\nabla v_{\delta_k}|^2 \, dx \, dt + C_{T_3} \sum_{k=1}^{l_2} \int_0^T \int_\Omega |v_{\delta_k}|^2 \, dx \, dt,
\]
(4.2.292)

where \(C_{T_2}\) and \(C_{T_3}\) are independent of \(v_{\delta}\) and \(\delta\). Therefore invoking the estimates (4.2.284), (4.2.285), (4.2.284), (4.2.287), (4.2.291) and (4.2.292) in (4.2.290) we see that the r.h.s. of (4.2.290) is finite and independent of \(\delta\), i.e.,
\[
\frac{\beta}{4} \sum_{k=1}^{l_2} \int_0^T \int_\Omega |\nabla v_{\delta_k}|^2 \, dx \, dt \leq C_{T_4},
\]
i.e.,
\[
|||v_{\delta}|||_{L^2((0,T);L^2(\Omega))^2} \leq C_{T_5},
\]
(4.2.293)

where \(C_{T_5} := \left( \frac{\beta}{4} C_{T_4} \right)^\frac{1}{2} \) is independent of \(\delta\).

(x) Testing (4.2.271) by \( \phi \in L^2((0,T);H^{1,2}(\Omega))^f\) and following the steps of part (ix) leads us to
\[
|||\partial \frac{\partial v_{\delta}}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq C_{T_6},
\]
(4.2.294)

where \(C_{T_6}\) is independent of \(\delta\). Arguments similar to steps (ix) and (x) will yield (xi)
\[
|||\nabla u_{\delta}|||_{L^2((0,T);L^2(\Omega))^f} \leq C_{T_7}
\]
(4.2.295)

and

(xii)
\[
|||\partial \frac{\partial u_{\delta}}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^*)} \leq C_{T_8},
\]
(4.2.296)

where \(C_{T_7}\) and \(C_{T_8}\) are independent of \(\delta\). Therefore combining the above estimates, we obtain
\[
|||u_{\delta}|||_{L^2((0,T);L^2(\Omega))^f} + |||u_{\delta}|||_{L^\infty((0,T);L^\infty(\Omega))^f} + |||\nabla u_{\delta}|||_{L^2((0,T);L^2(\Omega))^f} + |||\partial \frac{\partial u_{\delta}}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^*)} + |||\nabla u_{\delta}|||_{L^2((0,T);L^2(\Omega))^f} + |||\partial \frac{\partial u_{\delta}}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^*)}
\]
\[
+ |||w_{\delta}|||_{L^2((0,T);L^\infty(\Omega)^f)} + |||\partial \frac{\partial w_{\delta}}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^*)} + |||\nabla w_{\delta}|||_{L^2((0,T);L^\infty(\Omega)^f)} + |||\partial \frac{\partial w_{\delta}}{\partial t}|||_{L^2((0,T);H^{1,2}(\Omega)^*)}
\]
\[
\leq C_{T_3} + \sum_{p=65}^{71} C_p + \sum_{p=75}^{78} C_p = C_{T_4} < \infty,
\]

where \(C_{T_4} := \sum_{p=65}^{71} C_p + \sum_{p=75}^{78} C_p\) is independent of \(\delta\).
Now we have sufficient tools to send $\delta \to 0$. We follow the idea shown in [vDP04]. Let $z_\delta \in L^\infty((0,T) \times \Omega \times \Gamma)^{\delta}$ be defined by

$$z_\delta(t,x,y) = \psi_\delta(w_\delta(t,x,y)) \quad \text{for a.e.} \ (t,x,y) \in (0,T) \times \Omega \times \Gamma.$$ 

Due to estimate (4.2.280), there exists a triple

$$\langle u, v, w \rangle \in \mathcal{F}_2^w \times \mathcal{G}_2^w \times \mathcal{H}_2^w$$

such that the following convergences holds:

(i) $u_\delta \rightharpoonup u$ in $L^2((0,T);H^{1,2}(\Omega))^{I_1}$.

(ii) $\frac{\partial u_\delta}{\partial t} \rightharpoonup \frac{\partial u}{\partial t}$ in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}$.

(iii) $u_\delta \rightharpoonup u$ in $L^2((0,T);L^2(\Omega))^{I_1}$.

(iv) $v_\delta \rightharpoonup v$ in $L^2((0,T);H^{1,2}(\Omega))^{I_2}$.

(v) $\frac{\partial v_\delta}{\partial t} \rightharpoonup \frac{\partial v}{\partial t}$ in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}$.

(vi) $v_\delta \rightharpoonup v$ in $L^2((0,T);L^2(\Omega))^{I_2}$.

(vii) $w_\delta \rightharpoonup w$ in $L^2((0,T) \times \Omega \times \Gamma)^{I_2}$.

(viii) $\frac{\partial w_\delta}{\partial t} \rightharpoonup \frac{\partial w}{\partial t}$ in $L^2((0,T) \times \Omega \times \Gamma)^{I_2}$.

(ix) $\int_\Gamma \frac{\partial u_\delta}{\partial t} d\sigma_y \rightharpoonup \int_\Gamma \frac{\partial w}{\partial t} d\sigma_y$ in $L^2((0,T) \times \Omega)^{I_2}$.

(x) $z_\delta \rightharpoonup z$ in $L^\infty((0,T) \times \Omega \times \Gamma)^{I_2}$.

**Theorem 4.2.3.2.** The weak limits u and v belong to $L^\infty((0,T) \times \Omega)^{I_1}$ and $L^\infty((0,T) \times \Omega)^{I_2}$ respectively.

**Proof.** Investing the knowledge of strong convergences and $L^\infty$ - estimates of $(u_\delta)_{\delta>0}$ and $(v_\delta)_{\delta>0}$ and replicating the steps of theorem 4.1.2.2.4 will yield the proof.

**Theorem 4.2.3.3.** The source terms $(S_1 R(u_\delta, v_\delta))_{\delta>0}$ and $(S_2 R(u_\delta, v_\delta))_{\delta>0}$ are strongly convergent to $S_1 R(u,v)$ and to $S_2 R(u,v)$ in $L^2((0,T) \times \Omega)^{I_1}$ and $L^2((0,T) \times \Omega)^{I_2}$ respectively.

**Proof.** The strong convergences of $(u_\delta)_{\delta>0}$ and $(v_\delta)_{\delta>0}$ and the $L^\infty$ - estimates of $u_\delta$, $v_\delta$, $u$ and $v$ finish off the proof. Follow the steps of theorem 4.2.2.2.4.

**Remark 4.2.3.4.** Note that the strong convergence of $(S_1 R(u_\delta, v_\delta))_{\delta>0}$ in $L^2((0,T) \times \Omega)^{I_1}$ implies its strong convergence in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}$ and this shows its weak convergence in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_1}$. Similarly $(S_2 R(u_\delta, v_\delta))_{\delta>0}$ is weakly convergent in $L^2((0,T);H^{1,2}(\Omega)^*)^{I_2}$.

**Theorem 4.2.3.5.** There exists a unique weak solution

$$u \in \mathcal{F}_2^w \cap L^\infty((0,T) \times \Omega)^{I_1}, \ v \in \mathcal{G}_2^w \cap L^\infty((0,T) \times \Omega)^{I_2}, \ w \in \mathcal{H}_2^w \text{ and } z \in \mathcal{M}_2^\infty$$

of the problem

$$\frac{\partial u}{\partial t} - \nabla \left( P \nabla u - \frac{1}{|Y|^p} (q - q_0) u \right) = S_1 R(u,v) \quad \text{in} \ (0,T) \times \Omega,$$ 

$$- \left( P \nabla u - \frac{1}{|Y|^p} (q - q_0) \right) \cdot \vec{n} = d \quad \text{on} \ (0,T) \times \partial \Omega_{in},$$ 

$$- (P \nabla u + \frac{1}{|Y|^p} \vec{q}_0) \cdot \vec{n} = 0 \quad \text{on} \ (0,T) \times \partial \Omega_{out},$$ 

$$u(0,x) = u_0(x) \quad \text{in} \ \Omega,$$
\[
\frac{\partial v}{\partial t} - \nabla \left( PV - \frac{1}{|Y|} (\bar{q} - \bar{q}_0) v \right) = S_2 R(u,v)
\]
\[
- \frac{1}{|Y|} \int_{\Gamma} \frac{\partial w}{\partial t} \, dy \text{ in } (0, T) \times \Omega, \tag{4.2.304}
\]
\[
- \left( PV - \frac{1}{|Y|} (\bar{q} - \bar{q}_0) v \right) \cdot \vec{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_m, \tag{4.2.305}
\]
\[
- (PV + \frac{1}{|Y|} \bar{q}_0) \cdot \vec{n} = 0 \quad \text{on } (0, T) \times \partial \Omega_{out}, \tag{4.2.306}
\]
\[
v(0, x) = v_0(x) \quad \text{in } \Omega, \tag{4.2.307}
\]
\[
\frac{\partial w}{\partial t} = -k_d z \quad \text{on } (0, T) \times \Omega \times \Gamma, \tag{4.2.308}
\]
\[
w(0, x, y) = w_0(x, y) \quad \text{on } \Omega \times \Gamma, \tag{4.2.309}
\]
\[
z \in \psi(w) \quad \text{on } (0, T) \times \Omega \times \Gamma, \tag{4.2.310}
\]

where
\[
\psi(w_m) = \begin{cases} 
0 & \text{if } w_m < 0, \\
[0, 1] & \text{if } w_m = 0, \\
1 & \text{if } w_m > 0,
\end{cases} \tag{4.2.311}
\]

which satisfies the estimate
\[
|||v|||_{L^2((0,T);L^2(\Omega))} + |||v|||_{L^\infty((0,T);L^\infty(\Omega))} \leq |||\nabla u|||_{L^2((0,T);L^2(\Omega))} \tag{4.2.312}
\]

where \(C_70\) is independent of \(\delta\).

**Proof.** The estimate (4.2.312) follows immediately from the weak convergences in (4.2.299). Moreover, \((u, v, w)\) satisfies the equation (4.2.300)-(4.2.309). Here special attention needs to be paid to prove (4.2.310). This part is shown in theorem 2.21 in [vDP04].

**Lemma 4.2.3.6.** Suppose that \(p > n + 2\) and \(\bar{q}_0 \in L^2((0, T) \times \Omega)\). If we define a map \(\Lambda_{\bar{q}_0} : L^\infty((0, T) \times \Omega)^{l_2} \to \mathcal{L}^p((0, T); H^{1,q}(\Omega))^{l_2}\) by
\[
\langle \Lambda_{\bar{q}_0} \phi, \zeta \rangle := \frac{1}{|Y|} \sum_{k=1}^{l_2} \int_0^T \int_{\Omega} \phi_k \bar{q}_0 \cdot \nabla \zeta \, dx \, dt, \quad \zeta \in L^q((0, T); H^{1,q}(\Omega))^{l_2},
\]
then the map \(\Lambda_{\bar{q}_0}\) is well defined and continuous.

**Proof.** The proof is similar as the one for lemma 4.2.1.3.1.

**Theorem 4.2.3.7.** There exists a unique positive global weak solution
\[
u \in \mathcal{F}_p^u \cap L^\infty((0, T) \times \Omega)^{l_1}, v \in \mathcal{G}_p^v \cap L^\infty((0, T) \times \Omega)^{l_2}, w \in \mathcal{H}_p^w, z \in \mathcal{M}_\infty^z \tag{4.2.313}
\]
of the problem (4.2.300)-(4.2.311).
Proof. Step 1. (a) Multiplying both sides of the $m$-th ODE of (4.2.308) by $\frac{\partial w_m}{\partial t} \left| \frac{\partial w_m}{\partial t} \right|^{p-2}$ and integrating, we obtain

$$\int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_m}{\partial t} \right|^p d x d \sigma_y d t = -k_d \int_0^T \int_{\Omega} \int_{\Gamma} \psi(w_m) \left| \frac{\partial w_m}{\partial t} \right|^{p-2} d x d \sigma_y d t,$$

i.e.,

$$\int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_m}{\partial t} \right|^p d x d \sigma_y d t \leq \int_0^T \int_{\Omega} \int_{\Gamma} k_d |\psi(w_m)| \left| \frac{\partial w_m}{\partial t} \right|^{p-1} d x d \sigma_y d t,$$

i.e.,

$$\int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_m}{\partial t} \right|^p d x d \sigma_y d t \leq \int_0^T \int_{\Omega} \int_{\Gamma} \left[ \frac{p-1}{p} \left| \frac{\partial w_m}{\partial t} \right|^p + \frac{1}{p} k_d^p \right] d x d \sigma_y d t,$$

i.e.,

$$\frac{1}{p} \int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_m}{\partial t} \right|^p d x d \sigma_y d t \leq \frac{k_d^p}{p} \int_0^T \int_{\Omega} \int_{\Gamma} d x d \sigma_y d t,$$

i.e.,

$$\sum_{m=1}^{I_2} \int_0^T \int_{\Omega} \int_{\Gamma} \left| \frac{\partial w_m}{\partial t} \right|^p d x d \sigma_y d t \leq k_d^p T |\Omega| |\Gamma| I_2,$$

i.e.,

$$\left\| \frac{\partial w}{\partial t} \right\|_{L^p((0,T) \times \Omega \times \Gamma)} < \infty. \quad (4.2.314)$$

(b) Multiplying both side of the $m$-th ODE of (4.2.308) by $w_m |w_m|^{p-2}$ and integrating, we obtain

$$\int_0^t \int_{\Omega} \int_{\Gamma} w_m |w_m|^{p-2} \frac{\partial w_m}{\partial \theta} d x d \sigma_y d \theta = -k_d \int_0^t \int_{\Omega} \int_{\Gamma} \psi(w_m) |w_m|^{p-2} w_m d x d \sigma_y d \theta,$$

i.e.,

$$\frac{1}{p} \int_0^t \frac{\partial}{\partial \theta} \left| w_m(\theta) \right|_{L^p(\Omega \times \Gamma)}^p d \theta \leq \int_0^t \int_{\Omega} \int_{\Gamma} \left[ \frac{p-1}{p} |w_m|^p + \frac{1}{p} k_d^p \right] d x d \sigma_y d \theta,$$

i.e.,

$$\left| w_m(t) \right|_{L^p(\Omega \times \Gamma)}^p \leq \left| w_m(0) \right|_{L^p(\Omega \times \Gamma)}^p + k_d^p T |\Omega| |\Gamma| + (p-1) \int_0^t \left| w_m(\theta) \right|_{L^p(\Omega \times \Gamma)}^p d \theta.$$
Note that $||w_m(0)||_{L^p(\Omega \times \Gamma)} < \infty$ by footnote 43. A straightforward application of Gronwall’s inequality and integration from 0 to $T$ leaves

$$|||w|||_{L^p((0,T) \times \Omega \times \Gamma)} < \infty.$$  \hspace{1cm} (4.2.315)

Therefore (4.2.314)-(4.2.315) shows that $w \in H^w_p$.

Step 2. The abstract formulation of the problem (4.2.304)-(4.2.307) is given by

$$\frac{\partial v}{\partial t} + Av = f(v) + f_{\text{bound}}(v),$$  \hspace{1cm} (4.2.316)

$$v(0,x) = v_0(x),$$  \hspace{1cm} (4.2.317)

where $f(v) = S_2 R(u,v) + \kappa v - \frac{1}{|\Gamma|} \int_\Gamma \frac{\partial w}{\partial t} d\sigma_y - \bar{q} \cdot \nabla v - \Lambda \bar{q}_0 v$, $f_{\text{bound}}(v) = Q_{\partial \Omega^m}(v)$ and the operator $A : H^{1,p}(\Omega^p)^I \to [H^{1,q}(\Omega^p)^*)^I$ is defined as $A v := (A_1 v_1, A_2 v_2, \ldots, A_{I_2} v_{I_2})$ such that for $1 \leq k \leq I_2$,

$$(A_k v_k, \zeta_k) := \int_\Omega P \nabla v_k(x) \cdot \nabla \zeta_k(x) dx + \kappa \int_\Omega v_k(x) \zeta_k(x) dx \text{ for } v_k \in H^{1,p}(\Omega) \text{ and } \zeta_k \in H^{1,q}(\Omega),$$  \hspace{1cm} (4.2.318)

where $\kappa > 0$. The estimate (4.2.312) and lemma 4.2.3.6 imply $f \in L^p((0,T);H^{1,q}(\Omega)^*)^I$. From lemma 4.2.1.3.1 it follows that $f_{\text{bound}} \in L^p((0,T);H^{1,q}(\Omega)^*)^I$. Moreover the initial condition $v_0 \in X^w_p$. Therefore by theorem 3.3.1, there exists a unique solution $v \in G^w_p$ of (4.2.304)-(4.2.307), i.e., a unique solution of (4.2.300)-(4.2.303) and it satisfies the estimate

$$||v||_{G^w_p} \leq C_{S_0} \left(||v_0||_{X^w_p} + ||f + f_{\text{bound}}||_{L^p((0,T);H^{1,q}(\Omega)^*)^I} \right).$$  \hspace{1cm} (4.2.319)

Step 3. Again using the estimate (4.2.312) and following the arguments of step 2, we obtain the existence of a unique weak solution $u \in F^w_p$ of (4.2.300)-(4.2.303) This completes the proof. ♦
In this chapter, the models M1 and M2 are investigated numerically. For the sake of illustration, we restrict ourselves to relatively simple 2-dimensional situations. For the numerical simulations, COMSOL Multiphysics 4.3a (see [Com10]) is used. In section 5.1 model M1 and in section 5.2 model M2 are examined. For both the models, we solve the micro problem, the cell-problems and the macro problem respectively. The scaling parameter $\varepsilon$ and the regularization parameter $\delta$ are chosen as 0.2 and 0.001 respectively. All the figures in this chapter are generated by the author using COMSOL Multiphysics 4.3a.

5.1 Simulation of Model M1

The model M1 at the micro scale (see section 2.5.2) is given by

$$\frac{\partial u_\varepsilon}{\partial t} - \nabla \cdot D \nabla u_\varepsilon = SR(u_\varepsilon) \quad \text{in} \quad (0,T) \times \Omega^p_\varepsilon, \quad (5.1.1)$$

$$u_\varepsilon(0,x) = u_0(x) \quad \text{in} \quad \Omega^p_\varepsilon, \quad (5.1.2)$$

$$-D \nabla u_\varepsilon \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \partial \Omega, \quad (5.1.3)$$

$$-D \nabla u_\varepsilon \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \Gamma_\varepsilon. \quad (5.1.4)$$

The homogenized form of (5.1.1)-(5.1.4) is given by

$$\frac{\partial u}{\partial t} - \nabla \cdot P \nabla u = SR(u) \quad \text{in} \quad (0,T) \times \Omega, \quad (5.1.5)$$

$$-P \nabla u \cdot \vec{n} = 0 \quad \text{on} \quad (0,T) \times \partial \Omega, \quad (5.1.6)$$

$$u(0,x) = u_0(x) \quad \text{in} \quad \Omega. \quad (5.1.7)$$

Here $P = (p_{jk})_{1 \leq j,k \leq 2}$ is a second order tensor with components

$$p_{jk} = \int_{Y^p} \frac{D}{|Y^p|} \left( \delta_{jk} + \frac{\partial a_j}{\partial y_k} \right) dy \quad \text{for all} \quad j,k = 1,2, \quad (5.1.8)$$

where for all $j = 1,2$, $(a_j)$ is the solution of the cell-problem

$$-\nabla_y \cdot (D(\nabla_y a_j(y) + e_j)) = 0 \quad \text{for} \quad y \in Y^p, \quad (5.1.9)$$

$$-D(\nabla_y a_j(y) + e_j) \cdot \vec{n} = 0 \quad \text{for} \quad y \in \Gamma, \quad (5.1.10)$$

$y \mapsto a_j(y)$ is $Y$-periodic. \quad (5.1.11)

**The physics setting:** Let us consider a domain $\Omega := [0,1.2] \times [0,1]$ in $\mathbb{R}^2$. Assume that $Y = [0,1] \times [0,1] \subset \mathbb{R}^2$ is the representative cell with $Y^s := B((0.5,0.5),0.15)$ as the solid inclusion\footnote{For $r \in \mathbb{R}^n$, $B(r,\epsilon)$ denotes an open ball centered at $r$ and radius $\epsilon$.}. Suppose that four mobile species $A$, $B$, $M$ and $N$ are present inside $\Omega$. The chemical species diffuse and react with each other (cf. figure 5.1.1).
The reaction is reversible and is given by

\[ 2A + 3B \rightleftharpoons M + 2N. \]  

(5.1.12)

The stoichiometric coefficients are -2, -3, 1 and 2, and the reaction rates for each species can be given by (2.4.7). Here \( I = 4 \) and \( J = 1 \).

### 5.1.1 Simulation at the Micro Scale

Let \( u_{\varepsilon_i} \) denote the concentration of \( i\)-th species for \( 1 \leq i \leq 4 \). We choose the scaling parameter \( \varepsilon = 0.2 \). Also, let \( D = 1.0, k_f^j = 1.8, k_b^j = 12.2 \). Initially, let us assume 

\[ u_{\varepsilon_1}(0,x) = 5x, \quad u_{\varepsilon_2}(0,x) = 2(x + 3), \quad u_{\varepsilon_3}(0,x) = 5x \quad \text{and} \quad u_{\varepsilon_4}(0,x) = 2x. \]

We “choose coarser option” mesh available in COMSOL to discretize the domain \( \Omega^\varepsilon \). The triangulation of the domain \( \Omega^\varepsilon \) is depicted in the following figure:

![Fig. 5.1.2: The triangulation of \( \Omega^\varepsilon \) for \( \varepsilon = 0.2 \).](image)

We solved the system of diffusion-reaction equations at the micro scale for \( t = 10 \) secs. We notice: the number of elements for mesh \( = 4930 \), the number of degrees of freedom \( = 10640 \) and the time taken by the solver \( = 104 \) secs. However, here we compare the solution for species A only at the micro and the macro scale, since the comparison of the solutions for the rest of the species can be done analogously. The concentration of species A is depicted in the following pictures for \( t = 0.25 \) secs, \( t = 0.35 \) secs, \( t = 3 \) secs and \( t = 10 \) secs respectively:
5.1. Simulation of Model M1

5.1.1. Simulation of Model M1

5.1.2. Solution of the Cell-Problems

In figure 5.1.3, we see the change in concentration of species A at different time. As time progresses, the concentration of species A increases and due to reversible reaction after $t = 3$ secs, the reaction reaches equilibrium. This is also shown in the figure 5.1.4 where the concentration of species A at a point (top left point of the domain) is plotted. Now we compute the effective diffusion tensor for species A. We commence by solving the cell-problems (5.1.9)-(5.1.11) in $Y$. 

**Figure 5.1.3**: Concentration of species $A$ in $\Omega_p$ at different time.

**Figure 5.1.4**: Concentration of species $A$ at the top left point of $\Omega_p$ in 10 secs.

5.1.2 Solution of the Cell-Problems

We choose the "finer mesh option" (available in COMSOL) for the triangulization of the cell $Y$. The triangulization of $Y$ is depicted below:
Chapter 5. Numerical Simulations and Outlook

Figure 5.1.5: The triangulation of cell $Y$.

In the following figure, we see the solutions of the cell-problems.

Figure 5.1.6: Solution $a_j$ of the cell-problem for $j = 1, 2$.

With the help of 'Derived Values' feature in COMSOL, we compute the diffusive tensor by the formula (5.1.8). Thus we obtain

$$P = (p_{jk})_{1 \leq j, k \leq 2} = \begin{bmatrix}
0.93409 & 4.19 \times 10^{-7} \\
4.19 \times 10^{-7} & 0.93409
\end{bmatrix}. \tag{5.1.13}$$

5.1.3 Simulation at the Macro Scale

For the simulation of upscaled model, we choose $P$ from (5.1.13), $k_f^j = 1.8$, $k_b^j = 12.2$. Initially, $u_1(0, x) = 5x$, $u_2(0, x) = 2(x + 3)$, $u_3(0, x) = 5x$ and $u_4(0, x) = 2x$. We choose the coarser mesh (in COMSOL) for $\Omega$ with 144 elements. We also notice that: the number of degrees of freedom = 352 and the time taken by the solver = 11 secs. The numerical simulation is shown in the following pictures:
5.1. Simulation of Model M1

Conclusions: Firstly, we notice that for the same type of mesh the solver takes less time to solve the macro problem than to solve the micro problem. Therefore the upscaled model is computationally efficient. Secondly, the upscaled model gives us the global information of the properties related to our porous medium. In figure 5.1.7, it is shown that as time progresses there is an increase in the concentration of species $A$ and after $t = 2.4$ secs the reaction reaches equilibrium as expected. By comparing the figures 5.1.3 and 5.1.7, we can notice that the upscaled model (5.1.5)-(5.1.7) is a good approximation to our original micro problem (5.1.1)-(5.1.4). This can also be seen by comparing the figures 5.1.4 and 5.1.8.

Figure 5.1.7: Concentration of species $A$ in $\Omega$ at different time scales.

Figure 5.1.8: Concentration of species $A$ at the top left point of $\Omega$ in 10 secs.
5.2 Simulation of Model M2

In this section, we replicate the process of section 5.1 for model M2. The micro and the macro problems are given by (2.5.21)-(2.5.36) and (4.2.300)-(4.2.311) respectively.

The physics setting: Let $\Omega$, $Y$, $Y^*$ and $\Gamma$ be like in section 5.1. Suppose that a chemical species $A$ is present in the fluid which enters in domain $\Omega$. The dissolution of immobile species (present on the surface of the solid parts) occurs on $\Gamma$. An another mobile chemical species $B$ is supplied via dissolution. The mobile species $A$ and $B$ react under the following reversible reaction (see also figure 5.2.1):

$$2A \rightleftharpoons 3B. \quad (5.2.1)$$

![Figure 5.2.1: Presence of mobile species A and B in the pore space and immobile species on \( \Gamma \).](image)

The stoichiometric coefficients of $A$ and $B$ are -2 and 3 respectively and the reaction rates of these species can be given by (2.4.7). By choosing an appropriate $\vec{q}_e$ which satisfies (2.5.36), the numerical simulations for mobile and immobile species (both at the micro and the macro scale) can be conducted and the results can be compared as we did in section 5.1.
Chapter 6

Summary and Outlook

6.1 Summary

In chapter 4, we proved the positivity, existence and uniqueness of the global solution for the models I and II respectively. At first, both the models are considered at the micro scale. Model M1 is considered without advection. In section 4.1, we proved the existence of a unique positive global weak solution of model M1. In section 4.2, we showed the existence of a unique positive global weak solution for model M2. We considered a complex scenario by incorporating dissolution in model M2. In order to prove the existence of the solution for both models, with the help of a Lyapunov functional, we obtained inequalities like (4.1.34), (4.2.78) and (4.2.152). These inequalities gave us global a-priori estimates of the solution. The inequalities of this type can also be found in the works of Glitzky, Gröger and Hünlich (cf. theorems 3.1 and 3.2 in [GGH92]) to solve nonlinear parabolic equations. These inequalities are proved to be a very efficient tool in order to show the existence of the global solution. After proving the existence of the solution, we upscaled the models (M1 and M2) from the micro to the macro scale using two-scale convergence and periodic unfolding. The homogenization (upsaling) of models M1 and M2 are shown in sections 4.1.2 and 4.2.2 respectively. We performed the numerical computations at the micro scale and at the macro scale for both the models in sections 5.1 and 5.2 respectively. From the conclusions of sections 5.1 and 5.2, we see that the upscaled models for M1 and M2 at the macro scale are a good approximation to the models M1 and M2 considered at the micro scale. For the future work, following continuations can be made.

6.2 Outlook

- **Use of different types of scaling:** For the sake of simplicity, in this work natural scale has been chosen. It is already shown in the works of Peter and Böhm (cf. [PB08], [PB05], see also [Pet06]) that with different choices of scaling factor at the micro scale one obtains different types of upscaled models at the macro scale. Thus it would be very interesting to have a different scaling in (2.5.16)-(2.5.19) or in (2.5.21)-(2.5.35). The idea to use other types of scaling can be motivated from: how much our porous medium is perforated ? or, how do the parameters (e.g. diffusion coefficient) involved in the equations oscillate ? or, what kind of flux conditions are needed on the surface of the solid parts ? etc. For a brief explanation see [All92]. Also see [PB08], [NR92], [Dob12], [Fre11] and references therein.

- **Different types of diffusion coefficients:** The following generalizations can be made for the diffusion coefficients:
  
  ▶ In this work, we considered the same diffusion coefficient for all the mobile species. We required this assumption to establish the inequality (4.1.49). Pierre has proved the existence of the global solution of a parabolic system with two different diffusion coefficients but, to our knowledge, the existence results for the
global solution in case of $I$ ($> 2$) different types of diffusion coefficients is still unknown. It would be captivating to prove the existence of the global solution of the systems considered in this work for $I$ ($> 2$) types of diffusion coefficients.

- We also considered constant diffusion coefficient in both models. In several real world problems the diffusion coefficients are piecewise continuous or in $L^\infty(\Omega)$, in such cases we may need to impose some condition on $p$. The problems with $D \in L^\infty(\Omega)$ are addressed in the works of Rehberg and Dintelmann and references therein (cf. [RDR09]). One can consider essentially bounded diffusion coefficients in the equations proposed in this work.

- **Lipschitz domains:** Our analysis is predicated to the domains which has sufficiently smooth boundaries but many real world problems involve Lipschitz domains. Rehberg and Dintelmann have considered such type of problems in [RDR09] (see also the references therein). Interested readers should conduct the investigations of chapter 4 for Lipschitz domains.

- **Different boundary conditions:** Following generalizations can be made for the boundary conditions:

  - The construction of Lyapunov functional also depends on the boundary conditions. In section 3.4 in [Krä11], Kräutle has indicated how one can construct the Lyapunov functional in the presence of Dirichlet BCs. Thus one can try to obtain the existence of solution in the presence of Dirichlet BCs in $H^{1,p}$ - setting.

  - To have nonlinear inflow-outflow boundary conditions.

  - The problems incorporated with mixed BCs, i.e., both Dirichlet and Neumann BCs has drawn a great attention of mathematician as they fit perfectly to many real world situations (see [RDR09] and references therein). In the literature, it is shown that due to the presence of mixed BCs we loose the regularity on $p$, i.e., $p \leq 4$. Therefore it would be very interesting to incorporate our systems with mixed type of BCs and obtain the existence of the global solution.

- **Including precipitation in the model:** Both precipitation and dissolution are widely explored in the fields chemical engineering, pharmaceutical industry and several others. In our work, we paid attention to dissolution process only, however, considering precipitation of immobile species (crystals) on the surface of the solid parts is definitely worth to inspect. See the works of Knabner, Duijn, Noorden, Böhm, Peter, Muntean and references therein (cf. [Kna86], [KvDH95], [vDP04], [vNP08], [vN09b], [vN09a], [PB08], [MB09b], [BJDR98], [FM12] etc.) for a detailed overview.

- **Moving boundary and variable geometry:** The dissolution and precipitation inside a porous medium can also lead to the problems with moving boundary. Since these processes occur on the surface of the solid parts (see figure 6.2.1), they may affect the size of the matrices and this could lead to the change in geometry of the domain (cf. [vN09a], [Pet06]).
In our work the boundary $\Gamma$ is considered as fixed but one can consider the model proposed in this work with moving boundary $\Gamma$, i.e., Stefan like problem (cf. [vN09b], [vN09a], [vNP08], [Pet06]).

- **Including diffusion-reaction on the surface** $\Gamma$: Other than dissolution and precipitation, one can consider diffusion and reaction of chemical species on the surface of the solid parts. Such type of models has been studied in [HJ91], [NR92], [Pet03], [Dob12] etc for the linear reaction rates on $\Gamma$ but one can modify such models by incorporating nonlinear reaction rates on $\Gamma$. To our knowledge the existence of the global solution and homogenization of such models are still open.

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47 This picture is taken from Qualichemlab.com.
Appendices

A. Inequalities

Here we state some elementary inequalities which we have used frequently throughout this work. The proofs of all these inequalities can be found in the appendix B.2 of [Eva98].

**Lemma A.1** (Young’s inequality with \( \epsilon \)). Let \( 1 \leq p, q \leq \infty \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Assume that \( \epsilon, a \) and \( b > 0 \), then

\[
ab \leq \epsilon a^p + C(\epsilon) b^q,
\]

where \( C(\epsilon) = (\epsilon p)^{-\frac{q}{p}} q^{-1} \).

**Lemma A.2** (Hölder’s inequality). Let \( 1 \leq p, q \leq \infty \) be such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Suppose further that \( u \in L^p(\Omega) \) and \( v \in L^q(\Omega) \), then

\[
\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.
\]

**Lemma A.3** (Generalized Hölder’s inequality). Let \( 1 \leq p_1, p_2, \ldots, p_r \leq \infty \) be such that \( \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_r} = 1 \). Assume that \( u_s \in L^{p_s}(\Omega) \) for \( s = 1, 2, \ldots, r \), then

\[
\|u_1 u_2 \ldots u_r\|_{L^1(\Omega)} \leq \prod_{s=1}^{r} \|u_s\|_{L^{p_s}(\Omega)}.
\]

**Lemma A.4** (Minkowski’s inequality). Let \( 1 \leq p \leq \infty \) and \( u, v \in L^p(\Omega) \), then

\[
\|u + v\|_{L^p(\Omega)} \leq \|u\|_{L^p(\Omega)} + \|v\|_{L^p(\Omega)}.
\]

**Lemma A.5** (Generalized Minkowski’s inequality). Let \( 1 \leq p \leq \infty \) and \( u_s \in L^p(\Omega) \) for \( s = 1, 2, \ldots, r \), then

\[
\left\| \sum_{s=1}^{r} u_s \right\|_{L^p(\Omega)} \leq \sum_{s=1}^{r} \|u_s\|_{L^p(\Omega)}.
\]

**Lemma A.6** (Lyapunov’s interpolation inequality). Let \( 1 \leq p \leq q \leq r \leq \infty \) and \( 0 < \theta < 1 \) be such that \( \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r} \). Assume also that \( u \in L^p(\Omega) \cap L^r(\Omega) \). Then \( u \in L^q(\Omega) \) and satisfies

\[
\|u\|_{L^q(\Omega)} \leq \|u\|_{L^p(\Omega)}^\theta \|u\|_{L^r(\Omega)}^{1-\theta}.
\]

**Lemma A.7** (Gronwall’s inequality in differential form). Let \( u(.) \) be a nonnegative, absolutely continuous function on \([0, T] \) which satisfies the following inequality

\[
\frac{\partial u(t)}{\partial t} \leq \phi(t) u(t) + \psi(t) \quad \text{for a.e. } t,
\]

where \( \phi(t) \) and \( \psi(t) \) are nonnegative, summable functions on \([0, T] \). Then

\[
u(t) \leq e^{\int_0^t \phi(s) ds} \left[ u(0) + \int_0^t \psi(s) ds \right], \quad \text{for all } t \in [0, T].
\]

In particular, if \( \frac{\partial u(t)}{\partial t} \leq \phi(t) u(t) \) and \( u(0) = 0 \), then \( u(t) = 0 \) on \([0, T] \).
Lemma A.8 (Gronwall’s inequality in integral form). Let $C_1$ and $C_2 \geq 0$. Assume that $v(.)$ is a nonnegative, summable function on $[0, T]$ which satisfies the following integral inequality

$$v(t) \leq C_1 \int_0^t v(s) \, ds + C_2$$

for a.e. $t$. \hspace{1cm} (A.9)

Then

$$v(t) \leq C_2 \left(1 + tC_1 e^{C_1 t}\right)$$

for a.e. $t \in [0, T]$. \hspace{1cm} (A.10)

In particular, if $v(t) \leq C_1 \int_0^t v(s) \, ds$ for a.e. $t$, then $v(t) = 0$ a.e. $t \in [0, T]$.

Lemma A.9 (Discrete Hölder’s inequality). Let $1 \leq p, q \leq \infty$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. If $a_k \geq 0, b_k \geq 0$ for $k = 1, 2, \ldots, n$, then the following inequality holds:

$$\sum_{k=1}^n a_k b_k \leq \left( \sum_{k=1}^n a_k^p \right)^{\frac{1}{p}} \left( \sum_{k=1}^n b_k^q \right)^{\frac{1}{q}}.$$

(B.11)

B. Some Important Theorems and Lemmas

Theorem B.1 (Schaefer’s fixed point theorem). Let $\Xi$ be a Banach space. Assume that $Z : \Xi \rightarrow \Xi$ is a continuous and compact map. Suppose further that the set

$$\{u \in \Xi : u = \lambda Z(u) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then $Z$ has a fixed point.

Proof. See theorem 4 in section 9.2.2 in [Eva98].

Lemma B.2. For $l = 1, 2, \ldots, s$, let $a_l$ and $\bar{a}_l \in \mathbb{R}$, then

$$a_1 a_2 \ldots a_s - \bar{a}_1 \bar{a}_2 \ldots \bar{a}_s = \sum_{l=1}^s a_1 \ldots a_{l-1} (a_l - \bar{a}_l) \bar{a}_{l+1} \ldots \bar{a}_s.$$

(B.2)

Proof. It is a mere calculation.

Theorem B.3 (Sobolev continuous embedding theorem). Let $1 \leq p \leq \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial \Omega$.

(i) If $s_1 < s_2$, where $s_1$ and $s_2$ are any two positive real number, then

$$H^{s_2,p}(\Omega) \hookrightarrow H^{s_1,p}(\Omega).$$

(B.3)

(ii) If $1 \leq p_1 \leq p_2 \leq \infty$ and $s \in \mathbb{R}$, then

$$H^{s,p_2}(\Omega) \hookrightarrow H^{s,p_1}(\Omega).$$

(B.4)

(iii) If $k$ be any positive integer, then

$$H^{k,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & 1 \leq q \leq p^* = \frac{np}{n-kp}, \quad kp < n, \\ L^q(\Omega), & 1 \leq q < \infty, \quad kp = n, \\ C^{\alpha}(\Omega), & 0 < \alpha \leq 1 - \frac{n}{kp}, \quad kp > n. \end{cases}$$

(B.5)

Proof. (i) See theorem 6.2.3 in [BL76].

(ii) Follows directly from $L^{p_2}(\Omega) \hookrightarrow L^{p_1}(\Omega)$, if $p_1 \leq p_2$.

(iii) See corollary 1.3.1 [WYW06]. See also [AF03], [Eva98].

\hfill\Box
Theorem B.4 (Sobolev compact embedding theorem). Let $1 \leq p \leq \infty$. Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary $\partial \Omega$. Then

$$H^{1,p}(\Omega) \hookrightarrow \begin{cases} L^q(\Omega), & 1 \leq q \leq p^* < \frac{np}{n-p}, \ p < n, \\ L^q(\Omega), & 1 \leq q < \infty, \ p = n, \\ C^\alpha(\bar{\Omega}), & 0 < \alpha < 1 - \frac{n}{p}, \ p > n. \end{cases} \quad (B.6)$$

Proof. Cf. theorem 1.3.3 in [WYW06]. See also [AF03], [Eva98].

Theorem B.5 (Trace theorem). Let $1 \leq p < \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial \Omega$. Then there exists a bounded linear operator $T : H^{1,p}(\Omega) \rightarrow L^p(\partial \Omega)$ such that

(i) $Tu := u|_{\partial \Omega}$ if $u \in H^{1,p}(\Omega) \cap C(\bar{\Omega})$

and

(ii) $||Tu||_{L^p(\partial \Omega)} \leq C ||u||_{H^{1,p}(\Omega)}$, for each $u \in H^{1,p}(\Omega)$,

where $C$ depends on $p$ and $\Omega$ but it is independent of $u$.

Proof. See theorem 1 in section 5.5 in [Eva98].

Theorem B.6 (Extension theorem). Let $1 \leq p \leq \infty$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial \Omega$. Suppose that $V$ is a bounded open set such that $\Omega \subset V$. Then there exists a bounded linear operator $E : H^{1,p}(\Omega) \rightarrow H^{1,p}(\mathbb{R}^n)$ such that for each $u \in H^{1,p}(\Omega)$:

(i) $Eu := u$ a.e. in $\Omega$,

(ii) $Eu$ has a support in $V$,

and

(iii) $||Eu||_{H^{1,p}(\mathbb{R}^n)} \leq C ||u||_{H^{1,p}(\Omega)}$, \quad (B.8)

where $C$ depends only on $p$, $\Omega$, and $V$.

Proof. See theorem 1 in section 5.4 in [Eva98].

Theorem B.7. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary, then for all $u \in H^{1,2}(\Omega)$ the following estimate hold:

$$||u||_{L^2(\partial \Omega)}^2 \leq C ||u||_{H^{1,2}(\Omega)} ||u||_{L^2(\Omega)}^2, \quad (B.9)$$

where the constant $C$ is independent of $u$.

Proof. See lemma 5.6 in [Krä08].

Lemma B.8. Let $\mu^0$ be given as in (4.1.19). Then $\langle \mu^0 + \log u_\varepsilon, SR(u_\varepsilon) \rangle_I \leq 0$.

Proof. See pages 71 - 72 in [Krä08].
References


References


References


References


References


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The transport process in a porous medium is a complex phenomena. In this thesis, the heterogeneities inside a porous medium are assumed to be periodically distributed and diffusion-reaction of a finite number of chemical species are investigated. Two different models are proposed in this work. In model M1, diffusion-reaction of mobile chemical species are considered. The chemical processes are modeled via mass action kinetics and the modeling leads to a system of multi-species diffusion-reaction equations (nonlinear partial differential equations) at the micro scale. For this system of equations, existence of a unique positive global weak solution is proved by the help of a Lyapunov functional and Schaefer’s fixed point theorem. The upscaled model of this system is obtained using periodic homogenization which is an averaging method.

In model M2, we consider diffusion-advection-reaction of two different types of mobile species (type I and type II). The type II species are supplied via dissolution process due to the presence of immobile species on the surface of the solid parts. The presence of mobile and the immobile species make the model complex and the modeling yields a coupled system of nonlinear partial differential equations. The existence of a unique positive global weak solution of this coupled system is shown. Finally, with the help of periodic homogenization, model M2 is upscaled from the micro scale to the macro scale.

Numerical simulations are conducted for both models separately. For the purpose of illustration, we restrict ourselves to relatively simple 2-dimensional situations. For models M1 and M2, simulation results at the micro scale and at the macro scale are compared.