Multi-Mechanism Models
Theory and Applications

Nils Hendrik Kröger
Multi-Mechanism Models
Theory and Applications

Nils Hendrik Kröger

Dissertation

zur Erlangung des Grades eines Doktors der Naturwissenschaften
Dr. rer. nat.

Vorgelegt im Fachbereich 3 (Mathematik & Informatik)
der Universität Bremen
im Juli 2013
Datum des Promotionskolloquiums: 18. November 2013

Gutachter: PD. Dr. Michael Wolff (Universität Bremen)
Prof. Dr. Ing. habil. Dr. h.c. Holm Altenbach (Otto-von-Guericke-Universität Magdeburg)
Abstract

Multi-mechanism models (MM models) are used studying various materials and mechanical effects.

In this work, a general concept of modeling with MM models of serial type is introduced within the framework of continuum mechanics. Contrary to many authors, the thermoelastic strain is not regarded as a special separated strain. The modular principle of MM models is illustrated by several basic, advanced and generalized mechanisms and MM models. We specify the modeling of linear viscoelasticity with MM models and apply the concept of MM models to the phenomena of transformation-induced plasticity.

In the case of viscoelasticity, we present the 3d mathematical problems of a thermoelastic-(N)-coupled-Kelvin-Voigt-element model and an isothermal-(N)-coupled-Kelvin-Voigt-element model. The latter model is mathematically analyzed according to its weak solvability as a mixed boundary value problem by transforming the original problem into an equivalent system of integro-differential equations. Therefore, the presented approach of proving an existence and uniqueness result is nonstandard. In case of a 1d rod, the modeling and the mathematical treatment are provided related to the isothermal-(2)-coupled-Kelvin-Voigt-element model. We conduct numerical simulations of the isothermal-(2)-coupled-Kelvin-Voigt-element model for 3d and 1d situations. The results underline the quality of the introduced viscoelastic model covering material effects like ratcheting without predicting instantaneous elasticity.

The phenomena of transformation-induced plasticity can occur for materials which undergo phase transformations when moderate stresses are applied. A detailed test evaluation for experiments characterizing the interaction of classic plasticity and transformation-induced plasticity of the steel 100Cr6 (SAE 52100) is presented. A MM model considering the interaction of classic plasticity and transformation-induced plasticity is verified by these experiments. Therefore, we develop numerical schemes of the model suited for uniaxial situations and perform systematical parameter identifications for some selected experiments.
Zusammenfassung

Mehr-Mechanismen-Modelle (MM-Modelle) werden für verschiedenste Materialien und mechanische Effekte studiert.


Danksagung (Acknowledgement)


Entscheidend zum erfolgreichen Gelingen beigetragen hat auch das äußerst herzliche kollegiale Miteinander am ZeTeM (Zentrum für Technomathematik) und in der Arbeitsgruppe Modellierung & Partielle Differentialgleichungen. Im Speziellen bedanke ich mich für die vielen hilfreichen Anmerkungen und anregenden Diskussionen rundum die Arbeit und weit darüber hinaus bei Dr. Sören Boettcher, Dr. Sören Dobberschütz, Dr. Hari Shankar Mahato, Dr. Jonathan Montalvo-Urquizo, Michael Eden und ganz besonders bei Simone Bökenheide.

Weiter danke ich Karsten Kruse und Dirk Heinemann für die schöne Zeit während des Studiums und allen Freunden, welche mich stets in meinem Unterfangen unterstützt beziehungsweise in richtigen Momenten davon abgelenkt haben.

Und nicht zuletzt möchte ich meiner Familie für die Unterstützung in aller Hinsicht danken!

Oder um es mit einem Wort zu sagen:

Danke!

Bremen, Juli 2013

Nils Hendrik Kröger
## Contents

1. Introduction .............................. 1  
   1.1. Notation ........................................ 3  
      1.1.1. Nomenclature .................................. 3  
      1.1.1.1. Variables and Parameters ..................... 3  
      1.1.1.2. Symbols .................................. 6  
      1.1.2. Abbreviations .................................. 7  
      1.1.3. Voigt-Notation for Material Laws ................. 7  
      1.1.4. Function Spaces .................................. 8  

2. Multi-Mechanism Models ................. 11  
   2.1. State of the Art ......................... 11  
   2.2. General Model of Serial Type ............... 12  
   2.3. Modular Principle - Examples of Mechanisms and Models 14  
      2.3.1. Thermoelastic Elements ......................... 16  
      2.3.1.1. Basic Thermoelastic Element .................... 16  
      2.3.1.2. Coupled Thermoelastic Mechanisms with Hardening 17  
      2.3.2. Viscous Mechanism ............................... 18  
      2.3.3. Viscelastic Mechanisms ............................. 18  
      2.3.3.1. Based on Rheological Maxwell Model ............... 18  
      2.3.3.2. Based on Rheological Kelvin-Voigt Model .......... 19  
      2.3.4. Plastic Mechanisms ............................... 20  
      2.3.4.1. Rheological Ideal Plastic Mechanism .............. 20  
      2.3.4.2. Plasticity with Back Stress – Chaboche Model .... 21  
      2.3.4.3. mMnC: MM-Model for Plasticity .................. 22  
      2.3.5. Creep Mechanism ................................ 27  
      2.3.6. Viscoplastic Mechanisms ....................... 28  
      2.3.7. TRIP Mechanism ................................ 28  
      2.3.7.1. Thermoelastic-TRIP Model ....................... 29  
      2.3.8. MM Model with Chaboche Ansatz ................. 31  
   2.4. Some Remarks on Thermomechanical Consistency ........... 32  

3. Viscoelasticity within the Framework of MM Models ........... 39  
   3.1. Thermoelastic-Viscoelastic MM Model .................. 39  
   3.2. Isothermal Viscoelasticity (3d) ................. 42  
      3.2.1. Modeling – (N)-coupled-KV Model ................. 42  
      3.2.2. Weak Solvability of the Transformed Transient Problem 44  
      3.2.3. Numerical Simulations with COMSOL 4.3a – 3d Narrow Rod 57  
          3.2.3.1. Cyclic Tension Test with Relaxation .......... 58  
          3.2.3.2. Results .................................. 60  
   3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod .... 61  
      3.3.1. Modeling – (2)-coupled-KV Model ................. 61  
      3.3.2. Initial and Boundary Conditions ................. 64  
          3.3.2.1. Initial Conditions .......................... 64  
          3.3.2.2. Boundary Conditions ....................... 64
3.3.3. Weak Solvability of the Transient Problem ................................ 67
3.3.3.1. Homogeneous Dirichlet Boundary Value Problem .................. 68
3.3.3.2. Inhomogeneous Dirichlet Boundary Value Problem ............... 81
3.3.3.3. Mixed Boundary Value Problem ................................ 83
3.3.3.4. Problem of Consistent Inhomogeneous Dirichlet Boundary Con-
ditions ....................................................... 87
3.3.4. Nondimensionalization and Scaling of the Mixed Boundary Value Problem 89
3.3.4.1. Nondimensionalization ................................ 90
3.3.4.2. Scaling ........................................... 91
3.3.4.3. Nondimensionalized and Scaled Problem ...................... 91
3.3.5. Numerical Simulations with COMSOL 4.3a – 1d Rod ................. 92
3.3.5.1. Creep Test ........................................ 93
3.3.5.2. Cyclic Tension Test with Relaxation ........................ 98
3.3.5.3. Results ............................................ 104

4. Transformation-Induced Plasticity – Experiments and Model Verifications 105
4.1. Experiments and Data Post Processing ........................................ 106
4.1.1. Experimental Set-Up and Procedure .................................... 106
4.1.2. Measured Data ..................................................................... 109
4.1.2.1. Engineering Stress ........................................... 109
4.1.2.2. Engineering Strains ........................................... 109
4.1.3. Material Data .................................................................... 109
4.1.3.1. Austenite .................................................... 109
4.1.3.2. Bainite ..................................................... 110
4.1.4. Derived Data .................................................................... 110
4.1.4.1. Phase Fractions ............................................. 111
4.1.5. Post Processing ............................................................. 113
4.1.5.1. Restriction to Subparts of the Experiment ...................... 113
4.1.5.2. Smoothing .................................................................. 113
4.1.5.3. Reducing ......................................................... 114
4.1.6. Data Analysis and Conclusions .......................................... 114
4.1.6.1. Part 1: Austenitic Transformation .................................. 114
4.1.6.2. Part 2: Predeformation at Austenitizing Temperature ........ 115
4.1.6.3. Part 3: Quenching from Austenitizing to Bainitizing Temperature 118
4.1.6.4. Part 4: Predeformation at Bainitizing Temperature ............ 118
4.1.6.5. Part 5: Bainitic Transformation .................................... 122
4.1.6.6. Part 6: Cooling to Room Temperature ........................... 131
4.2. Thermoelastic-Plastic-TRIP Model ............................................. 131
4.2.1. MM Model for Interaction of Classic Plasticity and TRIP ........... 132
4.2.2. 1d Model for Uniaxial Experiments ................................... 135
4.2.2.1. Numerical Scheme of the 1d Model for Uniaxial Experiments 137
4.3. Parameter Identification .......................................................... 140
4.3.1. Identification for Predeformation ....................................... 140
4.3.1.1. Results for 850°C .............................................. 141
4.3.1.2. Results for 280°C .............................................. 145
4.3.1.3. Results for 360°C .............................................. 147
4.3.1.4. Discussion of the Conducted Parameter Identifications for Pre-
deformations ...................................................... 149
4.3.2. Identification for Bainitic Transformation .............................. 150
4.3.2.1. Results for transformation temperature 280°C with predeforma-
ation at 850°C: Group (SA ±50MPa) .................................. 153
4.3.2.2. Results for transformation temperature 280°C with predeformation at 280°C: Group (UA ±50MPa) .................. 155
4.3.3. Conclusions of the Parameter Identifications .................. 156

5. Summary and Outlook 159

A. Concept of True Stress and True Strain 161
   A.1. True Stress ........................................ 161
   A.2. True Strain ........................................ 161

Bibliography 162
List of Figures

2.1. Example: Scheme of a MM model with \((N+1)\) partial strains ........................................ 13
2.2. Example: Alternative scheme of a \((N+1)\)-M model including \((N+1)\)-inner variables – inner variables illustrated in a second plane ........................................ 14
2.3. Schemes of the fundamental rheological elements, cf. [4, 46] ........................................ 15
2.4. Scheme of a thermoelastc element ..................................................................................... 17
2.5. Series connection of \((N+1)\)-coupled Maxwell mechanisms – The dashed element represents the Maxwell mechanisms \(1, \ldots, N - 1\) ........................................ 18
2.6. Series connection of \((N+1)\)-coupled Kelvin-Voigt mechanisms – The dashed element represents the Kelvin-Voigt mechanisms \(1, \ldots, N - 1\) ........................................ 20
2.7. Scheme for a plastic Chaboche model with five inner variables (resp. back stresses) .... 21
2.8. Scheme of a 3M2C with approaches \(A\) and \(B\) ......................................................... 25
3.1. Geometry of the 3d rod ................................................................................................. 57
3.2. Series connection of two coupled Kelvin-Voigt elements .............................................. 57
3.3. Responses of a 3d rod in \(x\) direction at the rod’s center point of the right edge as \((2)\)-coupled KV model to a asymmetric cyclic body load for various coupling parameters .................................................................................................................... 59
3.4. Von-Mises stress response of a 3d rod at the midsection \((z = 0.05)\) as \((2)\)-coupled KV model to a asymmetric cyclic body load for \(p = 1\) ........................................ 60
3.5. Strain response \((\varepsilon_{11})\) of a 3d rod in \(x\) direction at the rod’s midpoint \((x = 0.5, y = z = 0.05)\) as \((2)\)-coupled KV model to a asymmetric cyclic body, \(p = 0\) \((-\cdots)\), \(p = 0.98\) \((\cdots)\), \(p = 1\) \((-\cdots)\) ........................................ 61
3.6. Rod as \((1d, 1d)\)-model .................................................................................................. 62
3.7. Dirichlet boundary conditions ....................................................................................... 64
3.8. Robin boundary conditions .......................................................................................... 66
3.9. Inhomogeneous Robin boundary at the right edge ........................................................ 67
3.10. Strain response \((\partial_x u)\) of a 1d rod at the right edge \((x = 1)\) as \((2)\)-coupled KV model to a constant boundary stress for varying \(p\) ........................................ 94
3.11. Stress \((\sigma)\) and strain \((\partial_x u)\) response of a 1d rod at the mid point \((x = 0.5)\) as \((2)\)-coupled KV model to a constant boundary stress for varying \(p\) ........................................ 95
3.12. Stress \((\sigma)\) and strain \((\partial_x u)\) response of a 1d rod at the left edge \((x = 0)\) as \((2)\)-coupled KV model to a constant boundary stress for varying \(p\) ........................................ 96
3.13. Stress \((\sigma)\) and strain \((\partial_x u)\) response of a 1d rod as \((2)\)-KV model to a constant boundary stress (without coupling \(p = 0\)) ........................................ 97
3.14. Stress \((\sigma)\) and strain \((\partial_x u)\) response of a 1d rod as \((2)\)-coupled KV model to a constant boundary stress for positive singular coupling \(p = 1\) ........................................ 97
3.15. Stress \((\sigma)\) and strain \((\partial_x u)\) response of a 1d rod as \((2)\)-coupled KV model to a constant boundary stress for negative singular coupling \(p = -1\) ........................................ 98
3.16. Strain response \((\partial_x u)\) of a 1d rod at the right edge \((x = 1)\) as \((2)\)-coupled KV model to a cyclic boundary stress for varying \(p\) ........................................ 99
3.17. Stress \((\sigma)\) and strain \((\partial_x u)\) response of a 1d rod at the mid point \((x = 0.5)\) as \((2)\)-coupled KV model to a cyclic boundary stress for varying positive \(p\) ........................................ 100
3.18. Stress \((\sigma)\) and strain \((\partial_x u)\) response of a 1d rod at the mid point \((x = 0.5)\) as \((2)\)-coupled KV model to a cyclic boundary stress for varying negative \(p\) ........................................ 100
3.19. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for varying positive $p$ ....... 101
3.20. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for varying negative $p$ ....... 101
3.21. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-KV model (without coupling $p = 0$) to a cyclic boundary stress ............... 102
3.22. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for positive singular coupling $p = 1$ ........................................ 102
3.23. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for negative singular coupling $p = -1$ .......................................................... 102
3.24. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the right edge ($x = 1$) as (2)-coupled KV model to a constant boundary stress for a regular case $p = 0.5$ and varying $\eta = \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^{4}\}$ ................................................................. 103
3.25. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the right edge ($x = 1$) as (2)-coupled KV model to a constant boundary stress for a regular case $p = 0.5$ and $\eta = 10^{-4}$ ........................................... 103
3.26. Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the right edge ($x = 1$) as (2)-coupled KV model to a constant boundary stress for a regular case $p = 0.5$ and $\eta = 10^{4}$ ............................................. 104

4.1. Experimental Set-Up .......................................................... 107
4.2. General scheme of the experimental procedure .................................................. 108
4.3. Heating and austenitization for all considered experiments (Q103 excluded) ........ 115
4.4. Predeformation of stable austenite: Group +4% SA ........................................ 116
4.5. Predeformation of stable austenite: Group +6% SA ........................................ 116
4.6. Predeformation of stable austenite: Group -4% SA ........................................ 117
4.7. Predeformation of stable austenite: Group -6% SA ........................................ 117
4.8. Predeformation of stable austenite: Exemplarily comparison of experiments vs. literature values (cf. [1] and Section 4.1.3) .................................................. 117
4.9. Quenching from $\theta_A = 850^\circ$C: Comparison of experiments vs. literature values (Q138 excluded) .......................................................... 119
4.10. Predeformation of undercooled austenite: Group +2% UA$_{280^\circ}$C .................. 119
4.11. Predeformation of undercooled austenite: Group +4% UA$_{280^\circ}$C .................. 120
4.12. Predeformation of undercooled austenite: Group +6% UA$_{280^\circ}$C .................. 120
4.13. Predeformation of undercooled austenite: Group -2% UA$_{280^\circ}$C .................. 121
4.14. Predeformation of undercooled austenite: Group +4% UA$_{360^\circ}$C .................. 121
4.15. Predeformation of undercooled austenite: Comparison of experiments vs. literature values (cf. [1] and Section 4.1.3) .................................................. 121
4.16. Predeformation of undercooled austenite: Comparison of experiments vs. literature values (cf. [1] and Section 4.1.3) .................................................. 122
4.17. Transformation at $\theta_B = 280^\circ$C: $e_V$-evolution after quenching for all experiments with predeformation of stable austenite ......................... 123
4.18. Transformation at $\theta_B = 280^\circ$C: After quenching for all experiments with predeformation of stable austenite and without applied stress during transformation .............. 123
4.19. Transformation at $\theta_B$: $p_B$ and $e_V^{S5}$ evolutions for all considered experiments .......................................................... 124
4.20. Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_i^{S5}$ and the phase fractions $p_B$ for the Group ±0 MPa$_{280^\circ}$C ................................. 126
4.21. Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_i^{S5}$ and the phase fractions $p_B$ for the Group +25 MPa$_{280^\circ}$C ................................. 126
4.22. Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_i^{S5}$ and the phase fractions $p_B$ for the Group $+50$ MPa$\_280^\circ$C .................................................. 127
4.23. Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_i^{S5}$ and the phase fractions $p_B$ for the Group $-25$ MPa$\_280^\circ$C .................................................. 127
4.24. Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_i^{S5}$ and the phase fractions $p_B$ for the Group $-50$ MPa$\_280^\circ$C .................................................. 128
4.25. Transformation at $\theta_B = 360^\circ$C: Evolutions of the (crawled) inelastic (TRIP) strains $e_i^{S5}$ and the phase fractions $p_B$ for the Groups $\pm 0$ MPa$\_360^\circ$C and $\pm 25$ MPa$\_360^\circ$C . 128
4.26. Transformation at $\theta_B = 360^\circ$C: Evolutions of the (crawled) inelastic (TRIP) strains $e_i^{S5}$ and the phase fractions $p_B$ for the Group $\pm 50$ MPa$\_360^\circ$C .......................................... 129
4.27. Transformation at $\theta_B = 280^\circ$C: Average applied stresses vs. inelastic (TRIP) strains $e_i^{S5}$ at the end of transformation .................................................. 129
4.28. Transformation at $\theta_B = 360^\circ$C: Average applied stresses vs. inelastic (TRIP) strains $e_i^{S5}$ at the end of transformation .................................................. 130
4.29. Transformation at $\theta_B = 280^\circ$C: Inelastic (plastic) strains $e_i^{S2}(iE_2)$ of predeformation at $850^\circ$C vs. inelastic (TRIP) strains $e_i^{S5}$ at the end of transformation . 130
4.30. Transformation at $\theta_B = 280^\circ$C: Inelastic (plastic) strains $e_i^{S4}(iE_4)$ of predeformation at $280^\circ$C vs. inelastic (TRIP) strains $e_i^{S5}$ at the end of transformation . 130
4.31. Transformation at $\theta_B = 360^\circ$C: Inelastic (plastic) strains $e_i^{S2}(iE_2)$ resp. $e_i^{S4}(iE_4)$ of predeformation at $360^\circ$C vs. inelastic (TRIP) strains $e_i^{S5}$ at the end of transformation 131
4.32. Cooling to room temperature: Comparison of experiments vs. literature values (cf. [1] and Section 4.1.1) ........................................................................... 132
4.33. Starting with Group $+4\%$ SA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg.1 using the resulting parameter set of Step 4 ($a_p = 239.10$, $c_p = 8218.27$ MPa, $\beta_p = 61.07$, $\gamma_p = 14.61$ MPa) .......................................................... 143
4.34. Starting with Group $-6\%$ SA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg.1 using the resulting parameter set of Step 4 ($a_p = 43.78$, $c_p = 855.73$ MPa, $\beta_p = 500.67$, $\gamma_p = 12360.50$ MPa) .......................................................... 144
4.35. Starting with Group $-2\%$ UA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg.1 using the resulting parameter set of Step 4 ($a_p = 31.86$, $c_p = 4367.14$ MPa, $\beta_p = 1023.71$, $\gamma_p = 276219.31$ MPa) .......................................................... 146
4.36. Starting with Group $+4\%$ UA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg.1 using the resulting parameter set of Step 2 ($a_p = 40.35$, $c_p = 3834.25$ MPa, $\beta_p = 796.26$, $\gamma_p = 192267.23$ MPa) .......................................................... 148
4.37. Predeformation at $280^\circ$C: Comparison of $\kappa_{w}$ for Q99 ($\pm0\%$, $+50$ MPa) and Q100 ($\pm0\%$, $-50$ MPa) .................................................. 152
4.38. Transformation at $280^\circ$C: Results of step 4 of the optimization procedure using Tanaka’s approach and the experiments of Group (SA $\pm50$MPa) ................................................. 154
4.39. Transformation at $280^\circ$C: Results of step 4 of the optimization procedure using the approach by Desalos and Denis and the experiments of Group (SA $\pm50$MPa) 154
4.40. Transformation at $280^\circ$C: Results of step 4 of the optimization procedure using Tanaka’s approach and the experiments of Group (UA $\pm50$MPa) ................................................. 156
1. Introduction

“The whole is greater than the sum of its parts.”

Aristotle (384 B.C. - 322 B.C.)

Describing material phenomena in materials science is a problem of mathematical modeling. The goal is to formulate an abstract mathematical model such that we are able to make statements on the observed material behavior of the original problem. One approach of modeling complex material behavior of solids is the approach of multi-mechanism models (MM models). The approach by MM models is a generalization of classic approaches like rheology. In the last two decades, MM models -- precisely 2M models -- have been used to describe the inelastic material behavior of steel. Important applications of 2M models are the modeling of phenomena like cyclic plasticity (in relation to ratcheting), interaction of classic plasticity and transformation-induced plasticity (TRIP), or interaction of classic plasticity and creep.

The main goals of this thesis are

- to present a detailed and general model approach of MM models covering a wide range of material behaviors,
- to provide results of mathematical investigations on related boundary value problems, and
- to verify exemplarily a specific MM model by experiments.

In this work, our model approach of MM models is embedded in the concept of continuum mechanics. Our main scope is not the understanding of the physics involved in material modeling on the micro scale, but the comprehensive theoretical and mathematical treatment of the phenomena of solid materials describing many effects on the macro scale.

MM models are introduced for small deformations in Chapter 2 as a generalization of series connection of rheological elements. We present basic, advanced and generalized concepts to model phenomena like elasticity, plasticity, viscoplasticity, viscoelasticity (including creep), and transformation-induced plasticity.

Contrary to many authors, our main concept of a MM model does not separate the thermoeelastic strain as a special strain and we allow coupling between the scalar and tensorial inner variables. As generalized mechanisms and models, we present e.g. coupled thermoelastic mechanisms including hardening and two plastic MM models with \( m \) mechanisms with \( n \) yield criteria and coupled kinematic and coupled isotropic hardening -- the \( m\text{MnC}-A \) and \( m\text{MnC}-B \) model. Furthermore, all presented example mechanisms and models (except TRIP) are collected under one MM model with Chaboche ansatz. Moreover, we give some remarks and clarify some issues on the thermomechanical consistency based on these generalizations. The concept of MM models is applied to viscoelasticity and to the interaction of classic plasticity and TRIP.

A detailed mathematical treatment of (isothermal) linear viscoelasticity within the framework of MM models is addressed in Chapter 3 for 3\( d \) and 1\( d \) problems. The main mathematical result of this work is the mathematical investigation of the related 3\( d \) mixed boundary value problem of the isothermal-(N)-coupled Kelvin-Voigt elements model. Our approach of proving an existence
and uniqueness result is nonstandard by transforming the original problem into an equivalent system of integro-differential equations. In order to underline the quality of the isothermal-(N)-coupled Kelvin-Voigt elements model covering material effects like ratcheting, we provide numerical simulations performed with COMSOL Multiphysics® 4.3a. By simplifications to the 1d case, new problems arise in form of a singular mass matrix and inconsistent Dirichlet boundary conditions. We give a detailed derivation and motivation of the model and treat several related boundary value problems – suitable to describe displacement and stress driven test situations – concerning their weak solvability. A nondimensionalization and scaling of the 1d mixed boundary value problem is used to conduct qualitative numerical simulations with COMSOL Multiphysics® 4.3a.

In Chapter 4, we focus on the material phenomena of interaction between classic plasticity and TRIP. In the framework of the German Research Foundation (DFG) project BO 1144/4-1 “Multi-Mechanism Models: Theory and its Application to some Phenomena in the Material Behavior of Steel”, experiments characterizing the interaction of classic plasticity and transformation-induced plasticity of the steel 100Cr6 (SAE 52100) were conducted. We provide a detailed test evaluation. We embed the MM model of [68] for the interaction between classic plasticity and TRIP in our concept. Numerical schemes of the model suited for uniaxial situations are provided. For some selected experiments, we perform systematical parameter identifications using our numerical algorithm.

A summary and outlook of this thesis is given in Chapter 5 with regard to possible continuations and extensions of this work.
1.1. Notation

In this section, we introduce the notations for symbols, variables and parameters, abbreviations and some function spaces which are used in this thesis.

1.1.1. Nomenclature

In general, we use the bold-faced format for vectors and matrices (resp. tensors).

1.1.1.1. Variables and Parameters

If not stated otherwise, we use the following nomenclature for variables and parameters.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Unit (SI)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$</td>
<td>m</td>
<td>displacement vector</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>Green's strain tensor</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\frac{kg}{ms^2}$</td>
<td>Cauchy's stress tensor</td>
</tr>
<tr>
<td>$\theta$</td>
<td>K</td>
<td>absolute temperature</td>
</tr>
<tr>
<td>$\psi$</td>
<td>$\frac{m^2}{s^2}$</td>
<td>mass density of free energy</td>
</tr>
<tr>
<td>$e$</td>
<td>$\frac{m^2}{s^2}$</td>
<td>mass density of internal energy</td>
</tr>
<tr>
<td>$\eta$</td>
<td>$\frac{m^2}{Ks^2}$</td>
<td>mass density of internal entropy</td>
</tr>
<tr>
<td>$q$</td>
<td>$\frac{kg}{m^2s}$</td>
<td>heat-flux density vector</td>
</tr>
<tr>
<td>$\varepsilon_j$</td>
<td>1</td>
<td>partial strains</td>
</tr>
<tr>
<td>$\sigma_j$</td>
<td>$\frac{kg}{ms^2}$</td>
<td>partial stresses</td>
</tr>
<tr>
<td>$\alpha_j$</td>
<td>1</td>
<td>tensorial inner variables</td>
</tr>
<tr>
<td>$a_j$</td>
<td>1</td>
<td>scalar inner variables</td>
</tr>
<tr>
<td>$X_j$</td>
<td>$\frac{kg}{ms^2}$</td>
<td>back stresses</td>
</tr>
<tr>
<td>$C_\theta$</td>
<td>$\frac{m^2}{s^2}$</td>
<td>calorimetric function</td>
</tr>
<tr>
<td>$s_j$</td>
<td>1</td>
<td>accumulated strains</td>
</tr>
<tr>
<td>$p_j$</td>
<td>1</td>
<td>phase fractions</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit (SI)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho$</td>
<td>$\frac{kg}{m^3}$</td>
<td>mass density</td>
</tr>
</tbody>
</table>
1.1. Notation

\( f \) \( \frac{\text{kg}}{\text{m}^2 \text{s}^2} \) external volume forces

\( r \) \( \frac{\text{kg}}{\text{ms}} \) volume density of heat supply

\( A_j \) 1 weights of the mechanisms

\( E_{ij} \) \( \frac{\text{kg}}{\text{ms}^2} \) elasticity related tensors of 4-th order

\( V_{ij} \) \( \frac{\text{kg}}{\text{ms}} \) viscosity related tensors of 4-th order

\( \lambda_\theta \) \( \frac{\text{kg m}}{\text{s}^3 \text{K}} \) heat conductivity

\( c_d \) \( \frac{\text{m}^2}{\text{s}^2 \text{K}} \) specific heat

Especially in Chapter 4 (uniaxial case):

<table>
<thead>
<tr>
<th>Variable</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_L )</td>
<td>1</td>
<td>longitudinal strain</td>
</tr>
<tr>
<td>( e_D )</td>
<td>1</td>
<td>transversal strain</td>
</tr>
<tr>
<td>( e_V )</td>
<td>1</td>
<td>volume strain</td>
</tr>
<tr>
<td>( S )</td>
<td>MPa ( = \frac{\text{kg}}{\text{ms}^2} \times 10^6 )</td>
<td>uniaxial stress</td>
</tr>
<tr>
<td>( e_{in} )</td>
<td>1</td>
<td>inelastic strain</td>
</tr>
<tr>
<td>( x_p )</td>
<td>MPa</td>
<td>plastic back stress</td>
</tr>
<tr>
<td>( x_{tp} )</td>
<td>MPa</td>
<td>TRIP back stress</td>
</tr>
<tr>
<td>( e_p )</td>
<td>1</td>
<td>plastic strain</td>
</tr>
<tr>
<td>( e_{tp} )</td>
<td>1</td>
<td>TRIP strain</td>
</tr>
<tr>
<td>( \alpha_p )</td>
<td>1</td>
<td>plastic inner variable</td>
</tr>
<tr>
<td>( \alpha_{tp} )</td>
<td>1</td>
<td>TRIP inner variable</td>
</tr>
<tr>
<td>( s_p )</td>
<td>1</td>
<td>accumulated plastic strain</td>
</tr>
<tr>
<td>( s_{tp} )</td>
<td>1</td>
<td>accumulated TRIP strain</td>
</tr>
<tr>
<td>( R )</td>
<td>MPa</td>
<td>variable of isotropic hardening</td>
</tr>
<tr>
<td>( p_j )</td>
<td>1</td>
<td>phase fraction of ( j )-th phase</td>
</tr>
</tbody>
</table>
### 1. Introduction

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$</td>
<td>MPa</td>
<td>elastic modulus</td>
</tr>
<tr>
<td>$\nu$</td>
<td>1</td>
<td>Poisson’s ratio</td>
</tr>
<tr>
<td>$K$</td>
<td>MPa</td>
<td>compression modulus</td>
</tr>
<tr>
<td>$\lambda_L$</td>
<td>MPa</td>
<td>second Lamé coefficient</td>
</tr>
<tr>
<td>$\mu$</td>
<td>MPa</td>
<td>shearing modulus</td>
</tr>
<tr>
<td>$\lambda_p$</td>
<td>$\frac{1}{s}$</td>
<td>plastic multiplier</td>
</tr>
<tr>
<td>$R_{0j}$</td>
<td>MPa</td>
<td>yield stress of $j$-th phase</td>
</tr>
<tr>
<td>$\beta_p$</td>
<td>1</td>
<td>parameter of isotropic hardening</td>
</tr>
<tr>
<td>$\gamma_p$</td>
<td>MPa</td>
<td>parameter of isotropic hardening</td>
</tr>
<tr>
<td>$a_p$</td>
<td>1</td>
<td>parameter of kinematic (plastic) hardening</td>
</tr>
<tr>
<td>$c_p$</td>
<td>MPa</td>
<td>parameter of kinematic (plastic) hardening</td>
</tr>
<tr>
<td>$a_{tp}$</td>
<td>1</td>
<td>parameter of kinematic (TRIP) hardening</td>
</tr>
<tr>
<td>$c_{tp}$</td>
<td>MPa</td>
<td>parameter of kinematic (TRIP) hardening</td>
</tr>
<tr>
<td>$c_{int}$</td>
<td>MPa</td>
<td>parameter of interaction of plasticity and TRIP</td>
</tr>
<tr>
<td>$\kappa_{0j}$</td>
<td>$\frac{1}{\text{MPa}}$</td>
<td>Greenwood-Johnson parameter of $j$-th phase</td>
</tr>
<tr>
<td>$\Phi_j$</td>
<td>1</td>
<td>Saturation function of $j$-th phase</td>
</tr>
</tbody>
</table>

Related to the underlying mechanism, we denote associated quantities with the same subscript: $te$ - thermoelastic, $e$ - elastic, $v$ - viscous, $ve$ - viscoelastic, $p$ - plastic, $c$ - creep, $tp$ - TRIP. If only one material behavior is described, we sometimes abstain a subscript.
1.1. Notation

1.1.1.2. Symbols

We fix the notations for several operations on functions, vectors and tensors.

Let be \( u \) a function \( u(t, x) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \).

Partial time derivative of \( u \) w.r.t. \( t \)
\[ \partial_t := \frac{\partial}{\partial t} \]

Partial spatial derivative of \( u \) w.r.t. \( x_i \)
\[ \partial_i := \frac{\partial}{\partial x_i} \]
if \( n = 1, \partial_x := \frac{\partial}{\partial x} \)

Mixed partial derivative of \( u \) w.r.t. \( t, x_i \) and \( x_j \)
\[ \partial_{ij} := \frac{\partial}{\partial t} \partial_{x_i} \partial_{x_j} \]

Positive part of \( u \) (McCauley brackets)
\[ \langle u(t, x) \rangle := \max\{u(t, x), 0\} \]

In the component by component notation \( \delta \) (Kronecker delta) is given by

\[ \delta_{ij} = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases} \] (1.1.1)

The determinant of a second order tensor \( A \in \mathbb{R}^{3 \times 3} \) is given by

\[ \det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}. \] (1.1.2)

Let be \( a, b \in \mathbb{R}^n \) vectors, \( A, B \in \mathbb{R}^{3 \times 3} \) tensors of second order.

<table>
<thead>
<tr>
<th>vector / tensor representation</th>
<th>component-by-component</th>
</tr>
</thead>
<tbody>
<tr>
<td>Divergence of ( a )</td>
<td>( \text{div } a )</td>
</tr>
<tr>
<td>Divergence of ( A )</td>
<td>( \text{Div } A )</td>
</tr>
<tr>
<td>Trace of ( A )</td>
<td>( \text{tr } (A) )</td>
</tr>
<tr>
<td>Determinant of ( A )</td>
<td>( \det(A) )</td>
</tr>
<tr>
<td>Gradient of ( a )</td>
<td>( \nabla a )</td>
</tr>
<tr>
<td>Scalar product of vectors ( a ) : ( b )</td>
<td>( \sum_{k=1}^{n} a_k b_k )</td>
</tr>
<tr>
<td>Scalar product of tensors ( A ) : ( B )</td>
<td>( \sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij} B_{ij} )</td>
</tr>
<tr>
<td>Multiplication by components ( a \cdot b )</td>
<td>( (a_i b_j)_{i=1,...,n} )</td>
</tr>
<tr>
<td>Deviator of ( A )</td>
<td>( A^* )</td>
</tr>
<tr>
<td>First invariant of ( A )</td>
<td>( \Gamma^1(A) = \text{tr } (A) )</td>
</tr>
</tbody>
</table>
Second invariant of $A$

$\mathcal{I}^2(A) = \frac{1}{2}(\text{tr}(A)^2 - \text{tr}(A^2))$

Third invariant of $A$

$\mathcal{I}^3(A) = \det(A)$

1.1.2. Abbreviations

We introduce some abbreviations for sets, sums and cases of equations.

**Sets** We define the index set

$$\mathcal{J} := \{0, \ldots, J\},$$

for $J \in \mathbb{N}$. With $\alpha_J$ we denote the tuple $\alpha_0, \ldots, \alpha_J$. The number of elements of $\mathcal{J}$ is given by

$$|\mathcal{J}| := J + 1.$$

**Sums** We use for multiple sums the abbreviation

$$\sum_{\{n,m,o,\ldots\}} := \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{o} \ldots$$

If we use the index set notation in sums, we understand it as

$$\sum_{j \in \mathcal{J}} := \sum_{j=0}^{J}.$$

**Equations** Instead of

$$f(x) = a(x) \cdot b(x) \quad \text{with} \quad b(x) := \begin{cases} 1, & x > 0, \\ 0, & \text{else}, \end{cases}$$

we write

$$f(x) = a(x) \cdot (x > 0),$$

where $(x > 0)$ is 1 if fulfilled, and 0 else.

1.1.3. Voigt-Notation for Material Laws

In general, material laws in the 3d space for small deformations are expressed by relations of Green’s strain tensor $\varepsilon$ and Cauchy’s stress tensor $\boldsymbol{\sigma}$ and material dependent tensors of fourth order (e.g., $T_k$ and $S_k$). For example, in case of rheological models, they are given by

$$\sum_{k=0}^{n} T_k : \frac{\partial}{\partial t^k} \varepsilon = \sum_{k=0}^{m} S_k : \frac{\partial}{\partial t^k} \varepsilon.$$

We motivate a vector-matrix notation (cf. [4], also Voigt notation) of the material laws by the example of a Kelvin-Voigt model, where $n = 0$ and $m = 1$. Anisotropic, linear viscoelastic materials can be described by

$$\sigma_{ij} = \sum_{\{k,l\} = 1}^{\{3,3\}} E_{ijkl} \varepsilon_{kl} + \sum_{\{k,l\} = 1}^{\{3,3\}} V_{ijkl} \partial_{l} \varepsilon_{kl} \quad i,j = 1, \ldots, 3.$$
The elastic tensor $E$ and the viscous tensor $V$ are symmetric, cf. [4]

(1.1.8a) \[ E_{ijkl} = E_{jikl} = E_{ijlk} = E_{klij} \]
(1.1.8b) \[ V_{ijkl} = V_{jikl} = V_{ijlk} = V_{klij}. \]

In case of a fully anisotropic material, the material tensors have 21 linear independent coordinates. Since the strain and stress tensor are symmetric, they possess six independent entries. Therefore, the material law can be rewritten in vector-matrix notation as we map the indices of the tensor notation to the indices of the vector-matrix notation in the following manner: $11 \rightarrow 1$, $22 \rightarrow 2$, $33 \rightarrow 3$, $23 \rightarrow 4$, $13 \rightarrow 5$ and $12 \rightarrow 6$. If we apply it on (1.1.7), we obtain

(1.1.9) \[ \bar{\sigma}_i = (M_E)_{ij} \bar{\varepsilon}_j + (M_V)_{ij} \partial_t \bar{\varepsilon}_j, \]

where

(1.1.10) \[ \bar{\varepsilon} := \begin{pmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{pmatrix}, \quad \begin{pmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{13} \\ \varepsilon_{12} \end{pmatrix}, \quad \bar{\varepsilon} := \begin{pmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{pmatrix}, \quad \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{pmatrix}. \]

If we define

(1.1.11) \[ M_E := \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & E_{15} & E_{16} \\ E_{22} & E_{23} & E_{24} & E_{25} & E_{26} \\ E_{33} & E_{34} & E_{35} & E_{36} \\ S & E_{44} & E_{45} & E_{46} \\ Y & E_{55} & E_{56} \\ M & E_{66} \end{pmatrix} \]

and similarly $M_V$, the short form of the material law in Voigt notations is

(1.1.12) \[ \bar{\sigma} = M_E \bar{\varepsilon} + M_V \partial_t \bar{\varepsilon}. \]

**Remark 1.1.1.** If $M_E$ (resp. $M_V$) are invertible, $\det M_E \neq 0$ (resp. $\det M_V \neq 0$), there exists by retransformation of the indices a related inverse tensor of fourth order $E^{-1}$ to $E$ (resp. $V^{-1}$ to $V$).

1.1.4. **Function Spaces**

Let $S := [0, T]$ be a time interval and $\Omega \subset \mathbb{R}^d$ a space domain, here usually $d = 1$ or $3$. We recall some common notations for functions spaces, cf. e.g. [25], [62] or [79, 80, 81]. Let be $X$ a (real) Banach space.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C^k(\Omega)$</td>
<td>set of continuous functions with continuous derivatives up to order $k \in \mathbb{N}$</td>
</tr>
<tr>
<td>$C_0^\infty(\Omega)$</td>
<td>set of infinitely continuously differentiable functions with compact support in $\Omega$</td>
</tr>
<tr>
<td>$AC^k(S)$</td>
<td>set of absolutely continuous functions with absolutely continuous derivatives up to order $k \in \mathbb{N}$</td>
</tr>
<tr>
<td>$L^p(\Omega)$</td>
<td>standard Lebesgue space over $\Omega$, $p \in [1, \infty]$</td>
</tr>
<tr>
<td>$W^{k,p}(\Omega)$</td>
<td>standard Sobolev space over $\Omega$, $k \in \mathbb{N}$, $p \in [1, \infty]$</td>
</tr>
<tr>
<td>$W^0_{k,p}(\Omega)$</td>
<td>closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$</td>
</tr>
<tr>
<td>$C^k(S,X)$</td>
<td>set of $X$-valued functions with continuous derivatives up to order $k \in \mathbb{N}$</td>
</tr>
<tr>
<td>$AC^k(S,X)$</td>
<td>set of absolutely continuous, $X$-valued functions with absolutely continuous derivatives up to order $k \in \mathbb{N}$</td>
</tr>
<tr>
<td>$L^p(\Omega, X)$</td>
<td>standard Bochner-Lebesgue space, $p \in [1, \infty]$</td>
</tr>
<tr>
<td>$C^{k,p}$</td>
<td>class of domains whose boundary is locally representable as a graph of a $C^{k,p}$-function (cf. [62])</td>
</tr>
</tbody>
</table>

**Definition 1.1.2.** Let $X$ be a function space and $n \in \mathbb{N}$. We define the vector set

\[
X^n := \{ x = (x_1, \ldots, x_n) \mid x_i \in X \ \forall \ i = 1, \ldots, n \}
\]

and the matrix set

\[
X^{n \times n} := \{ x = (x_{ij})_{i,j=1,\ldots,n} \mid x_{ij} \in X \ \forall \ i,j = 1, \ldots, n \}.
\]

A subset of $X^{n \times n}$ is

\[
X_{\text{sym}}^{n \times n} := \{ x \in X^{n \times n} \mid x_{ij} = x_{ji} \ \forall \ i,j = 1, \ldots, n \}.
\]

$(\cdot \mid \cdot)_Y$ denotes the scalar product in $Y \in \{X, X^n, X^{n \times n}\}$.

**Definition 1.1.3.** Let $f, g \in L^2(\Omega)$ and $\varrho \in L^\infty(\Omega)$ with $\varrho(x) \geq \varrho_0 > 0$ f.a.a. $x \in \Omega$. We define the weighted scalar product in $L^2(\Omega)$

\[
(f, g)_\varrho := \int_\Omega \varrho(x)f(x)g(x)\,dx.
\]

This scalar product generates the weighted $L^2$-norm

\[
\|f\|_{\varrho,L^2} := \left( \int_\Omega \varrho(x)|f(x)|^2\,dx \right)^{1/2}.
\]

Let $f \in [L^2(\Omega)]^n$, $n \in \mathbb{N}$, then we define

\[
(f \cdot g)_{\varrho,[L^2]^n} := \int_\Omega \sum_{i=1}^n \varrho(x)f_i(x)g_i(x)\,dx.
\]
1.1. Notation

This scalar product generates the weighted $[L^2]^n$-norm

\[(1.1.19) \quad \| f \|_{e[\mathbb{L}^2]^n} := \left( \sum_{i=1}^{n} \int_{\Omega} g(x) |f_i(x)|^2 \, dx \right)^{1/2}.\]

Let $f \in [L^2(\Omega)]^{n \times n}$, $n \in \mathbb{N}$, then we define

\[(1.1.20) \quad (f, g)_{e[\mathbb{L}^2]^{n \times n}} := \int_{\Omega} \sum_{\{i,j\}=1}^{\{n,n\}} g(x) f_{ij}(x) g_{ij}(x) \, dx.\]

This scalar product generates the weighted $[L^2]^{n \times n}$-norm

\[(1.1.21) \quad \| f \|_{e[\mathbb{L}^2]^{n \times n}} := \left( \sum_{\{i,j\}=1}^{\{n,n\}} \int_{\Omega} g(x) |f_{ij}(x)|^2 \, dx \right)^{1/2}.\]
2. Multi-Mechanism Models

In this chapter, we give a short overview of the origin and past as well as recent applications of multi-mechanism models (MM models) for small deformations. We introduce the basic setting of our model approach for MM models of serial type while we generalize the concept of the past approaches by not separating the thermoelastic strain as a special strain. Our approach is based on the concept of rheological models connected in series. The modular principle is illustrated by several basic, advanced and generalized mechanisms and MM models. Especially, we confer to our two generalized approaches of \textit{mMnC}-models for plasticity based on [77]. Concluding the chapter, we give some remarks and clarify some issues due to thermomechanical consistency based on the generalizations.

2.1. State of the Art

MM models\textsuperscript{1} are introduced by Cailletaud and Saï (1995) in their paper [17] based on a model by Contesti and Cailletaud ([21]) and on a model by Khabou et al. ([34]) for a elastic-plastic-viscoplastic material model in the framework of small deformations.\textsuperscript{2} The main scope of a MM model is to combine several material effects (or similar material effects), to model their interactions and to obtain in that way a better description of the material behavior.

The common application for MM models is the approximation of the material behavior of steels (cf. [50] and reference within), especially their ratcheting behavior. In [17], Cailletaud and Saï present two 2M models (with one (1C) and two yield criteria (2C)) generalizing their underlying models by permitting an interaction between two plastic, two viscoplastic or one plastic and one viscoplastic mechanisms. The ansatz of Cailletaud and Saï is used as a basis to further extensions and modifications, see [52] for general plastic 2M1C and 2M2C models and for their thermomechanical consistency [72, 77], for an update of the original model by Taleb [57] or for a more general viscoplastic 2M2C model [51] and [53]. In [50], Saï gives an overview of the current state of plastic and viscoplastic MM models. He recalls the unified Chaboche model, 2M1C and 2M2C types of the MM model. Applications of MM models besides steel are the plastic and viscoplastic materials mortal-rubber ([60]), semi-crystalline polymers ([48]) and ductile materials ([8]). More recent model approaches of MM models consider also transformation-induced plasticity, e.g. [67, 72], and viscoelasticity, e.g. [35, 37, 36, 65], as mechanisms. Recent publications consider MM models for large deformation as well, e.g. [43]. A rather new approach are MM models of parallel type, see [63].

The list of provided examples of MM models is neither fully exhaustive nor complete since every series connection of rheological models (see e.g. [4, 27]) is a MM model and many model approaches for a varying number of material behaviors are not yet collected under the patronage of MM models.

\textsuperscript{1}The original proposed MM model is of serial type, cf. Section 2.2. Serial should be understood as for a series connection of rheological models where the partial strains add up. In a general setting, a MM model of parallel and clearly of mixed type can be proposed as well. We do not discuss the issue here.

\textsuperscript{2}In the majority of publications concerning plastic or viscoplastic material behavior, a (thermo)elastic element in series is considered, but it is often drooped in the naming of the models.
2.2. General Model of Serial Type

We provide a general approach for MM models in the framework of continuum mechanics for solids in the case of small deformations. The principles of continuum mechanics suppose that the following balance equations hold in a time-space domain $[0,T] \times \Omega$:

\begin{align}
(2.2.1a) \quad & \rho \partial_t u - \text{Div } \sigma = f, \\
(2.2.1b) \quad & \rho \partial_t e + \text{div } q = \sigma : \partial_t \varepsilon + r, \\
(2.2.1c) \quad & -\rho \partial_t \psi - \rho \eta \partial_t \theta + \sigma : \partial_t \varepsilon - \frac{1}{\theta} q \cdot \nabla \theta \geq 0.
\end{align}

The relations (2.2.1) describe the balance of momentum (2.2.1a), the balance of energy (2.2.1b) and the Clausius-Duhem inequality (2.2.1c) ensures thermomechanical consistency. The notations are:

- $\varepsilon$ describes the displacement,
- $\varepsilon$ the linearized Green strain tensor,
- $\sigma$ the stress Cauchy tensor,
- $\theta$ the temperature,
- $\rho$ the mass density,
- $f$ external volume forces,
- $e$ the mass density of the internal energy,
- $q$ the heat-flux density vector,
- $r$ the volume density of heat supply,
- $\psi$ the mass density of the free (or Helmholtz) energy and
- $\eta$ the mass density of entropy.

In the case of small deformations, the linearized Green strain tensor is given by

\begin{equation}
(2.2.2) \quad \varepsilon = \frac{1}{2} (\nabla u + \nabla u^T).
\end{equation}

We state the well-known relation between the mass densities of internal energy, free energy and free entropy\(^3\)

\begin{equation}
(2.2.3) \quad e = \psi + \theta \eta.
\end{equation}

In the further development of the general MM model of serial type, several assumptions are made. The most crucial one is the additive split of the strain tensor into $N + 1$ partial strains $\varepsilon_N$, $\varepsilon_N := \{\varepsilon_0, \ldots, \varepsilon_N\}$,

\begin{equation}
(2.2.4) \quad \varepsilon = \sum_{j \in N} A_j \varepsilon_j.
\end{equation}

$A_j, j \in N$, are positive real numbers. The 0-th strain is mostly identified with the thermoelastic strain, whereas the remaining partial strains represent the inelastic material behavior. We stick to the most general case. For a graphical illustration, we suggest as example the scheme of Fig. 2.1 and more detailed (noting the dependency of inner variables) the scheme of Fig. 2.2.

In order to describe the full material behavior, we have to pose further constitutive equations and evolution equations on $\psi$, $e$, $q$, $\varepsilon_N$ and all defined inner variables. In the framework of MM models of serial type, we assume a free energy function $\psi$ depending on tensorial and scalar inner variables $(\alpha_{J_0} \text{ and } a_{J_0})^5$ and the temperature $\theta$, where $J_\alpha := \{0, \ldots, J_\alpha\}$ (in general $J_\alpha \geq N$) and $J_\alpha := \{0, \ldots, J_\alpha\}$,

\begin{equation}
(2.2.5) \quad \psi = \psi(\theta, \alpha_{J_0}, a_{J_0}).
\end{equation}

Since the temperature will be determined by the energy equation, we will not assume $\theta$ to be an inner variable. Before stating additional governing equations for the remaining variables, we

\(^3\)Instead of mass density of internal energy, free energy resp. free entropy, we use the short terms internal energy, free energy resp. free entropy.

\(^4\)For a comment on dependencies on further quantities see Remark 2.4.1.

\(^5\)If later defined evolutions equation are $\alpha_j = \varepsilon_k$ for some $j \in J_\alpha$, $k \in N$, we sometimes replace the dependency on $\alpha_j$ by the dependency on $\varepsilon_k$ and drop the evolution equation $\alpha_j = \varepsilon_k$. Note that in this general approach, the (thermo)elastic strain does not play a special role and it may even be dropped in the model process. This is contrary to most of the authors dealing with MM models.
Figure 2.1.: Example: Scheme of a MM model with \((N + 1)\) partial strains

take a look at the Clausius-Duhem inequality. With

\[
\partial_t \psi = \frac{\partial \psi}{\partial \theta} \partial_t \theta + \sum_{j \in J_a} \frac{\partial \psi}{\partial \alpha_j} : \partial_t \alpha_j + \sum_{j \in J_a} \frac{\partial \psi}{\partial a_j} \partial_t a_j,
\]

the inequality (2.2.1c) reads as

\[
(2.2.7) \quad - \rho \rho \theta \left( \frac{\partial \psi}{\partial \theta} + \eta \right) - \rho \left( \sum_{j \in J_a} \frac{\partial \psi}{\partial \alpha_j} : \partial_t \alpha_j + \sum_{j \in J_a} \frac{\partial \psi}{\partial a_j} \partial_t a_j \right) + \sigma : \partial_t \varepsilon - \frac{1}{\theta} q \cdot \nabla \theta \geq 0.
\]

We propose the Fourier law of heat conduction

\[
(2.2.8) \quad q = -\lambda_\theta \nabla \theta,
\]

where \(\lambda_\theta \geq 0\) is the heat conductivity.\(^5\) Following arguments of Coleman and Gurtin (20) and assuming that the free energy does not depend on \(\partial_t \theta\) or \(\nabla \theta\), we obtain by (2.2.7) and (2.2.8)

\[
(2.2.9) \quad \eta = -\frac{\partial \psi}{\partial \theta}.
\]

Introducing the thermomechanical forces

\[
(2.2.10) \quad X_j := \rho \frac{\partial \psi}{\partial \alpha_j}, \quad j \in J_a, \quad \text{and} \quad Q_j := \rho \frac{\partial \psi}{\partial a_j}, \quad j \in J_a,
\]

where we refer to \(X_{J_a}\) as back stresses, and partial stresses \(\sigma_{J_a}\),

\[
(2.2.11) \quad \sigma_j := A_j \sigma, \quad j \in N,
\]

we can state the remaining governing equations related to the free energy \(\psi\), the partial strains \(\varepsilon_{J_a}\) and the inner variables \(\alpha_{J_a}\) and \(a_{J_a}\).

The partial strains \(\varepsilon_{J_a}\) and the inner variables \(\alpha_{J_a}\) and \(a_{J_a}\) are determined by the evolution equations\(^7\)

\[^5\] In the most general case, \(q\) depends on further quantities, \(q = q(\theta, \sigma_{J_a}, \varepsilon_{J_a}, \alpha_{J_a}, a_{J_a})\).

\[^7\] The dependency of a variable \(\xi \in \{\sigma_{J_a}, \varepsilon_{J_a}, \alpha_{J_a}, a_{J_a}\}\) stands for the possible dependency of \(O(\xi)\), where \(O\)

is an operator, \(e.g.\) Id., polynomial, \(\partial_t\), \(\nabla\), \(\text{tr}\) or integral operator and combinations of them. Obviously, a dependency on \(X_{J_a}\) and \(Q_{J_a}\) is always a dependency on \(\theta, \alpha_{J_a}\) and \(a_{J_a}\), but since the thermodynamic forces play a special role, we denote the dependency. Generalizations concerning further dependencies of time and space derivatives of \(\theta\) are for future work. We are very general in our assumption to cover a wide range of
(2.2.12a) \[ F^N_\alpha (\theta, \sigma_N, \varepsilon_N', \alpha_{J_{\alpha}}, X_{J_{\alpha}}, a_{J_{\alpha}}, Q_{J_{\alpha}}) = 0_N, \]

(2.2.12b) \[ F^3_{J_{\alpha}} (\theta, \sigma_N, \varepsilon_N', \alpha_{J_{\alpha}}, X_{J_{\alpha}}, a_{J_{\alpha}}, Q_{J_{\alpha}}) = 0_{J_{\alpha}}, \]

(2.2.12c) \[ F^3_{J_{\alpha}} (\theta, \sigma_N, \varepsilon_N', \alpha_{J_{\alpha}}, X_{J_{\alpha}}, a_{J_{\alpha}}, Q_{J_{\alpha}}) = 0_{J_{\alpha}}. \]

Hereby, the evolution functionals \( F^N_\alpha, F^3_{J_{\alpha}} \) and \( F^3_{J_{\alpha}} \) have to complete the existing set of equations so that the full model is

- solvable
- and thermomechanically consistent.

In other words, the material model has to fulfill the Clausius-Duhem inequality (2.2.1c) for all imaginable thermomechanical processes, see [27, Definition 13.3]. General statements due to thermomechanical consistency cannot be given at this point and the Clausius-Duhem inequality has to be checked individually for the proposed set of evolution equations. The use of nonstandard evolution equations, e.g. integro-differential equations, is unusual and we have to ensure in that case that the model is still deterministic. If needed, the model is completed with suitable initial values for the inner variables and boundary conditions.

**Remark 2.2.1.** We abstain the concept of dissipation potentials (cf. [42, p. 61] or [14] for a application on viscoelasticity) in defining our evolution equations. This allows more freedom in modeling, but restricts in conclusions on thermomechanically consistency before specifying the governing functionals \( F^N_\alpha, F^3_{J_{\alpha}} \) and \( F^3_{J_{\alpha}} \).

### 2.3. Modular Principle - Examples of Mechanisms and Models

The underlying principle of the idea of MM models is a modular composition of several material effects and their interactions. In this section, we provide some widely-used mechanisms for thermoelast, viscous, viscoelastic, plastic, creep, TRIP material behavior and embed them in the previous setting. In case of the thermoelastic mechanism, we generalize the basic concept to kinematic hardening and coupling. The viscoelastic rheological Maxwell and Kelvin-Voigt are extended within the context of MM models. For plasticity, we present a new approach of mMnC model approaches. E.g. fractional models and highly nonlinear ones with possible implicit equations can be included and would be a interesting topic to investigate as well.
models.
All presented mechanisms can be combined. If the mechanisms describing different material behavior are uncoupled, the (partial) free energies will simply add up like the partial strains. In a coupled case, we cannot additively split up the free energy due to the coupling terms. The coupled case of combinations of all presented mechanisms and example models is covered by a MM model with Chaboche ansatz. The MM model with Chaboche ansatz collects all presented other mechanisms in an abstract way.

A set of mechanisms (or one mechanism) related to one material phenomenon is also called element, analogously to the concept of rheology. Many mechanisms are based on the three fundamental rheological elements – the elastic Hooke element, the viscous Newton element and the plastic St. Venant element, cf. Fig. 2.3. The free energy functional is assumed to be an approximation of quadratic order with respect to the inner variables $\alpha_{\mathcal{J}_a}$ and $a_{\mathcal{J}_a}$:

\[
\psi(\theta, \alpha_{\mathcal{J}_a}, a_{\mathcal{J}_a}) := \frac{1}{2\theta} \left( \sum_{i \in \mathcal{J}_a} \alpha_i : C_{ij} : \alpha_j + \sum_{i,j \in \mathcal{J}_a} c_{ij} a_i a_j + \sum_{i \in \mathcal{J}_a, j \in \{1,2,3\}} d_{ij} a_j \mathcal{J}_a^k(\alpha_i) + \sum_{j \in \mathcal{J}_a} \gamma_j^a : \alpha_j + \sum_{j \in \mathcal{J}_a} \gamma_j^a a_j \right) - C_\theta(\theta).
\]

The inner variables of tensorial type $\alpha_{\mathcal{J}_a}$ are related to the kinematic hardening\(^{6}\) of the mechanisms (e.g., elastic, viscoelastic, plastic) of the model and the inner variables of scalar type $a_{\mathcal{J}_a}$ are mostly related to effects like phase transformation or isotropic hardening. Hereby, $C_{ij}$, $c_{ij}$, $d_{ij}$, $\gamma_j^a$, $\gamma_j^a$ denote material parameters and may depend on $\theta$, $\sigma_N$, $\varepsilon_N$, $\alpha_{\mathcal{J}_a}$, $X_{\mathcal{J}_a}$, $a_{\mathcal{J}_a}$ or $Q_{\mathcal{J}_a}$,\(^{10}\) $C_{ij}$, for $i \neq j$, will denote the coupling of the tensorial inner variables and therefore of the mechanisms $\varepsilon_j$, $c_{ij}$, for $i \neq j$, the coupling of the scalar variables, and $d_{ik}$ the coupling of tensorial variables (in terms of invariants) with scalar ones.\(^{11}\) Often, $\gamma_j^a(\theta)$ are temperature dependent tensor functions of second order and describe the influence of the temperature on the inner variables, e.g., the thermal expansion (see Section 2.3.1). $C_\theta(\theta)$ is the calorimetric function. The typical approach is

\[
C_\theta(\theta) := c_d(\theta \ln \theta - \theta),
\]

\(^{6}\)The factor $\frac{1}{2\theta}$ is convenient for the general anisotropic approach, while we consider the material parameters as tensors. In the isotropic case with applications to uniaxial experiments, the factor $\frac{1}{2\theta}$ is used, cf. Chapter 4.

\(^{10}\)Here, we do not allow softening.

\(^{11}\)Obviously, if the material parameters depend on many quantities, the model becomes complex. Thus in most applications, we restrict ourselves to constant, temperature dependent or phase fraction dependent parameters.

\(^{11}\)This is a nonstandard approach inspired by coupling related to phase transformations, see e.g., [71].
2.3. Modular Principle - Examples of Mechanisms and Models

where \( c_d \) is the specific heat coefficient, cf. [14, 29].

In our concept, the free energy is supposed to be a convex function with respect to \( \alpha_J \) and \( a_J \) for all fixed temperature \( \theta \) and concave with respect to \( \theta \) for all fixed inner variables \( \alpha_J \) and \( a_J \). According to our quadratic ansatz, this is equivalent to \( \psi \) being a positive semidefinite function with respect to \( \alpha_J \) and \( a_J \). Hence, the choice of coefficients underlies certain restrictions. We will specify these restrictions adjusted to the example problems.

2.3.1. Thermoelastic Elements

In most applications of MM models, a thermoelastic element is included which is uncoupled from the remaining mechanisms. The thermoelastic element is usually not regarded as a mechanism. However, in order to achieve a consistent model kit, we do. We denote the thermoelastic-related functions, variables and parameters with the subscript \( \text{te} \). If we neglect the temperature dependency, we use the subscript \( e \). In this section, we recall the basic thermoelastic element and introduce a thermoelastic element with hardening consisting of several coupled thermoelastic mechanisms.

2.3.1.1. Basic Thermoelastic Element

The basic thermoelastic element is determined by one mechanism related to the free energy (cf. [27, Section 13.2.3] for detailed derivation excluding inner variables)

\[
(2.3.3) \quad \psi := \psi_{\text{te}}(\theta, \alpha_{\text{te}}) := \frac{1}{2\varrho} (\alpha_{\text{te}} : E_{\text{te}} : \alpha_{\text{te}} + \gamma_{\text{te}}(\theta) : \alpha_{\text{te}}) - C_\theta(\theta),
\]

where \( E_{\text{te}} \) is the positive definite elastic tensor of fourth order, \( \gamma_{\text{te}} \) a second order tensor related to thermal expansion and \( C_\theta \) the calorimetric function defined in (2.3.2). If the material model contains a thermoelastic element, the calorimetric function is usually included in the thermoelastic part of the free energy. The evolution functional related to \( \varepsilon_{\text{te}} \) is defined by

\[
(2.3.4) \quad F_{\text{te}}^\varepsilon(\sigma_{\text{te}}, X_{\text{te}}) := \sigma_{\text{te}} - X_{\text{te}}.
\]

and the evolution functional related to thermoelastic inner variable \( \alpha_{\text{te}} \) by

\[
(2.3.5) \quad F_{\text{te}}^\alpha(\varepsilon_{\text{te}}, \alpha_{\text{te}}) := \alpha_{\text{te}} - \varepsilon_{\text{te}}.
\]

With (2.3.4) and (2.3.5), we obtain

\[
(2.3.6) \quad \sigma_{\text{te}} = X_{\text{te}} = \varrho \frac{\partial \psi}{\partial \alpha_{\text{te}}} = E_{\text{te}} : \alpha_{\text{te}} + \gamma_{\text{te}}(\theta) : \alpha_{\text{te}} = E_{\text{te}} : \varepsilon_{\text{te}} + \gamma_{\text{te}}(\theta) : I.
\]

Without loss of generality, we set the weight of the mechanism \( A_{\text{te}} \) to 1, otherwise \( A_{\text{te}} \) can be considered in \( E_{\text{te}} \), so that

\[
(2.3.7) \quad \varepsilon = A_{\text{te}} \varepsilon_{\text{te}} = \varepsilon_{\text{te}} \quad \text{and} \quad \sigma = A_{\text{te}} \sigma = \sigma.
\]

We obtain for a stand-alone basic thermoelastic element

\[
(2.3.8) \quad \sigma = E_{\text{te}} : \varepsilon + \gamma_{\text{te}}(\theta) : I.
\]

---

12 Eventually, we split \( C_\theta \) into a sum of \( C_{\theta,j} \) so that \( C_\theta = \sum_{j \in J_G} C_{\theta,j} = \sum_{j \in J_G} c_{\theta,j}(\theta \ln \theta - \theta) \).

13 In the isotropic case, we usually set \( \gamma_{\text{te}}(\theta) = \gamma_{\text{te}} : I \) with \( \gamma_{\text{te}} < 0 \).

14 Although, \( \varepsilon_{\text{te}} \) is not stated explicitly in the evolution functional \( F_{\text{te}}^\varepsilon \), we identify it as the evolution functional related to thermoelastic strain \( \varepsilon_{\text{te}} \).
Thermoelastic element besides inelastic mechanisms  Assuming a (N+1)-mechanism model, \( N := \{0, \ldots, N\} \), in case of one uncoupled, standard thermoelastic element besides inelastic mechanisms, we identify \( \varepsilon_0 \) with \( \varepsilon_{te} \) and \( \varepsilon_{in} \) with \( \sum_{j \in N^1} \varepsilon_{j} \), \( N^1 := N \setminus \{0\} \). Moreover, the free energy is an additive split of two potentials, the thermoelastic one, \( \psi_{te} \), and the inelastic one, \( \psi_{in} \),

\[
\psi = \psi_{te}(\theta, \varepsilon_{te}) + \psi_{in}(\alpha_{J_n}, a_{J_m}).
\]

Since we have no coupling in this case, we obtain the basic thermoelastic evolution functional, cf. e.g., [17, 65],

\[
F_{te}^e(\sigma_{te}, X_{te}) = \sigma_{te} - X_{te} = \sigma_{te} - E_{te} : \varepsilon_{te} - \gamma_{te}(\theta) : I.
\]

The inelastic strains \( \varepsilon_{N^1} \) and their related inner variables \( \alpha_{J_n} \) can be set by one of the approaches in the following sections.

2.3.1.2. Coupled Thermoelastic Mechanisms with Hardening

In our model approach, it seems reasonable to include hardening and coupling as well for several thermoelastic mechanisms since hardening and their coupling are features of many used mechanisms. Hardening results in a viscelastic-like model behavior in this case.

Let us consider a material with \( N + 1 \) thermoelastic mechanisms \( \varepsilon_{te,j}, j \in N := \{0, \ldots, N\} \), where each mechanism has one related inner variable \( \alpha_{te,j} \) with hardening behavior, see Section 2.3.4 for details. The free energy is defined as

\[
\psi = \psi_{te}(\theta, \alpha_{te,N}) := \frac{1}{2\theta} \left( \sum_{i,j \in N} \alpha_{te,i} : E_{te,ij} : \alpha_{te,j} + \sum_{j \in N} \gamma_{te,j}(\theta) : \alpha_{te,j} \right) - \sum_{j \in N} C_{\theta,j}(\theta).
\]

The evolution functionals related to \( \varepsilon_{te,N} \) are defined by

\[
F_{te}^e(\sigma_{te,j}, X_{te,j}) = \sigma_{te,j} - X_{te,j}, \ j \in N,
\]

and the evolution functionals related to \( \alpha_{te,N} \) by

\[
F_{te}^\alpha(\theta, \varepsilon_{te,N}, \alpha_{te,N}, X_{te,N}) := \partial_t \alpha_{te,j} - \partial_t \varepsilon_{te,j} + \Phi_{te,j}(\theta, \sigma_{te,N}, \alpha_{te,N}, X_{te,N}),
\]

\[
\alpha_{te,j}(0) = \alpha_{te,j}^0, \ j \in N.
\]

\( \Phi_{te,N}^\alpha \) are tensor functions of second order and have to fulfill restrictions due to thermomechanical consistency,

\[
\sum_{j \in N} X_{te,j} : \Phi_{te,j}^\alpha(\theta, \sigma_N, \alpha_{te,N}, X_{te,N}) \geq 0 \text{ for all admissible arguments.}
\]
We will justify this later in Section 2.4. Examples for $\Phi_{te,N}$ are given in Section 2.3.4.

### 2.3.2. Viscous Mechanism

A viscous Newton mechanism (cf. Fig. 2.3(b)) denoted by the subscript $v$, is featured by total dissipation\(^ {15} \) of all free energy (cf. [4, Chapter 9]). Therefore, the free energy is given by

\[
(2.3.15) \quad \psi = \psi_v = 0.
\]

The evolution of $\varepsilon_v$ in the simplest (rheological) case, with $A_v = 1$, is determined by

\[
(2.3.16) \quad F_v^e(\sigma_v, \varepsilon_v) := \sigma_v - V : \varepsilon_v = A_v \sigma - V : \varepsilon_v = \sigma - V : \varepsilon_v(0) = \varepsilon_v^0.
\]

$V$ denotes the positive definite viscosity tensor. Nonlinear approaches can be covered when $V$ depends on further quantities but still remains a positive definite tensor.

### 2.3.3. Viscoelastic Mechanisms

Linear viscoelasticity based on rheological models is modeled by a combination of rheological elastic and rheological viscous elements. The basic combinations are the Maxwell, Kelvin-Voigt, Poynting-Thomson (also: Standard) or Burgers model, cf. [4]. We present the two basic viscoelastic mechanisms – based on the Maxwell and the Kelvin-Voigt model – and generalizations of them. It seems reasonable to restrict us to these two models since the Poynting-Thomson is a combination of a Kelvin-Voigt model and an elastic element, whereas the Burgers model is a series connection of a Maxwell and Kelvin-Voigt model.

#### 2.3.3.1. Based on Rheological Maxwell Model

Neglecting thermal effects, viscoelastic material behavior can be modeled by combining $N+1$ elastic and $N+1$ viscous mechanisms as a series connection, see Fig. 2.5. As a feature of the multi-mechanism ansatz, we allow a coupling between the elastic mechanisms. We obtain a

![Series connection of (N+1)-coupled Maxwell mechanisms](image)

Figure 2.5.: Series connection of (N+1)-coupled Maxwell mechanisms – The dashed element represents the Maxwell mechanisms 1, ..., $N - 1$

\(^{15}\) Total dissipation might not be observed in experiments, cf. e.g., [50]. E.g. Helm uses an additional term in the free energy to represent “the thermomechanical coupling phenomenon that only a certain part of the performed inelastic work is converted into heat and the remaining part is stored in the material body” ([30]). This applies also to the ideal plastic mechanism, see Section 2.3.4.1.
rheological-based \((N+1)\)-Maxwell model (assuming \(\varepsilon_j = \alpha_j, j \in \mathcal{N}\), cf. Fig. 2.5), where

\[
\psi := \psi_e + \psi_v := \psi_e(\varepsilon_{e,N}) := \frac{1}{2\rho} \left( \sum_{i,j \in \mathcal{N}} \varepsilon_{e,i} : E_{ij} : \varepsilon_{e,j} \right).
\]

Since we excluded inner variables, the back stresses \(X_{e,N}\) are directly related to the partial strains \(\varepsilon_{e,N}\)

\[
X_{e,j} := \frac{\partial \psi}{\partial \varepsilon_{e,j}} = \sum_{i \in \mathcal{N}} E_{ij} : \varepsilon_{e,i}, j \in \mathcal{N}.
\]

The evolution functionals related to the partial strains \(\varepsilon_{e,N}\) and \(\varepsilon_{v,N}\) are given by

\[
F_{e,j}^e(\sigma_{e,j}, \varepsilon_{e,N}) := \sigma_{e,j} - X_{e,j} = \sigma_{e,j} - \sum_{i \in \mathcal{N}} E_{ij} : \varepsilon_{e,i}, j \in \mathcal{N},
\]

\[
F_{v,j}^e(\sigma_{v,j}, \varepsilon_{v,j}) := \sigma_{v,j} - V_j : \partial \varepsilon_{v,j}, \varepsilon_{v,j}(0) = 0, j \in \mathcal{N}.
\]

Using the assumption of the additive split of the strain and transforming (2.3.20) into an integral form, we recover the well-known stress-strain relation of a \((N+1)\)-Maxwell chain. Without loss of generality, we set the weights of the mechanisms \(A_{e,j}\) and \(A_{v,j}\) equal to 1.16

\[
F^e(\sigma, \varepsilon) := \varepsilon - \sum_{j \in \mathcal{N}} E_j^{-1} : \sigma - \sum_{j \in \mathcal{N}} \int_0^t V_j^{-1} : \sigma(s) \, ds.
\]

This example shows that integral evolution laws may also be included in the setting of MM models; more general integro-differential equations.

### 2.3.3.2. Based on Rheological Kelvin-Voigt Model

An alternative to a series connection of elastic and viscous elements to model viscoelasticity is a series connection of Kelvin-Voigt elements (short: KV; one Kelvin-Voigt element is a parallel connection of one elastic and one viscous element), cf. Fig. 2.6. The related functions, variables and parameters are denoted by the subscript \(ve\). The free energy is defined analogously to the Maxwell model,

\[
\psi := \psi_{ve}(\varepsilon_{ve,N}) := \frac{1}{2\rho} \left( \sum_{i,j \in \mathcal{N}} \varepsilon_{ve,i} : E_{ij} : \varepsilon_{ve,j} \right).
\]

The evolution functionals for a Kelvin-Voigt based model are given by

\[
F_{e,j}^v(\sigma_{e,j}, \varepsilon_{e,N}) := \partial \varepsilon_{ve,j} - H_j : (\sigma_{ve,j} - X_{ve,j}), j \in \mathcal{N}.
\]

\(H_j, j \in \mathcal{N}\), are positive definite tensors of 4th order and the inverses of the viscosity tensors \(V_j, j \in \mathcal{N}\), cf. [65], and might depend on further quantities if the model gets more complicated; see Sections 2.3.5 and 2.3.7 for further details. A detailed modeling of linear viscoelasticity with MM models based on the rheological Kelvin-Voigt model is presented in Chapter 3.

---

16A coupling between the elastic elements leads back to a description without coupling. The evolution equations \(\sum_j E_{e,j} \varepsilon_{e,j} = A_{e,i} \sigma, i \in \mathcal{N}\), can be linearly transformed in \(\varepsilon_i = (E_{e,i})^{-1} \sigma, i \in \mathcal{N}\), if the coupling is not singular. In other words, if the matrix representation of \((E_{e,j})_{ij}\) is regular. More specific, the series connection of several elastic and viscous mechanisms can be reduced to a single series connection of one elastic and one viscous element if and only if the free energy is a positive definite bilinear form with respect to \(\varepsilon_{e,N}\).
2.3.4. Plastic Mechanisms

In this section, we present three approaches to model plasticity, the classic approach by rheology, an approach by Chaboche and an approach by MM models consisting of several coupled plastic mechanisms. Related functions, variables and parameters to plastic mechanisms are denoted by the subscript \( p \). In case of plastic material behavior, the related mechanisms \( \varepsilon_{p,j} \) are assumed to be traceless,

\[
(2.3.24) \quad \text{tr} (\varepsilon_{p,j}) = 0.
\]

In applications of plastic material models, a (thermo)elastic part is usually included in the model but dropped in the naming. Here, we concentrate on the pure plastic modeling and neglect a (thermo)elastic part for the moment – uncoupled it can be included without difficulties to obtain a (thermo)elasto-plastic model, see Section 2.3.1[Thermoeelastic element besides inelastic mechanisms].

2.3.4.1. Rheological Ideal Plastic Mechanism

The ideal plastic mechanism is the last of the three fundamental rheological mechanism besides the elastic and viscous mechanisms, cf. [4]. Similar to the rheological viscous mechanism, the free energy functional of the ideal plastic mechanism equals zero,

\[
(2.3.25) \quad \psi = \psi_p = 0.
\]

The basic evolution functional for the plastic strain is

\[
(2.3.26) \quad F_p^\varepsilon(\sigma_p, \varepsilon_p) := \partial_t \varepsilon_p - \left( \lambda_p \frac{\sigma_p^*}{\mathfrak{N}(\sigma_p^*)} \right) \cdot (f(\sigma_p^*) = 0).
\]

The functions \( \mathfrak{N} \) and \( f \) have to be proposed. \( f = 0 \), the yield condition, ensures that the plastic strain only evolves if the yield condition is fulfilled. \( \lambda_p \) denotes the plastic multiplier. A typical approach for \( \mathfrak{N} \) is the von-Mises norm. For a tensor \( \mathbf{A} \), we define

\[
(2.3.27) \quad \mathfrak{N}(\mathbf{A}) := ||A^*||_M := \sqrt{\frac{3}{2} A^* : A^*}.
\]
The corresponding yield function of von-Mises law type is defined by

\begin{equation}
(2.3.28) \quad f(\sigma^*) := \mathcal{R}(\sigma^*) - \sigma_0,
\end{equation}

where $\sigma_0 \geq 0$ denotes the initial yield stress. The plastic multiplier $\lambda_p$ can be obtained by exploiting the yield condition. We define the accumulated plastic strain as

\begin{equation}
(2.3.29) \quad s_p(t) := \int_0^t \sqrt{\frac{2}{3} \mathcal{D}(\mathbf{E}_p(s)) : \mathbf{E}_p(s)} \, ds.
\end{equation}

If the yield condition holds, $\mathcal{R}(\sigma) = \sigma_0$, it yields

\begin{equation}
(2.3.30) \quad \lambda_p = \mathcal{R}(\partial_s \mathbf{E}_p) = \frac{3}{2} \partial_t s_p.
\end{equation}

Obviously, this approach does not cover any hardening effects which will be addressed in the following sections.

### 2.3.4.2. Plasticity with Back Stress - Chaboche Model

In [19], the concept of several back stresses related to one plastic mechanism is reviewed. Embedded in our setting, the mechanism is related to $J_\alpha + 1$ inner variables, $J_\alpha = \{0, \ldots, J_\alpha\}$. Covering only kinematic hardening for the moment and assuming the inner variables are uncoupled (cf. [19]), we define for positive definite $C_j$,

\begin{equation}
(2.3.31) \quad \psi := \psi_p(\alpha_{J_\alpha}) := \frac{1}{2\lambda} \sum_{j \in J_\alpha} \alpha_j : C_j : \alpha_j
\end{equation}

and an overall back stress $X$,

\begin{equation}
(2.3.32) \quad X := \sum_{j \in J_\alpha} X_j \text{ with } X_j = \frac{\partial \psi_p}{\partial \alpha_j} = C_j : \alpha_j.
\end{equation}

Due to the definition of the free energy, there is no coupling included in the Chaboche model. The evolution functional of the plastic strain using a von-Mises criterion is given by

\begin{equation}
(2.3.33) \quad F_p(\sigma_p, \mathbf{E}_p, X_{J_\alpha}) := \partial_t \mathbf{E}_p - \left( \lambda_p \frac{\sigma_p^* - X^*}{\|\sigma_p^* - X^*\|_M} \right) \cdot (f(\sigma_p^* - X^*) = 0).
\end{equation}

\textsuperscript{17}In the original approach, the factor $\frac{1}{2\sqrt{2}}$ is used in the free energy functional for isotropic cases. Since we formulate our material law anisotropically with material tensors, it is convenient to use $\frac{1}{2\sqrt{2}}$ instead.
2.3. Modular Principle - Examples of Mechanisms and Models

Analogously, to Section 2.3.4.1 the yield function is

$$f(\sigma^*_p - X^*) := \|\sigma^*_p - X^*\|_M - \sigma_0.$$  (2.3.34)

Chaboche considers in [19] two approaches for kinematic hardening, a Prager and an Armstrong-Frederick approach. The evolution equations are given for the back stresses and have to be embedded in the concept of inner variables. The inner variables $\alpha_{\mathcal{J}_a}$ are divided in three subsets, $\alpha_{\mathcal{J}_{AP}}$ for considering linear Prager kinematic hardening, $\alpha_{\mathcal{J}_{AF}}$ and $\alpha_{\mathcal{J}_{AFc}}$ for considering the classic Armstrong-Frederick approach and an Armstrong-Frederick approach in back stress notation, where $\mathcal{J}_a = \mathcal{J}_{AP} \cup \mathcal{J}_{AF} \cup \mathcal{J}_{AFc}$ and $\mathcal{J}_{AP}, \mathcal{J}_{AF}, \mathcal{J}_{AFc}$ are pairwise disjoint. The evolution functionals of the inner variables $\alpha_{\mathcal{J}_{AP}}$ for Prager’s non-saturation hardening are defined as

$$F^\alpha_{p,j}(\varepsilon_p, \alpha_j) := \partial_t \alpha_j - \partial_t \varepsilon_p, \quad \alpha_{p,j}(0) = 0, \quad j \in \mathcal{J}_{AP} \subset \mathcal{J}_a.$$  (2.3.35)

A hardening law considering saturation is a classic Armstrong-Frederick approach. $A_{p,j}$ are positive definite tensors of 4-th order, such that for all $\alpha_{\mathcal{J}_{AFc}}$ the evolution functionals are given by

$$F^\alpha_{p,j}(\varepsilon_p, \alpha_j) := \partial_t \alpha_j - \partial_t \varepsilon_p + A_{p,j} : \alpha_j \partial_t s_p, \quad \alpha_{p,j}(0) = 0, \quad j \in \mathcal{J}_{AFc} \subset \mathcal{J}_a.$$  (2.3.36)

The classic Armstrong-Frederick approach does not cover coupling between the kinematic hardening evolutions. For later recognition, the Armstrong-Frederick evolution functionals can be given in a more general back stresses notation,

$$F^\alpha_{p,j}(\varepsilon_p, \alpha_j, X_j) := \partial_t \alpha_j - \partial_t \varepsilon_p + A_{p,j} : C_j^{-1} : X_j \partial_t s_p, \quad \alpha_{p,j}(0) = 0, \quad j \in \mathcal{J}_{AF} \subset \mathcal{J}_a.$$  (2.3.37)

With the generalized Armstrong-Frederick approach, coupling can be included by modifying the free energy functional, cf. e.g. (2.3.40). Chaboche includes in his review isotropic hardening as well as modifications to the Armstrong-Frederick approach to improve the modeling of ratcheting. We refer to the next Section 2.3.4.3 for more details on evolution functionals and realizations of isotropic and generalized kinematic hardening.

2.3.4.3. mMnC: MM-Model for Plasticity

We recollect an example of a MM model for plasticity from [71, 72] and extend it in some points. We define the set $\mathcal{N} := \{1, \ldots, m\}$ for $m$ inelastic partial mechanisms, and assign every mechanism one inner variable, so $\mathcal{J}_a = \mathcal{N}$. Depending on the number of yield criteria, we define $\mathcal{J}_a := \{1, \ldots, n\}$, $n \leq m$. The abbreviation $mMnC$ reads as $m$ plastic mechanisms with $n$ yield criteria. In [72], the model is proposed with an uncoupled, thermoelastic mechanism, which we will neglect for the moment. Two approaches for mMnC models are presented:

- $mMnC-A$ with one effective stress $J$ and one common plastic multiplier $\lambda_p$ for all $n$ yield criteria,
- $mMnC-B$ with one effective stress $J^Y_k$ and one common plastic multiplier $\lambda^Y_k$ for each of the $k$ yield criteria, $k \in \mathcal{J}_a$.

The effective stress(es) will indicate by the yield condition(s) if the mechanism is active in the plastic domain. The plastic multiplier(s) will contribute to the evolution of the partial plastic strains. Since the approach of the $mMnC-A$ model is a new approach, we will provide as an example for a comparison of the $mMnC-A$ and $mMnC-B$ models a $3M2C-A$ and a $3M2C-B$ model, see Figs. 2.8(a) and 2.8(b) for graphical schemes of these models.

18 Clearly, an extension to several inner variables for one mechanism is possible, cf. Section 2.3.4.2.
We include the special cases mM1C (here: mM1C-A is equivalent to mM1C-B) and mMmC-B of the mMnC-B model in our following model process to illustrate the embedding of the model approach of [72]. The presented models will include the 2M1C model and the 2M2C model of [72] as special cases of the mM1C and the mMmC-B model.

In order to distinguish which mechanism is featured by which yield criterion, we define the sets

\[ N^Y_k := \{ j \in \mathcal{N} \mid \text{\( j \)-th mechanism with \( k \)-th yield criterion} \}, \quad k \in \mathcal{J}_a, \]

and the index search function

\[ i(j) := \{ k \in \mathcal{J}_a \mid j \in N^Y_k \}, \quad j \in \mathcal{N}. \]

Clearly, it yields \( \bigcup_{k \in \mathcal{J}_a} N^Y_k = \mathcal{N} \) and all \( N^Y_k, k \in \mathcal{J}_a \), are pairwise disjoint. \( i(j) \) maps to just one index \( k \in \mathcal{J}_a \) for every \( j \in \mathcal{N} \). In both approaches, the free energy is defined by

\[ \psi := \psi(\alpha_N, a_{\mathcal{J}_a}) := \frac{1}{2\rho} \left( \sum_{i,j \in \mathcal{N}} \alpha_i : C_{ij} : \alpha_j \right) + \frac{1}{2\rho} \left( \sum_{i,j \in \mathcal{J}_a} c_{ij} a_i a_j \right). \]

Due to our restriction, \( \psi \) being a convex function for fixed temperature, the accompanying matrices of \((M_{C_{ij}})_{ij}\) and \((c_{ij})_{ij}\) have to be positive semidefinite. The coupling of the mechanisms through the kinematic hardening is realized by the back stresses,

\[ X_j = \rho \frac{\partial \psi}{\partial \alpha_j} = \sum_{i \in \mathcal{J}_a} C_{ij} : \alpha_i, \quad j \in \mathcal{J}_a, \]

and through the isotropic hardening by the thermomechanical forces

\[ R_j = \rho \frac{\partial \psi}{\partial a_j} = \sum_{i \in \mathcal{J}_a} c_{ij} : a_i, \quad j \in \mathcal{J}_a. \]

The effective stresses are given by the abbreviations

\[ J_j(\sigma_j, X_j) := \| \sigma_j^* - X_j^* \|_M, \quad j \in \mathcal{N}, \]

\[ J^Y_k(\sigma^Y_N, X^Y_N) := \left( \sum_{j \in \mathcal{J}^Y_k} J^M_j \right)^{\frac{1}{M_k}}, \quad M_k > 1, \quad k \in \mathcal{J}_a, \]

\[ J(\sigma_N, X_N) := \left( \sum_{j \in \mathcal{J}^M} J^M_j \right)^{\frac{1}{M}}, \quad M > 1, \]

and with them the yield functions are

\[ f_j(\sigma_N, X_N, R_{i(j)}) := J_j(\sigma_j, X_j) - R_{i(j)} - R_{0_i(j)}, \quad j \in \mathcal{J}, \quad (mMnC-A), \]

\[ f_j(\sigma_N, X_N, R_{i(j)}) := J^Y_k(\sigma^Y_N, X^Y_N) - R_{i(j)} - R_{0_i(j)}, \quad j \in \mathcal{J}, \quad (mMnC-B). \]

\( R_j \) describes the evolution of the yield surface(s) and \( R_{0j} \) their initial state.\(^{19}\)

\[ R_j(\alpha_{\mathcal{J}_a}) := \rho \frac{\partial \psi}{\partial a_j}, \quad j \in \mathcal{J}_a. \]

\(^{19}\)In plasticity, the thermodynamical forces \( Q_j \) mostly are referred to as \( R_j \) if related to isotropic hardening.
The remaining governing equations for the inner variables \( \alpha \) as the plastic multipliers can be determined by (2.3.49) and the definitions of the effective stresses (2.3.52a).

In the special cases of mM1C and mMmC-B, the yield functions are given by

\[
\begin{align*}
(2.3.46a) & \quad f(\sigma_N, X_N, R_1) := J - R_1 - R_{01}, \quad j \in \mathcal{N}, \quad {\text{mM1C}}, \\
(2.3.46b) & \quad f_j(\sigma_j, X_j, R_j) := J_j - R_j - R_{0j}, \quad j \in \mathcal{N}, \quad {\text{mMmC-B}}.
\end{align*}
\]

Moreover, we define

\[
\begin{align*}
(2.3.47a) & \quad \eta_j(\sigma_N, X_N) := \frac{\partial f_j}{\partial X_j} = \frac{3}{2} \frac{\sigma_j^* - X_j^*}{J_j} \left( \frac{J_j}{J} \right)^{M_j-1}, \quad j \in \mathcal{N}, \quad {\text{mMnC-A}}, \\
(2.3.47b) & \quad \eta_j(\sigma_N, X_N) := \frac{\partial f_j}{\partial X_j} = \frac{3}{2} \frac{\sigma_j^* - X_j^*}{J_j} \left( \frac{J_j}{J^{\mathcal{N}_j}(j)} \right)^{M_j(j)-1}, \quad j \in \mathcal{N}, \quad {\text{mMnC-B}}.
\end{align*}
\]

In case of the mMmC-B model, (2.3.47b) simplifies to

\[
(2.3.48) \quad \eta_j(\sigma_N, X_N) := \frac{\partial f_j}{\partial X_j} = \frac{3}{2} \frac{\sigma_j^* - X_j^*}{J_j}, \quad j \in \mathcal{N}.
\]

The different evolution functionals of the partial strains are given by

\[
\begin{align*}
(2.3.49a) & \quad F^j_j(\sigma_N, \varepsilon_N, X_N, R_{J_a}) := \partial_t \varepsilon_j - \lambda_p \eta_j \cdot (f_j = 0), \quad j \in \mathcal{N}, \quad {\text{mMnC-A}}, \\
(2.3.49b) & \quad F^j_j(\sigma_N, \varepsilon_N, X_N, R_{J_a}) := \partial_t \varepsilon_j - \lambda_{Y(j)}^{Y(j)} \eta_j \cdot (f_j = 0), \quad j \in \mathcal{N}, \quad {\text{mMnC-B}}.
\end{align*}
\]

\( \lambda_p \) and \( \lambda_{Y(j)}^{Y(j)} \) denote the plastic multipliers related to each yield criteria. Using the definition of the accumulated partial strains \( s_{p,j} \),

\[
(2.3.50) \quad s_{p,j}(t) := \int_0^t \sqrt{\frac{2}{3} \partial_s \varepsilon_j(s) : \partial_s \varepsilon_j(s)} \, ds, \quad j \in \mathcal{N}, \quad \text{with}
\]

\[
(2.3.51) \quad \partial_t s_{p,j}(t) = \sqrt{\frac{2}{3} \partial_s \varepsilon_j(s) : \partial_s \varepsilon_j(s)}, \quad j \in \mathcal{N},
\]

the plastic multipliers can be determined by (2.3.49) and the definitions of the effective stresses (2.3.43b), (2.3.43c) as

\[
\begin{align*}
(2.3.52a) & \quad \lambda_p = \left( \sum_{j \in \mathcal{N}} (\partial_t s_{p,j})^M_{M_j} \right)^{-1/M_j}, \quad {\text{mMnC-A}}, \\
(2.3.52b) & \quad \lambda_{Y(j)}^{Y(j)} = \left( \sum_{j \in \mathcal{N}_j^{Y(j)}} (\partial_t s_{p,j})^{M_{j_k}}_{M_k} \right)^{-1/M_k}, \quad k \in \mathcal{J}_a, \quad {\text{mMnC-B}}.
\end{align*}
\]

The remaining governing equations for the inner variables \( \alpha_N \) and \( a_{J_a} \) can be chosen according to the next paragraphs.

**Example** Assume we want to describe a material consisting of three different material phases. Each of the phases experience plastic material behavior and two of them yield at the same stress level. According to the modeling approach from above, we choose \( \mathcal{N} = \mathcal{J}_a := \{1, 2, 3\} \) and \( \mathcal{J}_a := \{1, 2\} \) for three mechanisms and two yield criteria, \( \mathcal{N}_1^{Y(j)} := \{1\} \) and \( \mathcal{N}_2^{Y(j)} := \{2, 3\} \).

In model approach 3M2C-A, the yield functions are given by

\[
(2.3.53) \quad f_1(\sigma_N, X_N, R) := J - R_1 - R_{01},
\]

24
(2.3.54) \[ f_2(\sigma_N, X_N, R) := J - R_2 - R_{02}, \]

and related to one effective stress level \( J \) for both criteria. The partial strains are determined by the evolution equations

(2.3.55) \[ \partial_t \varepsilon_1 = \lambda_p \frac{3 \sigma_1 - X_1}{J_1} \left( \frac{J_1}{J} \right)^{M-1} \cdot (f_1 = 0), \]

(2.3.56) \[ \partial_t \varepsilon_2 = \lambda_p \frac{3 \sigma_2 - X_2}{J_2} \left( \frac{J_2}{J} \right)^{M-1} \cdot (f_2 = 0), \]

(2.3.57) \[ \partial_t \varepsilon_3 = \lambda_p \frac{3 \sigma_3 - X_3}{J_3} \left( \frac{J_3}{J} \right)^{M-1} \cdot (f_2 = 0). \]

Due to the same effective stress, all evolutions are related to the same plastic multiplier.

In model approach 3M2C-B, the effective stresses differ such that

(2.3.58) \[ f_1(\sigma_N, X_N, R) := J_1^{YC} - R_1 - R_{01}, \]

(2.3.59) \[ f_2(\sigma_N, X_N, R) := J_2^{YC} - R_2 - R_{02}. \]

The evolution equations then differ in the plastic multipliers and the yield conditions,

(2.3.60) \[ \partial_t \varepsilon_1 = \lambda_1^{YC} \frac{3 \sigma_1 - X_1}{J_1^{YC}} \left( \frac{J_1}{J_1^{YC}} \right)^{M_1-1} \cdot (f_1 = 0), \]

(2.3.61) \[ \partial_t \varepsilon_2 = \lambda_2^{YC} \frac{3 \sigma_2 - X_2}{J_2^{YC}} \left( \frac{J_2}{J_2^{YC}} \right)^{M_2-1} \cdot (f_2 = 0), \]

(2.3.62) \[ \partial_t \varepsilon_3 = \lambda_2^{YC} \frac{3 \sigma_3 - X_3}{J_3^{YC}} \left( \frac{J_3}{J_3^{YC}} \right)^{M_2-1} \cdot (f_2 = 0). \]

**Evolution of Inner Variables** In the literature, different approaches are chosen for the evolution of an inner variable, cf. e.g.,[19, 22]. The evolutions of \( \alpha_{\mathcal{J}_n} \) are often related to the kinematic hardening of a mechanism, whereas \( \alpha_{\mathcal{J}_n} \) are often related to the isotropic hardening (the evolution of the yield surface). A detailed derivation of kinematic hardening of Armstrong-Frederick type and isotropic hardening concerning also the thermomechanical coupling is illustrated in [28].
2.3. Modular Principle - Examples of Mechanisms and Models

**Kinematic Hardening** We already recalled two types of kinematic hardening – the *Prager* and *Armstrong-Frederick* type for the Chaboche model, see Section 2.3.4.2. These two evolution laws are classic examples for a non-saturation kinematic hardening law (Prager) and a saturation kinematic hardening law (Armstrong-Frederick). We will collect a small selection of hardening laws for kinematic hardening.

A very general approach covering many descriptions of the kinematic hardening variables is

\[ F_{p,j}^\alpha(\theta, \sigma_N, \varepsilon_j, \alpha_N, X_N) := \partial_t \alpha_j - \partial_t \varepsilon_j + \Phi_j^\alpha(\theta, \sigma_N, \varepsilon_N, \alpha_N, X_N), \quad j \in N. \]  

(2.3.63)

\( \Phi_j^\alpha \) are tensor functions of second order and have to fulfill restrictions due to thermomechanical consistency,

\[ \sum_{j \in N} X_j : \Phi_j^\alpha(\theta, \sigma_N, \varepsilon_N, \alpha_N, X_N) \geq 0 \quad \text{for all admissible arguments.} \]  

(2.3.64)

We will justify this later in Section 2.4. Specifying \( \Phi_j^\alpha \), we give some well-known examples for kinematic evolution laws. Depending on the model ansatz \((m\!M\!n\!C-A)\) or \((m\!M\!n\!C-B)\), the evolution equations for the hardening differ in the dependency on the plastic multiplier(s),

\[ \Lambda_j^\alpha := \begin{cases} \lambda_p^Y & \text{(m\!M\!n\!C-A)}, \\ \lambda_{(ij)}^Y & \text{(m\!M\!n\!C-B)}, \end{cases} \quad j \in N. \]  

(2.3.65)

The evolution functionals of the inner variables of a (linear) Prager type \((N^{IP} \subset N)\) are given by

\[ F_{p,j}^\alpha(\varepsilon_j, \alpha_j) := \partial_t \alpha_j - \partial_t \varepsilon_j, \quad j \in N^{IP}, \]  

(2.3.66)

and the evolution functionals of an Armstrong-Frederick type \((N^{AF} \subset N)\) by

\[ F_{p,j}^\alpha(\varepsilon_N, X_j, \alpha_j) := \partial_t \alpha_j - \partial_t \varepsilon_j + A_j : C_{jj}^{-1} : X_j \Lambda_j^\alpha, \quad j \in N^{AF}. \]  

(2.3.67)

In \([22]\), three hardening laws filling the gap between the Prager and the Armstrong-Frederick approach are proposed by Desmorat. We slightly modify them for our purpose to cover and handle the coupling through the back stresses by positive semidefinite tensorial material parameters \(A_j\). With

\[ s_{\alpha,j} := \int_0^t \sqrt{\frac{2}{3} \alpha_j(s) : \alpha_j(s)} \, ds, \quad X_{eq,j} := \sqrt{\frac{2}{3} X_j : X_j} \]  

(2.3.68)

and \(K \geq 2\), we define

\[ F_{p,j}^\alpha(\varepsilon_j, \alpha_j, X_j) := \partial_t \alpha_j - \partial_t \varepsilon_j + A_j : C_{jj}^{-1} : X_j \partial_t s_{\alpha,j}, \quad j \in N^{D1}, \]  

(2.3.69)

\[ F_{p,j}^\alpha(\varepsilon_j, \alpha_j, X_j) := \partial_t \alpha_j - \partial_t \varepsilon_j + A_j : C_{jj}^{-1} : X_j \partial_t s_{eq,j} X_{eq,j}^{K-2}, \quad j \in N^{D2}, \]  

(2.3.70)

\[ F_{p,j}^\alpha(\varepsilon_j, \alpha_j, X_j) := \partial_t \alpha_j - \partial_t \varepsilon_j + A_j : C_{jj}^{-1} : X_j X_{eq,j}^{K-2} \partial_t X_{eq,j}, \quad j \in N^{D3}. \]  

(2.3.71)

Obviously, it yields \(N^{IP} \cup N^{AF} \cup N^{D1} \cup N^{D2} \cup N^{D3} = N\) and \(N^{IP}, N^{AF}, N^{D1}, N^{D2}, N^{D3}\) are pairwise disjoint. For all evolution equations, we pose the initial conditions \(\alpha_j(0) = 0, j \in N\). All five proposed evolution laws have a similar multiplicative form for \(\Phi_j^\alpha\).

\[ \Phi_j^\alpha = \Phi_j^\alpha : X_j, \quad j \in N. \]  

(2.3.72)

In all cases, \(\Phi_j^\alpha\) is a positive definite tensor function of fourth order.
Isotropic hardening  The isotropic hardening is modeled similarly to kinematic hardening. A general approach is
\[
F_{p,j}^\alpha(\theta, \sigma_N, \varepsilon_N, a_{Ja}, R_{Ja}) := \partial_t a_j - \Lambda_j^a + \Phi_j^\alpha(\theta, \sigma_N, \varepsilon_N, a_{Ja}, R_{Ja}), \quad j \in J_a,
\]
with
\[
\Lambda_j^a = \begin{cases} 
\lambda_p \text{ (mMnC-A)}, \\
\lambda_C \text{ (mMnC-B)} 
\end{cases}, \quad j \in J_a.
\]
Like $\Phi_j^\alpha, \Phi_j^\alpha_{Ja}$ have to fulfill certain restrictions due to thermomechanical consistency, see Section 2.4. A widely-used approach for isotropic hardening with saturation is an Armstrong-Frederick-like approach ($g_j$ and $b_j$ material parameters)
\[
F_{p,j}^\alpha(\varepsilon_N, a_j, R_j) := \partial_t a_j - \left( g_j - \frac{b_j}{\epsilon_{jj}} R_j \right) \Lambda_j^a, \quad j \in J_a,
\]
with $a_j(0) = 0$ for $j \in J_a$.

An extended approach of MM models applied to plasticity is also presented in [73].

### 2.3.5. Creep Mechanism

Materials like steel show creep behavior under moderate stress and high temperatures, see e.g. [45]. We recall one proposal for a material law concerning creep from [66]. Creep mechanism related functions, variables and parameter are denoted by the subscript $c$. We define the free energy by
\[
\psi := \psi_c(\alpha_c, a_c) := \frac{1}{2\theta} (\alpha_c : C_c : \alpha_c),
\]
where $C_c$ is a positive definite tensor of fourth order. The thermomechanical force
\[
X_c := \theta \frac{\partial \psi}{\partial \alpha_c}
\]
is called the creep back stress. The accumulated creep strain is defined by
\[
s_c(t) := \int_0^t \sqrt{\frac{2}{3}\partial_s \varepsilon_c(s) : \partial_s \varepsilon_c(s)} \, ds.
\]

defines the evolution functional for creep. $B_c, m_c, k$ and $D_c$ are material parameters. $B_c$ is a positive definite tensor of fourth order and $m \geq 0$. $\| \cdot \|_M$ denotes the von-Mises norm, see (2.3.27). $D_c$ is the (creep) drag stress. If $X_c \equiv 0$, $k$ is directly related to the three stages of creep. For $k < 0$, we have primary creep, for $k = 0$, secondary creep, and for $k > 0$, tertiary creep. The evolution of the inner variable $\alpha_c$ is often assumed as an Armstrong-Frederick approach, cf. also Section 2.3.4.3 [Kinematic Hardening],
\[
F_c^\alpha(\varepsilon_c, \sigma_c, X_c) := \partial_t \varepsilon_c - B_c : \frac{\sigma_c - X_c^*}{D_c} \left( \frac{\| \sigma_c - X_c \|_M}{D_c} \right)^{m-1} \mathbf{k}^c
\]
defines the evolution functional for creep. $B_c, m_c, k$ and $D_c$ are material parameters. $B_c$ is a positive definite tensor of fourth order and $m \geq 0$. $\| \cdot \|_M$ denotes the von-Mises norm, see (2.3.27). $D_c$ is the (creep) drag stress. If $X_c \equiv 0$, $k$ is directly related to the three stages of creep. For $k < 0$, we have primary creep, for $k = 0$, secondary creep, and for $k > 0$, tertiary creep. The evolution of the inner variable $\alpha_c$ is often assumed as an Armstrong-Frederick approach, cf. also Section 2.3.4.3 [Kinematic Hardening],
\[
F_c^\alpha(\varepsilon_c, \alpha_c, X_c) := \partial_t \alpha_c - \partial_t \varepsilon + A_c : C_c^{-1} : X_c (\partial_t s_c)^T.
\]
is a parameter to switch between an Armstrong-Frederick-like approach \((l = 1)\) or an approach by Robinson \((l = 0)\), cf. [6]. The drag stress is given by the evolution

\[
\partial_t D := (g_c - b_c D_c) \partial_t s_c, \quad D_c(0) = 1,
\]

cf. Section 2.3.4.3 [Isotropic Hardening]. It does not contribute to the free energy as an inner variable or depends on it as thermomechanical force. If we assume a different evolution for the drag stress, we have to ensure the positivity of \(D_c\). A MM model considering interaction between creep and plasticity is presented in [64].

**Remark 2.3.1.** Creep is clearly a generalization of viscoelastic material behavior (e.g. for Kelvin-Voigt viscoelasticity, see Section 2.3.3.2, if \(m = 1\), \(D_c \equiv 1\), \(k \equiv 0\) and cf. [65]).

### 2.3.6. Viscoplastic Mechanisms

Viscoplastic mechanisms can be modeled analogously to plasticity (Section 2.3.4). We state the modifications in the evolution functional(s) and give proposals for the viscoplastic multiplier(s) for the three presented viscoplastic models. The definitions of free energies and hardening laws can be used accordingly to Section 2.3.4.

The evolution functionals related to the viscoplastic strains are altered based on a rheological ansatz (cf. (2.3.26))

\[
F^p_{\varepsilon}(\sigma_p, \varepsilon_p) := \partial_t \varepsilon_p - \left(\lambda_p \frac{\sigma_p^*}{l(\sigma_p^*)} \right) \cdot (f(\sigma_p^*) \geq 0),
\]

based on a Chaboche ansatz (cf. (2.3.33))

\[
F^p_{\varepsilon}(\sigma_p, \varepsilon_p, X_{J_a}) := \partial_t \varepsilon_p - \left(\lambda_p \frac{\sigma_p^* - X^*}{\|\sigma_p^* - X^*\|_M} \right) \cdot (f(\sigma_p^* - X^*) \geq 0)
\]

or based on a MM ansatz (cf. (2.3.49))

\[
F^p_j(\sigma_{N}, \varepsilon_{N}, X_{N}, R_{J_a}) := \partial_t \varepsilon_j - \lambda_p \eta_j \cdot (f_j \geq 0), \quad j \in \mathcal{N}, \quad (mMnC-A),
\]

\[
F^p_j(\sigma_{N}, \varepsilon_{N}, X_{N}, R_{J_a}) := \partial_t \varepsilon_j - \lambda_j \eta_j \cdot (f_j \geq 0), \quad j \in \mathcal{N}, \quad (mMnC-B).
\]

Instead exploiting the yield condition(s) \((f = 0 \text{ resp. } f_j = 0)\) to calculate the plastic multiplier(s), we have to define the viscoplastic multiplier(s) separately, e.g. by (cf. [72])

\[
\lambda_p := \frac{2}{3\eta} \left\langle \frac{f}{D} \right\rangle^{n_\lambda},
\]

\[
\lambda_j := \frac{2}{3\eta_j} \left\langle \frac{f_j}{D_j} \right\rangle^{n_{\lambda,j}}.
\]

\(\eta > 0\) and \(\eta_j > 0\) denote the viscosities and \(D\) and \(D_j\) are drag stresses (cf. Section 2.3.5). \(n_\lambda > 0\) and \(n_{\lambda,j} > 0\) are additional material parameters.

### 2.3.7. TRIP Mechanism

Materials like steel undergo phase transformations depending on the temperature treatment. In order to take this into account, a subset \(p_{J_{ph}}\) of the scalar inner variables \(a_{J_a}\) will describe the different material phases. We will not separate between mass and volume fraction, and just refer to them as phase fractions, cf. [44].\(^20\) The general form of the free energy of a multi phase

\(^20\)“The difference between mass and volume fractions ("absolute error") is less than 0.02. The relative error is less than 4.5% in unfavorable cases.”([44])
material is defined by

\[
\psi(\theta, \alpha, J_\alpha, a, J_a) := \frac{1}{2\theta} \left( \sum_{i,j \in J_a} \alpha_i : C_{ij}(\theta, p_{J_{ph}}) : \alpha_j + \sum_{i,j \in J_a} c_{ij}(\theta, p_{J_{ph}}) a_i a_j \right.
\]
\[
+ \sum_{i \in J_a, j \in J_a} d_{kij}(\theta, p_{J_{ph}}) a_j \gamma^{k}(\alpha_i) + \sum_{j \in J_a} \gamma^j(\theta, p_{J_{ph}}) : \alpha_j
\]
\[
+ \sum_{j \in J_a} \gamma^j(\theta, p_{J_{ph}}) a_j \right) - C_0(\theta, p_{J_{ph}}).
\]

Each material parameter can be a function of the temperature and phase fractions. In complex cases, they may depend on further quantities. A common assumption is that a material parameter – valid for the whole mixed composite – is a composition of the material parameters of each phase.

We propose two mixture rules. Let \( C \) be a tensorial or scalar material parameter depending on the temperature \( \theta \) and the phase fractions \( p_{J_{ph}} \) and \( C_j, j \in J_{ph} \), the temperature-dependent material parameter of the \( j \)-th phase, then

\[
C(\theta, p_{J_{ph}}) := \sum_{j \in J_{ph}} p_j C_j(\theta),
\]

or

\[
C^{-1}(\theta, p_{J_{ph}}) := \sum_{j \in J_{ph}} p_j C^{-1}_j(\theta).
\]

Hereinafter, we will use mixture rule (2.3.87a) if not stated otherwise.

### 2.3.7.1. Thermoelastic-TRIP Model

We introduce the TRIP mechanism with back stress in a coupled model with thermoelastic material behavior, cf. [70]. A model for interaction with plasticity will be formulated in Chapter 4. TRIP related parameters and variables are denoted by the subscript \(tp\). We assume the following free energy of a \( J_{ph} \) phase material, \( J_{ph} := \{1, \ldots, J_{ph}\} \),

\[
\psi = \psi(\theta, \alpha_{te}, \alpha_{tp}, p_{J_{ph}}) = \frac{1}{2\theta} \left( \alpha_{te} : E_{te}(\theta, p_{J_{ph}}) : \alpha_{te} + \sum_{j \in J_{ph}} \beta_j p_j \alpha_{te} : I \right.
\]
\[
+ \gamma_{te}(\theta, p_{J_{ph}}) : \alpha_{te} + \alpha_{tp} : C_{tp}(\theta, p_{J_{ph}}) : \alpha_{tp} \right) - C_0(\theta, p_{J_{ph}}),
\]

\( \beta_{J_{ph}} \) are related to the density changes due to phase transformations and often approximated by

\[
\beta_j = \frac{1}{3} \frac{\varrho_0 - \varrho_{0,j}}{\varrho_{0,j}}, \quad j \in J_{ph},
\]

where \( \varrho_0 \) denotes the mixed reference density at \( t = 0 \) and \( \varrho_{0,j} \) the reference density of the \( j \)-th phase. The thermomechanical force

\[
X_{tp} := \varrho \frac{\partial \psi}{\partial \alpha_{tp}}
\]
is referred to as the TRIP back stress. We define the accumulated TRIP strain as

\[
s_{tp}(t) := \int_0^t \sqrt{\frac{2}{3} \partial_s \varepsilon_{tp}(s) : \partial_s \varepsilon_{tp}(s)} \, ds.
\]

**Thermoelastic Mechanism** The thermoelastic strain is determined by the classic approach

\[
F^e_{te}(\sigma_{te}, X_{te}) := \sigma_{te} - X_{te}
\]
and

\[
F^\alpha_{te}(\varepsilon_{te}, \alpha_{te}) := \varepsilon_{te} - \alpha_{te}
\]
such that the thermoelastic inner variable can be excluded, see Section 2.3.1.1.

**TRIP Mechanism** The evolution functional of the TRIP strain is similar to a viscoelastic and creep mechanism,

\[
F^e_{tp}(\theta, \sigma_{tp}, \varepsilon_{tp}, X_{tp}, p_{J_{ph}}) := \partial_t \varepsilon - \kappa_{tp}(\theta, p_{J_{ph}}) : (\sigma^*_{tp} - X^*_{tp}).
\]

\( \kappa_{tp} \) denotes the generalized Greenwood-Johnson parameter. For all admissible arguments, \( \kappa_{tp} \) is supposed to be a positive tensor of fourth order. \( \kappa_{tp} \) is defined by a generalization of the Franitza-Mitter-Leblond ansatz (cf. [13])

\[
\kappa_{tp}(\theta, p_{J_{ph}}) := \sum_{j \in J_{ph}} \kappa_{tp,j}(\theta) \frac{\partial \Phi_{sat,j}}{\partial p_j}(\partial_t p_j),
\]
where \( \kappa_{tp,j} \) are positive tensors of fourth order and \( \Phi_{sat,j}, j \in J_{ph}, \) are saturation functions which have to fulfill

\[
\Phi_{sat,j}(0) = 0, \quad \Phi_{sat,j}(1) = 1, \quad 0 \leq \Phi_{sat,j}(p_j) \leq 1, \quad \frac{\partial \Phi_{sat,j}}{\partial p_j} > 0 \quad \text{for all } p_j \in ]0,1[. \]

Typical approaches for the saturation functions are, cf. e.g. [12], \( j \in J_{ph}, \)

\[
\Phi_{sat,j}(p_j) = p_j, \quad (\text{Tanaka}),
\]
\[
\Phi_{sat,j}(p_j) = p_j(1 - \ln(p_j)), \quad (\text{Leblond}),
\]
\[
\Phi_{sat,j}(p_j) = p_j(2 - p_j), \quad (\text{Desalos / Denis}).
\]

The inner variable \( \alpha_{tp} \) can evolve according to the proposed kinematic hardening laws in Section 2.3.4.3[Kinematic Hardening], e.g. as an evolution according to Armstrong-Frederick,

\[
F^n_{tp}(\varepsilon_{tp}, \alpha_{tp}, X_{tp}) := \partial_t \alpha_{tp} - \partial_t \varepsilon_{tp} + A_{tp} : C_{tp}^{-1} : X_{tp},
\]
where \( A_{tp} \) is positive definite.

**Phase fractions** The evolution equations defining the phase fractions given in a differential form are defined by

\[
F^n_{ph,j}(\theta, \sigma_N, p_{J_{ph}}) := \partial_t p_j - \Phi_{ph,j}(\theta, \sigma_N, p_{J_{ph}}), \quad j \in J_{ph}.
\]
The phase fractions have to fulfill

\[
\sum_{j \in J_{ph}} p_j = 1, \quad p_j \geq 0, \ j \in J_{ph}.
\]

Due to this constraint, it has to hold

\[
\sum_{j \in J_{ph}} \Phi_{ph,j} = 0 \quad \text{and} \quad \sum_{j \in J_{ph}} p_j 0 = 1,
\]

with \( p_j 0 := p_j(0), \ j \in J_{ph} \). Clearly, the choices for \( \Phi_{ph,j}, j \in J_{ph} \), are restricted by (2.3.100) and (2.3.101). We will not specify the evolutions of the phase fractions any further and refer to the literature, cf. [74] for different approaches for multi phase material. For a statement on thermomechanical consistency see Remark 2.4.7.

In [9], the creep and TRIP behavior of the steel 100Cr6 is investigated. The presented material model is an example for an uncoupled 3M model considering a thermoelastic mechanism, a creep mechanism and a TRIP mechanism.

2.3.8. MM Model with Chaboche Ansatz

The evolutions of the partial strains described in Sections 2.3.1 - 2.3.6 follow an algebraic equation (thermoelastic mechanism) or a differential equation (viscous, viscoelastic (incl. creep) and plastic mechanisms). The general approach of Section 2.2 can be specified in an abstract way allowing only the used evolution equations of the partial strains for the presented example mechanisms of algebraic or differential kind. Moreover, instead of depending on one back stress, each mechanism can be related to a Chaboche-like back stress, cf. Section 2.3.4.2 and [73]. The TRIP mechanism of Section 2.3.7 is not covered in this model since we will not include evolution equations for scalar inner variables describing phase fraction evolutions.

First, we divide the index set of the mechanisms in algebraic (\( a \)) and differential (\( d \)) ones, hence

\[
N = N^a \cup N^d \quad \text{with} \quad N^a \cap N^d = \emptyset.
\]

Using only the presented mechanisms, each mechanism of \( N^a \) is a (thermo)elastic mechanism presented in Section 2.3.1. Each mechanism of \( N^d \) is one of the mechanisms of Sections 2.3.2 - 2.3.6, so that we cover viscous, viscoelastic, plastic, viscoplastic, and creep material behavior and their coupling. In order to distinguish the tensorial inner variables and, therefore, the back stresses related to one mechanism, the index set of tensorial inner variables \( J_\alpha \) is divided in subsets \( J_\alpha^j, j \in N \), where each index subset \( J_\alpha^j \) is related to the \( j \)-th mechanism,

\[
J_\alpha = \bigcup_{j \in N} J_\alpha^j \quad \text{with pairwise disjoint} \quad J_\alpha^j, j \in N.
\]

The free energy is given by (2.3.1),

\[
\psi(\theta, \alpha_{J_\alpha}, a_{J_\alpha}) := \frac{1}{2\varrho} \left( \sum_{i,j \in J_\alpha} \alpha_i : C_{ij} : \alpha_j + \sum_{i,j \in J_\alpha} c_{ij} a_i a_j + \sum_{i \in J_\alpha, j \in J_\alpha, k \in \{1,2,3\}} d_{kij} a_j J_k(\alpha_i) \right. \\
\left. + \sum_{j \in J_\alpha} \gamma_{j}^a : \alpha_j + \sum_{j \in J_\alpha} \gamma_{j}^a a_j \right) - C_\varrho(\theta).
\]
The thermomechanical forces \( X_{\mathcal{J}_a} \) and \( Q_{\mathcal{J}_a} \) are given by (2.2.10),
\[
X_j := \frac{\partial \psi}{\partial \alpha_j}, \quad j \in \mathcal{J}_a, \quad \text{and} \quad Q_j := \frac{\partial \psi}{\partial a_j}, \quad j \in \mathcal{J}_a.
\]

Like in the Chaboche model, we define for each mechanism a back stress,
\[
\tilde{X}_j := \sum_{k \in \mathcal{J}_d} X_k, \quad j \in \mathcal{N}.
\]

Using these back stresses, we modify the algebraic evolution functionals
\[
F^\varepsilon_j(\sigma_j, X_{\mathcal{J}_a}) = \sigma_j - \tilde{X}_j, \quad j \in \mathcal{N},
\]
and the differential evolution functionals
\[
F^\varepsilon_j(\theta, \sigma_N, \varepsilon_N, \alpha_{\mathcal{J}_a}, X_{\mathcal{J}_a}, a_{\mathcal{J}_a}, Q_{\mathcal{J}_a}) = \partial_t \varepsilon_j - H_j : (\sigma_j - \tilde{X}_j), \quad j \in \mathcal{N},
\]
with positive (semi)definite tensor functions \( H_j \) of fourth order,
\[
H_j = H_j(\theta, \sigma_N, \varepsilon_N, \alpha_{\mathcal{J}_a}, X_{\mathcal{J}_a}, a_{\mathcal{J}_a}, Q_{\mathcal{J}_a}).
\]

The positive (semi)definite tensor functions \( H_{\mathcal{N}_d} \) are given by the chosen mechanism for each \( j \in \mathcal{N}_d \), e.g. by
\[
H_j := \begin{cases} 
V^{-1}, & \text{(Viscoelastic, Section 2.3.3),} \\
\left( \frac{\lambda_{\mathcal{J}_a}}{\beta(\sigma_j)} \right) \cdot \left( f(\sigma_j) = 0 \right), & \text{(Ideal plastic, Section 2.3.4.1),} \\
B_j \left( \frac{\|\sigma_j - \tilde{X}_j\|}{D_j} \right) s_j, & \text{(Creep, Section 2.3.5).}
\end{cases}
\]

For plastic mechanisms, the plastic multipliers and the yield conditions are included in the related \( H_j \). In case of the plastic mechanisms related to a mMN model of Section 2.3.4.3, the plastic multipliers, effective stresses et cetera have to be modified with respect to \( \tilde{X}_j \).

We specify the evolution functionals of the inner variables with
\[
\Phi_k = \Phi_k : X_k, \quad k \in \mathcal{J}_a,
\]
where each \( \Phi_k \) is a positive (semi)definite tensor function of fourth order for all admissible arguments (including \( \Phi_k \equiv 0 \)), so
\[
F^\alpha_k(\theta, \sigma_N, \varepsilon_N, \alpha_{\mathcal{J}_a}, X_{\mathcal{J}_a}, a_{\mathcal{J}_a}, Q_{\mathcal{J}_a}) = \partial_t \alpha_k - \partial_t \varepsilon_j + \Phi_k : X_k, \quad k \in \mathcal{J}_a, \quad j \in \mathcal{N}.
\]

For simplicity, we just consider evolutions of saturation type for the scalar inner variables \( a_{\mathcal{J}_a} \),
\[
F^\alpha_j(\theta, \sigma_N, \varepsilon_N, \alpha_{\mathcal{J}_a}, X_{\mathcal{J}_a}, a_{\mathcal{J}_a}, Q_{\mathcal{J}_a}) = \partial_t a_j - (g_j - b_j Q_j), \quad j \in \mathcal{J}_a.
\]

\( g_{\mathcal{J}_a} \) and \( b_{\mathcal{J}_a} \) are positive functions and may depend on the quantities \( \theta, \sigma_N, \varepsilon_N, \alpha_{\mathcal{J}_a}, X_{\mathcal{J}_a}, a_{\mathcal{J}_a} \) and \( Q_{\mathcal{J}_a} \).

2.4. Some Remarks on Thermomechanical Consistency

In this section, we give statements concerning thermomechanical consistency of selected models of the previous section.

Assuming the Fourier law of heat conduction (2.2.8) holds, the remaining inequality to check
thermomechanical consistency is

\begin{equation}
\sigma : \partial_t \varepsilon - \varrho \left( \sum_{j \in J_a} \frac{\partial \psi}{\partial \alpha_j} : \partial_t \alpha_j + \sum_{j \in J_a} \frac{\partial \psi}{\partial a_j} \partial_t a_j \right) \geq 0.
\end{equation}

Specifying the evolution equations related to the partial strains and inner variables, we are able to give more precise statements. In case of an additive decomposition of the material model (resp. the free energy) into uncoupled mechanisms, it is sufficient to check each mechanism for thermomechanical consistency, cf. [46, p. 51, Theorem 1].

**Remark 2.4.1.** The weights of the mechanisms \( A_j \) are assumed to be positive real numbers independent on further quantities, e.g. temperature. This assumption results in

\begin{equation}
\partial_t A_j \equiv 0, \quad j \in \mathbb{N}.
\end{equation}

Otherwise, if \( \partial_t A_j \neq 0 \), we could not avoid additional terms in the remaining inequality and we would obtain

\begin{equation}
\sigma : \partial_t \varepsilon = \sum_{j \in \mathbb{N}} A_j^{-1} \varepsilon_j : \partial_t (A_j \varepsilon_j) = \sum_{j \in \mathbb{N}} \sigma_j : \partial_t \varepsilon_j + \sum_{j \in \mathbb{N}} A_j^{-1} \partial_t A_j \sigma_j : \varepsilon_j.
\end{equation}

Including \( \sum_{j \in \mathbb{N}} A_j^{-1} \partial_t A_j \sigma_j : \varepsilon_j \) into considerations, thermomechanical consistency requires much more effort to prove.

The **MM model with a Chaboche ansatz** of Section 2.3.8 covers the example mechanisms of Section 2.2 (excluding the TRIP mechanism) and their possible combinations. In the following theorem, we prove thermomechanical consistency of the **MM model with a Chaboche ansatz**.

**Theorem 2.4.2.** If all \( H_j, \quad j \in \mathbb{N}^d \), and all \( \Phi_k, \quad k \in J_a^d, \quad j \in \mathbb{N} \), are positive (semi)definite tensor functions of fourth order, all \( g_j \) and \( b_j \), \( j \in J_a \), are positive functions, and

\begin{equation}
\sum_{j \in J_a} \frac{g_j^2}{4b_j} \leq K
\end{equation}

with

\begin{equation}
K := \sum_{j \in \mathbb{N}^d} (\sigma_j - \bar{X}_j) : H_j : (\sigma_j - \bar{X}_j) + \sum_{k \in J_a} X_k : \Phi_k : X_k
\end{equation}

holds for all admissible arguments, then the MM model with a Chaboche ansatz presented in Section 2.3.8 is thermomechanically consistent.

**Proof.** In order to prove the thermomechanical consistency of the model, we have to show the validity of the remaining inequality (2.4.1). With the model assumptions of Section 2.3.8, we obtain

\begin{equation}
\sigma : \partial_t \varepsilon - \varrho \left( \sum_{j \in J_a} \frac{\partial \psi}{\partial \alpha_j} : \partial_t \alpha_j + \sum_{j \in J_a} \frac{\partial \psi}{\partial a_j} \partial_t a_j \right)
= \sum_{j \in \mathbb{N}} \sigma_j : \partial_t \varepsilon_j - \sum_{j \in J_a} X_j : \partial_t \alpha_j - \sum_{j \in J_a} Q_j \partial_t a_j
= \sum_{j \in \mathbb{N}} \sigma_j : \partial_t \varepsilon_j - \sum_{j \in \mathbb{N}, k \in J_a^d} X_k : (\partial_t \varepsilon_j - \Phi_k^\alpha : X_k) - \sum_{j \in J_a} Q_j \partial_t a_j.
\end{equation}
2.4. Some Remarks on Thermomechanical Consistency

\[ \begin{align*}
&= \sum_{j \in N} \sigma_j : \partial_t \varepsilon_j - \sum_{j \in N} \tilde{X}_j : \partial_t \varepsilon_j + \sum_{j \in J_a} X_j : \Phi_{\alpha}^j : X_j - \sum_{j \in J_a} Q_j \partial_t a_j \\
&= \sum_{j \in N^d} (\sigma_j - \tilde{\sigma}_j) : \partial_t \varepsilon_j + \sum_{j \in J_a} X_j : \Phi_{\alpha}^j : X_j - \sum_{j \in J_a} Q_j \partial_t a_j \\
&= \sum_{j \in N^d} (\sigma_j - \tilde{\sigma}_j) : H_j : (\sigma_j - \tilde{\sigma}_j) + \sum_{j \in J_a} X_j : \Phi_{\alpha}^j : X_j - \sum_{j \in J_a} Q_j \partial_t a_j.
\end{align*} \]

We define

\[ (2.4.7) \quad K := \sum_{j \in N^d} (\sigma_j - \tilde{\sigma}_j) : H_j : (\sigma_j - \tilde{\sigma}_j) + \sum_{j \in J_a} X_j : \Phi_{\alpha}^j : X_j. \]

\( K \) is nonnegative since all \( H_j, j \in N^d \), and \( \Phi_{\alpha}^j, j \in J_a \), are positive (semi)definite tensor functions for all admissible arguments. Exploiting the evolution equations for the scalar inner variables \( a_{J_a} \) to obtain thermomechanically consistency, the inequality

\[ (2.4.8) \quad K - \sum_{j \in J_a} Q_j \partial_t a_j = K - \sum_{j \in J_a} Q_j (g_j - b_j Q_j) = K + \sum_{j \in J_a} b_j \left( Q_j^2 - \frac{g_j}{b_j} Q_j \right) \]

\[ = K + \sum_{j \in J_a} b_j \left( Q_j - \frac{g_j}{2b_j} \right)^2 - \frac{g_j^2}{4b_j} \geq 0 \]

has to be valid. Since \( \sum_{j \in J_a} b_j \left( Q_j - \frac{g_j}{2b_j} \right)^2 \geq 0 \), it is sufficient, that

\[ (2.4.9) \quad K - \sum_{j \in J_a} \frac{g_j^2}{4b_j} \geq 0 \]

has to be fulfilled and this holds by assumption. \( \square \)

The conditions (2.4.4) might be difficult to prove, instead we consider the more general case

\[ (2.4.10) \quad K \geq K_Q, \]

where \( K_Q := \sum_{j \in J_a} Q_j \partial_t a_j \), to exploit a possible relation between \( K \) and \( K_Q \). If no scalar inner variables and their related thermomechanical forces appear in the model, \( K_Q \equiv 0 \), Theorem 2.4.2 is easily applicable, cf. Theorem 2.4.3 and Remark 2.4.4.

**Theorem 2.4.3.** The model of *coupled thermoelastic mechanisms with hardening* presented in Section 2.3.1.2 is thermomechanically consistent if

\[ (2.4.11) \quad \sum_{j \in N} X_{te,j} : \Phi_{\alpha}^j_{te,j}(\theta, \sigma_N, \alpha_{te,N}, X_{te,N}) \geq 0 \]

for all admissible arguments holds. As a special case, the basic thermoelastic mechanism of Section 2.3.1.1 is thermomechanically consistent too.

**Proof.** The model of coupled thermoelastic mechanisms with hardening is a special case of the MM model with Chaboche ansatz. Hence Theorem 2.4.2 is applicable. The remaining inequality is, note \( J_a = \emptyset \).
2. Multi-Mechanism Models

\[
\sigma : \partial_t \varepsilon - \varrho \left( \sum_{j \in J_a} \frac{\partial \psi}{\partial \alpha_j} \partial_t \alpha_j + \sum_{j \in J_a} \frac{\partial \psi}{\partial a_j} \partial_t a_j \right) = \sum_{j \in N} \sigma_{te,j} : \partial_t \varepsilon_{te,j} - \sum_{j \in N} X_{te,j} : \partial_t \alpha_{te,j}
\]

\[
= \sum_{j \in N} (\sigma_{te,j} - X_{te,j}) : \partial_t \varepsilon_j + \sum_{j \in N} X_{te,j} : \Phi_{te,j}^\alpha (\theta, \sigma_{te,N}, \alpha_{te,N}, X_{te,N}) \geq 0.
\]

Since \( \sigma_{te,j} - X_{te,j} = 0 \) for all \( j \in N \) by definition of the evolution equation, it remains the inequality

\[
\sum_{j \in N} X_{te,j} : \Phi_{te,j}^\alpha (\theta, \sigma_{te,N}, \alpha_{te,N}, X_{te,N}) \geq 0,
\]

which is fulfilled by assumption. In case of the basic thermoelastic element \( (N := \{0\}) \), it holds \( \Phi_{te}^\alpha = \Phi_{te,0} = 0 \). \( \Box \)

**Remark 2.4.4.** The presented models of the viscous mechanism (Section 2.3.2), the viscoelastic models (Section 2.3.3), the Chaboche model for plasticity (Section 2.3.4.2) and the creep model (Section 2.3.5) are thermomechanically consistent as special cases of the MM model with Chaboche ansatz without any inner variables of scalar type. Therefore, Theorem 2.4.2 is easily applicable without having to check (2.4.4).

Both mMnC models presented in Section 2.3.4.3 are special cases of the MM model with Chaboche ansatz (Section 2.3.8). And both models are also examples to exploit a possible relation between \( K \) and \( K_Q \) by the definitions of the evolution equation of the partial strains and their yield conditions.

**Theorem 2.4.5.** Let \( R_{0k}, c_{kk}, b_k, k \in J_a \), be positive functions, \( \Phi_j^\alpha \), \( j \in N \), positive semidefinite tensor functions of fourth order for all admissible arguments.

**A.)** If \( (n = |J_a|) \)

\[
\sum_{k \in J_a} \frac{R_{0k}}{n} \geq \sum_{k \in J_a} \frac{c_{kk}}{4b_k} \left( \frac{1}{n} - g_k \right)^2
\]

holds, the mMnC-A model of Section 2.3.4.3 is thermomechanically consistent.

**B.)** If

\[
\sum_{k \in J_a} R_{0k} \geq \sum_{k \in J_a} \frac{c_{kk}}{4b_k} (1 - g_k)^2
\]

holds, the mMnC-B model of Section 2.3.4.3 is thermomechanically consistent.

**Proof.** **A.)** In case of the mMnC-A model, the remaining inequality is

\[
\sigma : \partial_t \varepsilon - \varrho \left( \sum_{j \in J_a} \frac{\partial \psi}{\partial \alpha_j} \partial_t \alpha_j + \sum_{j \in J_a} \frac{\partial \psi}{\partial a_j} \partial_t a_j \right)
\]

\[
= \sum_{j \in N} (\sigma_j - X_j) : \partial_t \varepsilon_j + \sum_{j \in N} X_j : \Phi_j^\alpha (\theta, \sigma_{te,N}, \alpha_{te,N}, X_{te,N}) - \sum_{k \in J_a} \lambda_p R_k \left( g_k - \frac{b_k}{c_{kk}} R_k \right) \geq 0.
\]

Since all \( \Phi_j^\alpha \) are positive semidefinite, \( K_X \) is positive. The first sum \( K_{\sigma} \) can be reformulated to
2.4. Some Remarks on Thermomechanical Consistency

\[(2.4.17)\quad K_\sigma = \sum_{j \in N} (\sigma_j - X_j) : \left( \lambda_p \frac{3}{2} \frac{\sigma_j^* - X_j^*}{J_j} \left( \frac{J_j}{J} \right)^{M-1} \right) = \sum_{j \in N} \lambda_p \frac{1}{J_j} \left( \frac{J_j}{J} \right)^{M-1} \frac{3}{2} (\sigma_j^* - X_j^*) : (\sigma_j^* - X_j^*) = \lambda_p \frac{1}{J^{M-1}} \sum_{j \in N} J_j^{M-1} J_j^2 = \lambda_p J.\]

\(K_\sigma\) and \(K_R\) are only unequal 0 if the yield conditions is fulfilled, such that

\[(2.4.18)\quad J = R_k + R_{0k}, \quad k \in J_\alpha,\]

and

\[(2.4.19)\quad K_\sigma - K_R = \lambda_p \frac{1}{n} \sum_{k \in J_\alpha} J - \sum_{k \in J_\alpha} R_k \left( g_k - \frac{b_k}{c_{kk}} R_k \right) = \lambda_p \left( \sum_{k \in J_\alpha} \frac{R_k + R_{0k}}{n} - R_k \left( g_k - \frac{b_k}{c_{kk}} R_k \right) \right) = \lambda_p \left( \sum_{k \in J_\alpha} \frac{b_k}{c_{kk}} \left( R_k + \frac{(n-1-g_k)c_{kk}}{2b_k} \right)^2 - \frac{(n-1-g_k)^2 c_{kk}}{4b_k} + \frac{R_{0k}}{n} \right).\]

\(K_\sigma - K_R\) is positive for arbitrary \(R_k\)’s, if

\[(2.4.20)\quad \sum_{k \in J_\alpha} \frac{(n-1-g_k)^2 c_{kk}}{4b_k} \leq \sum_{k \in J_\alpha} \frac{R_{0k}}{n}\]

and this holds by assumption. Together with \(K_X\) being positive, the inequality \((2.4.16)\) holds. Therefore, the mMnC-A model is thermomechanically consistent.

B.) In case of the mMnC-B model, the remaining inequality is

\[(2.4.21)\quad \sigma : \partial_t \varepsilon - \varrho \left( \sum_{j \in J_\alpha} \frac{\partial \psi}{\partial \alpha_j} : \partial_t \alpha_j + \sum_{j \in J_\alpha} \frac{\partial \psi}{\partial a_j} \partial_t a_j \right) = \sum_{j \in N} (\sigma_j - X_j) : \partial_t \varepsilon_j + \sum_{j \in N} X_j : \overline{\Psi}_j^{\alpha} - \sum_{k \in J_a} \lambda_k^{YC} R_k \left( g_k - \frac{b_k}{c_{kk}} R_k \right) \geq 0.\]

Since all \(\overline{\Psi}_j^{\alpha}\) are positive semidefinite, \(K_X\) is positive. The first sum \(K_\sigma\) can be reformulated to

\[(2.4.22)\quad K_\sigma = \sum_{j \in N} (\sigma_j - X_j) : \left( \lambda_{Y_{(j)}}^{YC} \frac{3}{2} \frac{\sigma_j^* - X_j^*}{J_j} \left( \frac{J_j}{J_{Y_{(j)}}} \right)^{M_{(j)}-1} \right) = \sum_{k \in J_a} \lambda_k^{YC} \left( \sum_{j \in N} \left\{ \frac{1}{J_j} \left( \frac{J_j}{J_{Y_{(j)}}} \right)^{M_{(j)}-1} (\sigma_j - X_j) : (\sigma_j^* - X_j^*) \right\} \right) = \sum_{k \in J_a} \lambda_k^{YC} J_{k}^{YC}.\]
By exploiting the yield conditions $J^Y_C = R_k + R_{0k}$, $k \in J_\alpha$, we obtain for $K_. - K_R$

\begin{equation}
(2.4.23) \quad K_. - K_R = \sum_{k \in J_\alpha} \lambda_k^Y C \left( J_k^Y C - R_k \left( g_k - \frac{b_k}{c_{kk}} R_k \right) \right),
\end{equation}

\begin{equation}
= \sum_{k \in J_\alpha} \lambda_k^Y C \left( R_k + R_{0k} - R_k \left( g_k - \frac{b_k}{c_{kk}} R_k \right) \right),
\end{equation}

\begin{equation}
= \sum_{k \in J_\alpha} \lambda_k^Y C \left( \frac{b_k}{c_{kk}} \left( R_k - \frac{(1 - g_k)c_{kk}}{2b_k} \right)^2 - \frac{(1 - g_k)^2 c_{kk}}{4b_k} + R_{0k} \right). \tag{2.4.24}
\end{equation}

Since $K_X$ is positive, the remaining inequality (2.4.21) is fulfilled for arbitrary $R_k$’s if

\begin{equation}
\sum_{k \in J_\alpha} \frac{(1 - g_k)^2 c_{kk}}{4b_k} \leq \sum_{k \in J_\alpha} R_{0k}
\end{equation}

and this holds by assumption and so it yields thermomechanical consistency for the mMnC-B model.

**Remark 2.4.6.** In case of the mM1C model, we can specify the conditions (2.4.14) and (2.4.15).

If

\begin{equation}
(2.4.25) \quad g_1 \equiv 1, \quad g_1 < 1 : \quad \sqrt{4 \frac{b_1 R_{01}}{c_{11}}} + g_1 \geq 1 \quad \text{or} \quad g_1 > 1 : \quad \sqrt{4 \frac{b_1 R_{01}}{c_{11}}} + 1 \geq g_1
\end{equation}

holds, the mM1C model is thermomechanically consistent, see [72] for a similar proof of thermomechanical consistency for 2M1C and 2MC2 models (special cases of the mMnC-B model in Section 2.3.4.3).

**Remark 2.4.7.** The proof of thermomechanical consistency for TRIP models is not straightforward and may cause problems depending on the chosen evolution equations for the phase fractions. In [69], the authors present a thermomechanical consistent model for TRIP under special assumptions. In a more general context, the authors of [44] present a thermomechanical consistent model for TRIP for large deformation considering the chemical forces as approximations of the volume densities of the free enthalpy of the phases.

If we consider the phase fractions not as inner variables, but like additional parameters with evolutions, cf. the concept of drag stress (Section 2.3.5), Theorem 2.4.2 is applicable. One disadvantage of this simplification is that the phase transformations will not directly influence the temperature. The influence will only take place by the change of material parameters, thus no latent heat will evolve, cf. [12]. The situation of the phase fractions only being parameters is reasonable for isothermal phase transformations and thin geometries, cf. Chapter 4.
3. Viscoelasticity within the Framework of MM Models

Many materials, such as biological materials like blood and muscles, show viscoelastic material behavior, see [24]. Viscoelasticity is also a feature of synthetic polymers, like epoxy resin, which shows ratcheting effects under cyclic stress experiments without further damage, see [58]. Ratcheting cannot be modeled for linear viscoelasticity by the common rheological models like the Kelvin-Voigt model or the Poynting-Thomson model (also: Standard model). The Maxwell and Burgers model can describe ratcheting, but only by including the effect of instantaneous elasticity. However, there are some models for nonlinear viscoelasticity which take ratcheting into account, see e.g. [7, 18, 78]. Viscoelastic/viscoplastic-material models are investigated in order to predict the creep and recovery behavior of composite materials (cf. [3] e.g.) as well. Bonetti and Bonfanti, [14] and [15], present some mathematical results respectively existence and uniqueness of the corresponding mathematical model for thermoviscoelastic behavior. Results related to the description of linear and nonlinear viscoelasticity by integro-differential equations can be found in e.g. [11].

In this chapter, we specify the modeling of linear viscoelasticity with MM models based on a series connection of rheological Kelvin-Voigt elements (short: KV) (Section 2.3.3). We provide the 3d mathematical problems of a thermoelastic-(N)-coupled-KV model and isothermal-(N)-coupled-KV model. A mixed boundary value problem related to the latter model is analyzed according to its weak solvability – capable to describe stress driven experiments. In three special cases of coupling, we conduct 3d simulations of an isothermal-(2)-coupled-KV model. A detailed modeling and mathematical treatment is provided for the isothermal-(2)-coupled-KV model of a 1d rod. Moreover, we conduct a nondimensionalization and scaling of the mixed boundary value problem and numerical simulation for a creep test and a cyclic tension test with relaxation. The numerical simulations indicate that the introduced model approach for linear viscoelasticity within the framework of MM models is capable to reproduce qualitative effects like ratcheting without predicting instantaneous elasticity.

3.1. Thermoelastic-Viscoelastic MM Model

We present a thermoelastic-viscoelastic MM model as a series connection of \( N \) Kelvin-Voigt elements with interaction and a thermoelastic element, \( \mathcal{N} := \{0, \ldots, N\} \). Contrary to standard approaches for MM models, the thermoelastic mechanism is identified as the 0-th mechanism (cf. Section 2.3.1[Thermoelastic element besides inelastic mechanisms]), where the first to \( N \)-th are the Kelvin-Voigt elements. We base our model approach on the setting of Chapter 2 and especially Section 2.3.3. We recollect the basic equations – valid in a time-space domain \( [0, T] \times \Omega \) – of the balance of momentum (2.2.1a) and the balance of energy (2.2.1b)

\[
\begin{align*}
\rho \partial_t u - \text{Div} \sigma &= f, \\
\rho \partial_t \varepsilon + \text{div} q &= \sigma : \partial_t \varepsilon + r,
\end{align*}
\]

\(^1\)Ratcheting is the uniaxial directed accumulation of inelastic strain under cyclic loading, cf. [42].
3.1. Thermoelastic-Viscoelastic MM Model

the definition of the Green strain tensor (2.2.2), the assumption of the additive split of the strain tensor in a sum of partial strains\(^2\) (2.2.4) and the definition of partial stresses (2.2.11)

\[
\varepsilon = \frac{1}{2}(\nabla u + \nabla^T u), \quad \varepsilon = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \sigma_j = A_j \sigma, \quad j \in \mathcal{N}
\]

and the relations between the internal energy, free energy and entropy

\[
e = \psi + \theta \eta, \quad \eta = -\frac{\partial \psi}{\partial \theta}.
\]

We set \(A_0=1\) and define the free energy as the sum of a thermoelastic part and a viscoelastic part

\[
\psi(\theta, \varepsilon_{\mathcal{N}}) := \psi_{te}(\theta, \varepsilon_{te}) + \psi_{ve}(\theta, \varepsilon_{\mathcal{N}\setminus\{0\}})
\]

\[
:= \frac{1}{2\varrho} \left( \varepsilon_0 : E_{00} : \varepsilon_0 + \varepsilon_0 : \gamma_0 : \varepsilon_0 + \sum_{i,j \in \mathcal{N}\setminus\{0\}} \varepsilon_i : E_{ij} : \varepsilon_j \right) - C_0(\theta).
\]

\(E_{00}\) describes the fourth order elastic tensor of the thermoelastic mechanism, \(\gamma_0\) the second order tensor related to thermal expansion. The elastic tensors \(E_{ii}, i \in \mathcal{N}\setminus\{0\}\), are the fourth order elastic tensors of the viscoelastic mechanisms and \(E_{ij}, i \neq j, i, j \in \mathcal{N}\setminus\{0\}\), the coupling of the viscoelastic mechanisms. As usual, the thermodynamic forces (2.2.10) are

\[
X_j := \varrho \frac{\partial \psi}{\partial \varepsilon_j}, \quad j \in \mathcal{N}, \quad \text{such that}
\]

\[
X_0 = E_{00} : \varepsilon_0 + \gamma_0 : \varepsilon_0 + \sum_{i \in \mathcal{N}\setminus\{0\}} E_{ij} : \varepsilon_i, \quad j \in \mathcal{N}\setminus\{0\},
\]

and with these the evolution functionals read as

\[
F_0(\sigma_0, X_0) := \sigma_0 - X_0,
\]

\[
F_j(\sigma_j, \varepsilon_j, X_j) := \partial_t \varepsilon_j - V_j^{-1} : (\sigma_j - X_j), \quad j \in \mathcal{N}\setminus\{0\}.
\]

The evolution equations

\[
F_0(\sigma_0, X_0) = 0, \quad F_j(\sigma_j, \varepsilon_j, X_j) = 0, \quad j \in \mathcal{N}\setminus\{0\},
\]

complete the set of equations describing the thermoelastic-viscoelastic material behavior. In order to state a full mathematical problem, we have to pose initial and boundary conditions. Furthermore, the equations are reformulated. Noting

\[
\partial_t \psi = \frac{\partial \psi}{\partial \theta} \partial_t \theta + \sum_{j \in \mathcal{N}} \frac{\partial \psi}{\partial \varepsilon_j} : \partial_t \varepsilon_j, \quad \partial_t \eta = -\frac{\partial^2 \psi}{\partial \theta^2} \partial_t \theta - \sum_{j \in \mathcal{N}} \frac{\partial^2 \psi}{\partial \theta \partial \varepsilon_j} : \partial_t \varepsilon_j, \quad c_d = -\frac{\partial^2 \psi}{\partial \theta^2} \partial_t \theta,
\]

we obtain

\[
\varrho \partial_t e = \varrho (\partial_t \psi + \partial_t \theta \eta + \theta \partial_t \eta) = \varrho c_d \partial_t \theta + \sum_{j \in \mathcal{N}} X_j : \partial_t \varepsilon_j - \sum_{j \in \mathcal{N}} \theta \frac{\partial X_j}{\partial \theta} : \partial_t \varepsilon_j.
\]

\(^2\)Note again, \(\varepsilon = \sum_{j \in \mathcal{N}} A_j \varepsilon_j = \varepsilon_0 + \sum_{j \in \mathcal{N}\setminus\{0\}} A_j \varepsilon_j = \varepsilon_{te} + \sum_{j \in \mathcal{N}\setminus\{0\}} A_j \varepsilon_j.\)
With (3.1.8) and (3.1.9) it holds
\[
\sigma : \partial_t \varepsilon - \sum_{j \in \mathcal{N}} X_j : \partial_t \varepsilon_j = \sum_{j \in \mathcal{N} \setminus \{0\}} \partial_t \varepsilon_j : \sigma_j =: V_j : \partial_t \varepsilon_j
\]
and the equation of the energy balance can be reformulated to
\[
\rho c_d \partial_t \theta - \text{div} (\lambda_\theta \nabla \theta) = \sum_{j \in \mathcal{N} \setminus \{0\}} \partial_t \varepsilon_j : V_j : \partial_t \varepsilon_j + \sum_{j \in \mathcal{N}} \theta \frac{\partial X_j}{\partial \theta} : \partial_t \varepsilon_j + r.
\]

Eliminating the partial stresses and inserting the definition of the Green strain tensor, we can state the following mathematical problem assuming an undeformed material at \( t = 0 \).

**Problem 3.1.1.** Let \( S := [0, T], \; T > 0 \) be a time interval, \( \Omega \subset \mathbb{R}^3 \) a bounded open \( \mathcal{C}^{0,1} \)-domain with closed \( \Gamma_u \subset \partial \Omega \), where \( \Gamma_u \) has positive surface measure, and \( \Gamma_\sigma := \partial \Omega \setminus \Gamma_u \). Find functions
\[
\begin{align*}
(3.1.15a) & \quad u(t, x) : S \times \Omega \to \mathbb{R}^3, \\
(3.1.15b) & \quad \theta(t, x) : S \times \Omega \to \mathbb{R}, \\
(3.1.15c) & \quad \sigma(t, x) : S \times \Omega \to \mathbb{R}^{3 \times 3}, \\
(3.1.15d) & \quad \varepsilon_j(t, x) : S \times \Omega \to \mathbb{R}^{3 \times 3}, \; j \in \mathcal{N},
\end{align*}
\]
such that for given tensors \( E_{ij}, \; i, j \in \mathcal{N} \) with\(^3\)
\[
(3.1.16) \quad \sum_{\{p,q\}=0} \sum_{\{i,j,k,l\}=1} E_{pq,ijkl} T_{p,ijkl} T_{q,kl} \geq 0
\]
for arbitrary symmetric tensors \( T_{ij} \) of second order, positive definite \( V_j \) and second order tensor \( \gamma_\theta \), positive functions \( \epsilon_0, \lambda_\theta \), \( \rho \) and arbitrary, predetermined functions \( f, r, \gamma_u, \gamma_\sigma \), constant \( \theta_0 \), positive constants \( A_{ij} \),
\[
\begin{align*}
(3.1.17a) & \quad \rho \partial_t u - \text{Div} \sigma = f, \quad \text{in} \; S \times \Omega, \\
(3.1.17b) & \quad \rho c_d \partial_t \theta - \text{div} (\lambda_\theta \nabla \theta) = \sum_{j \in \mathcal{N} \setminus \{0\}} \partial_t \varepsilon_j : V_j : \partial_t \varepsilon_j + \sum_{j \in \mathcal{N}} \theta \frac{\partial X_j}{\partial \theta} : \partial_t \varepsilon_j + r, \quad \text{in} \; S \times \Omega, \\
(3.1.17c) & \quad \frac{1}{2} \langle \nabla u + \nabla u^T \rangle = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \text{in} \; S \times \Omega, \\
(3.1.17d) & \quad \sigma = X_0, \quad \text{in} \; S \times \Omega, \\
(3.1.17e) & \quad V_j : \partial_t \varepsilon_j = A_j \sigma - X_j, \quad j \in \mathcal{N} \setminus \{0\}, \quad \text{in} \; S \times \Omega, \\
(3.1.17f) & \quad X_0 = E_{00} : \varepsilon_0 + \gamma_0 : I, \quad \text{in} \; S \times \Omega, \\
(3.1.17g) & \quad X_j = \sum_{i \in \mathcal{N} \setminus \{0\}} E_{ij} : \varepsilon_i, j \in \mathcal{N} \setminus \{0\}, \quad \text{in} \; S \times \Omega,
\end{align*}
\]
is fulfilled for initial conditions with given \( u^{[0]}, u^{[1]}, \varepsilon_N^{[0]}, \theta^{[0]}, \theta_0, \theta_\Gamma \),
\[
\begin{align*}
(3.1.18a) & \quad u(0, x) = u^{[0]}, \quad \text{in} \; \Omega, \\
(3.1.18b) & \quad \partial_t u(0, x) = u^{[1]}(x), \quad \text{in} \; \Omega, \\
(3.1.18c) & \quad \theta(0, x) = \theta^{[0]}(x), \quad \text{in} \; \Omega,
\end{align*}
\]
\(^3\)This holds due to convexity of the free energy.
3.2. Isothermal Viscoelasticity (3d)

\(\varepsilon_j(0, x) = \varepsilon_j^0, \ j \in \mathcal{N}, \) in \(\Omega,\)

and boundary conditions (\(\nu\) unit normal vector on \(\partial\Omega\))

\[u(t, x) = \gamma_u(t, x), \quad \text{on } S \times \Gamma_u,\]

\[\sigma(t, x)\nu(x) = \gamma_\sigma(t, x), \quad \text{on } S \times \Gamma_\sigma,\]

\[-\lambda \frac{\partial \theta}{\partial \nu}(t, x) = \delta_\theta(x)(\theta(t, x) - \theta_\Gamma(t, x)), \quad \text{on } S \times \partial \Omega.\]

The mathematical treatment of problem 3.1.1 requires much effort, cf. thermoelasticity, due to the dissipation terms

\[\sum_{j \in \mathcal{N}\setminus\{0\}} \partial_t \varepsilon_j : V_j : \partial_t \varepsilon_j + \sum_{j \in \mathcal{N}} \theta \frac{\partial X_j}{\partial \theta} : \partial_t \varepsilon_j\]

related to the thermoelastic and viscoelastic mechanisms. Possible simplifications for treating the dissipation terms related to thermoelasticity can be found in [12]. Applying additional and similar linearizations for the viscoelastic dissipation could lead to a possible mathematical treatment. In [14], Bonetti and Bonfanti, for example, prove the existence and uniqueness of the solution for a simpler problem of a (thermo-)Kelvin-Voigt model with Dirichlet boundary conditions. We will concentrate in the following sections on the isothermal case neglecting the thermal effects.

3.2. Isothermal Viscoelasticity (3d)

The analytical treatment of Problem 3.1.1 requires much effort because of the coupling between heat transfer and mechanical equations. Therefore, we neglect all thermal effects and will present an isothermal model for viscoelasticity — a series connection of \(N\) coupled Kelvin-Voigt elements ((\(N\))-coupled-KV model).

We recall the model ansatz of Section 2.3.3 and state results concerning the weak solvability of the transformed transient problem — a system of integro-differential equations. The transformed transient problem is an auxiliary problem of the original \((N)\)-coupled-KV model. In this nonstandard approach, we are able to prove existence and uniqueness of a solution for a mixed boundary value problem via Galerkin’s method. Moreover, we present some qualitative numerical simulations for a \((2)\)-coupled-KV model with COMSOL Multiphysics® 4.3a.

3.2.1. Modeling \((N)\)-coupled-KV Model

Due to the neglect of the thermal effects, the modeling is restricted to the mechanical parts. We consider the equation of momentum and the evolution equations of the partial strains, all valid in a time-space domain \([0, T]\times\Omega\). We recall the balance of momentum (2.2.1a)

\[\rho \partial_t u - \text{Div } \sigma = f,\]

the definition of the Green strain tensor (2.2.2), the assumption of the additive split of the strain tensor in a sum of \(N\) partial strains (2.2.4) (\(\mathcal{N} := \{1, \ldots, N\}\)), and the definition of partial stresses (2.2.11)

\[\varepsilon = \frac{1}{2}(\nabla u + \nabla u^T), \quad \varepsilon = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \sigma_j = A_j \sigma, \ j \in \mathcal{N}.\]
The free energy is defined by

\[ \psi(\varepsilon_N) := \frac{1}{2\rho} \left( \sum_{i,j \in N} \varepsilon_i : E_{ij} : \varepsilon_j \right). \]

As usual, the thermodynamic forces (2.2.10) are

\[ X_j := \rho \frac{\partial \psi}{\partial \varepsilon_j} = \sum_{i \in N} E_{ij} : \varepsilon_i, \ j \in \mathcal{N}, \]

and with them the evolution functionals read as

\[ F_j(\sigma_j, \varepsilon_j, X_j) := \partial_t \varepsilon_j - V_j^{-1} : (\sigma_j - X_j), \ j \in \mathcal{N}. \]

The balance equation of energy and the relations between the internal energy, free energy and entropy are not necessary for the mechanical problem, but have to be considered when checking thermomechanical consistency, cf. Chapter 2. Adding suitable boundary condition, we formulate the following mathematical problem:

**Problem 3.2.1.** Let \( S := [0, T], T > 0 \) be a time interval, \( \Omega \subset \mathbb{R}^3 \) a bounded open \( C^{0,1} \)-domain with closed \( \Gamma_u \subset \partial \Omega \), where \( \Gamma_u \) has positive surface measure, and \( \Gamma_\sigma := \partial \Omega \setminus \Gamma_u \). Find functions

\[ u(t, x) : S \times \Omega \to \mathbb{R}^3, \]
\[ \sigma(t, x) : S \times \Omega \to \mathbb{R}^{3 \times 3}, \]
\[ \varepsilon_j(t, x) : S \times \Omega \to \mathbb{R}^{3 \times 3}, \ j \in \mathcal{N}, \]

such that for given tensors \( E_{ij}, \ i, j \in \mathcal{N} \) with

\[ \sum_{\{N,N\}} \sum_{\{p,q\}=1}^{3,3,3} E_{pq,ijkl} T_{p,ij} T_{q,kl} \geq 0 \]

for arbitrary symmetric tensors \( T_N \) of second order, positive definite \( V_N \) and \( \gamma_0 \), positive function \( \rho \) and arbitrary, predetermined functions \( f, \gamma_u, \gamma_\sigma \) and positive constants \( A_N \),

\[ \rho \partial_t u - \text{Div} \sigma = \rho f, \quad \text{in} \ S \times \Omega, \]
\[ \frac{1}{2} (\nabla u + \nabla u^T) = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \text{in} \ S \times \Omega, \]
\[ V_j : \partial_t \varepsilon_j = A_j \sigma - \sum_{i \in \mathcal{N}} E_{ij} : \varepsilon_i, \ j \in \mathcal{N}, \quad \text{in} \ S \times \Omega, \]

is fulfilled for initial conditions with given \( u^{(1)} \),

\[ u(0, x) = 0, \quad \text{in} \ \Omega, \]
\[ \partial_t u(0, x) = u^{[1]}(x), \quad \text{in} \ \Omega, \]
\[ \varepsilon_j(0, x) = 0, \ j \in \mathcal{N}, \quad \text{in} \ \Omega, \]

and boundary conditions (\( \nu \) unit normal vector on \( \partial \Omega \))

\[ u(t, x) = 0, \quad \text{on} \ S \times \Gamma_u, \]

\[ u(t, x) = 0, \quad \text{on} \ S \times \Gamma_u, \]

\[ \frac{1}{2} (\nabla u + \nabla u^T) = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \text{in} \ S \times \Omega, \]

\[ \varepsilon_j(0, x) = 0, \ j \in \mathcal{N}, \quad \text{in} \ \Omega, \]

\[ u(t, x) = 0, \quad \text{on} \ S \times \Gamma_u, \]

\[ \frac{1}{2} (\nabla u + \nabla u^T) = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \text{in} \ S \times \Omega, \]

\[ \varepsilon_j(0, x) = 0, \ j \in \mathcal{N}, \quad \text{in} \ \Omega, \]

\[ u(t, x) = 0, \quad \text{on} \ S \times \Gamma_u, \]

\[ \frac{1}{2} (\nabla u + \nabla u^T) = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \text{in} \ S \times \Omega, \]

\[ \varepsilon_j(0, x) = 0, \ j \in \mathcal{N}, \quad \text{in} \ \Omega, \]

\[ u(t, x) = 0, \quad \text{on} \ S \times \Gamma_u, \]
3.2. Isothermal Viscoelasticity (3d)

\begin{equation}
\sigma(t, x)\nu(x) = \gamma_\sigma(t, x), \quad \text{on } S \times \Gamma_\sigma.
\end{equation}

3.2.2. Weak Solvability of the Transformed Transient Problem

In the following, we only consider homogeneous Dirichlet boundary conditions, \( \gamma_u(t, x) = 0 \).\(^5\) We will search for a solution of Problem 3.2.1 in the Setting 3.2.2. The Problem 3.2.1 is transformed into the integro-differential Problem 3.2.5 and examined with respect to its weak solvability in the adapted Setting 3.2.6.

Setting 3.2.2.

Time & Space:

\begin{align}
(3.2.11a) & \quad S := [0, T] \subset \mathbb{R}, \ T > 0, \quad \text{(time)} \\
(3.2.11b) & \quad \Omega \subset \mathbb{R}^3, \text{bounded, open } C^{0,1}\text{-domain} \quad \text{(space)} \\
(3.2.11c) & \quad \Gamma_u \subset \partial \Omega \text{ closed with positive surface measure, } \Gamma_\sigma := \partial \Omega \setminus \Gamma_u \quad \text{(boundary)}
\end{align}

Parameter (-functions):

\begin{align}
(3.2.12a) & \quad \rho \in L^\infty(\Omega), \ \exists q_0 > 0 \text{ f.a.a. } x \in \Omega : \rho(x) > q_0 \quad \text{(mass density)} \\
(3.2.12b) & \quad \forall p, q \in \mathcal{N}, \ i, j, k, l \in \{1, 2, 3\} : \ E_{pq,ijkl} \in L^\infty(\Omega) \quad \text{(elasticities)} \\
(3.2.12c) & \quad \forall T_N \in \mathbb{R}^{3 \times 3}_{sym} \text{ f.a.a. } x \in \Omega : \sum_{\{p,q\}=0}^{\mathcal{N},\mathcal{N},\mathcal{N}} \sum_{(i,j,k,l)=1}^{3,3,3} E_{pq,ijkl}(x) T_{pi,ijkl} \geq 0 \quad \text{(restrictions)} \\
(3.2.12d) & \quad \text{f.a. } p \in \mathcal{N}, \ i, j, k, l \in \{1, 2, 3\} : \ V_{p,ijkl} \in L^\infty(\Omega) \quad \text{(viscosities)} \\
(3.2.12e) & \quad \exists c > 0 \text{ f.a. } p \in \mathcal{N}, \ i, j, k, l \in \{1, 2, 3\} \text{ f.a.a. } x \in \Omega : (V_p)_{ijkl}(x) > c \\
(3.2.12f) & \quad \text{f.a.a. } x \in \Omega : \mathbf{V}(x) \text{ positive definite}
\end{align}

Function spaces:

\begin{align}
(3.2.13a) & \quad V := W^{1,2}(\Omega), \ V_0 := \{v \in W^{1,2}(\Omega) \mid v(x)|_{\Gamma_0} = 0\}, \ H := L^2(\Omega) \\
(3.2.13b) & \quad \mathcal{V} := L^2(S, V), \ \mathcal{V}_0 := L^2(S, V_0), \ \mathcal{H} := L^2(S, H) \\
(3.2.13c) & \quad \mathcal{H}^{3 \times 3} := \left\{ \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} \mid h_{ij} \in \mathcal{H} \right\} \\
(3.2.13d) & \quad \mathcal{H}_{sym}^{3 \times 3} := \{ \mathbf{h} \in \mathcal{H}^{3 \times 3} | h_{ij} = h_{ji}, \ i, j = 1, \ldots, 3 \}
\end{align}

Data:

\begin{equation}
(3.2.14) \quad f \in \mathcal{H}^3, \ \mathbf{u}^{(0)} \in \mathcal{V}_0^3, \ \mathbf{u}^{(1)} \in \mathcal{H}^3, \ \gamma_\sigma \in [L^2(S, L^2(\Gamma_1))]^3
\end{equation}

Before proving the existence and uniqueness of a weak solution of Problem 3.2.1, we make use of some results of the \( C_0 \)-semigroup theory (cf. [5, 61]) to reformulate the problem, more accurate the evolution equations.

Remark 3.2.3. The evolution equations

\begin{equation}
(3.2.15) \quad \partial_t \mathbf{e}_j = V_j^{-1} : \left( A_j \sigma - \sum_{i \in \mathcal{N}} E_{ij} : \mathbf{e}_i \right), \ \mathbf{e}_j(0, x) = \mathbf{0}, \ j \in \mathcal{N},
\end{equation}

\(^5\)Inhomogeneous Dirichlet boundary conditions can be lead back to homogeneous Dirichlet boundary conditions by homogenization.
can be treated as ordinary differential equations in a Banach space $\mathcal{H}_{sym}^{3\times 3}$ via semigroup theory. We transform the evolution equations of tensorial notation in vector-matrix notation (cf. Section 1.1.3) and define

$$ y_\varepsilon := \begin{pmatrix} \vec{\varepsilon}_1 \\ \vdots \\ \vec{\varepsilon}_N \end{pmatrix}, \quad f_\sigma := \begin{pmatrix} M_{V_1}^{-1}A_1 \vec{\sigma} \\ \vdots \\ M_{V_N}^{-1}A_N \vec{\sigma} \end{pmatrix}, $$

$$ A := -\begin{pmatrix} M_{V_1}^{-1} & 0 \\ \vdots & \ddots \\ 0 & M_{V_N}^{-1} \end{pmatrix} \begin{pmatrix} M_{E_{11}} & \cdots & M_{E_{1N}} \\ \vdots & \ddots & \vdots \\ M_{E_{N1}} & \cdots & M_{E_{NN}} \end{pmatrix}. $$

The transformed system of evolution equations is then given as

$$ \frac{\partial}{\partial t} y_\varepsilon = Ay_\varepsilon + f_\sigma, \quad y_\varepsilon(0) = 0. \quad (3.2.16) $$

$A$ defines a linear, bounded operator on $H^{6N}$ as a matrix in $(L^\infty(\Omega))^{6N \times 6N}$. By [61, example (a) of Definition VII.4.4.], $A$ is the generator of the $C_0$-semigroup $e^{At}$ with

$$ e^{At} := \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}. \quad (3.2.17) $$

If we assume that at least $f_\sigma \in L^1([0,T],X)^{6N}$, by [5, Theorem 3.1.16] the Problem (3.2.16) has a unique mild solution $y_\varepsilon$ given by

$$ y_\varepsilon(t) = \int_0^t e^{A(t-s)} f_\sigma(s) \, ds, \quad t \in [0,T]. \quad (3.2.18) $$

The operator $A$ is a negative semidefinite matrix due to (3.2.12c) and (3.2.12f) for almost all $x \in \Omega$ and, therefore, there exists a eigendecomposition for all $x \in \Omega$ of $A$ with $6N$ non-positive eigenvalues $\lambda_q$. Transforming (3.2.18) back into a tensorial notation and using the eigendecomposition, we obtain the structure

$$ \varepsilon_j(t,x) = \int_0^t \sum_{\{p,q\}=1}^{6N,6N} a_{j,pq}(x)(t-s)^{p-1}e^{\lambda_q(x)(t-s)}D_j(x) : \sigma(s,x) \, ds. \quad (3.2.19) $$

$a_{j,pq}, D_j$ are dependent on the eigendecomposition, $A_j$ and $V_j$. With

$$ \Lambda(t-s,x) := \sum_{j \in \mathbb{N}} A_j \Lambda_j(t-s,x) \quad (3.2.20) $$

and the additive decomposition of the strain tensor (3.2.2), we obtain

$$ \varepsilon(t,x) = \int_0^t \Lambda(t-s,x) : \sigma(s,x) \, ds. \quad (3.2.21) $$

Since $e^{A(x)t}$ is measurable with respect to $x$ by [61, p. 490, Satz A.1.5(c)], $\Lambda(t,\cdot)$ is measurable.
Moreover, all entries of \( \Lambda(t, \cdot) \) belong to \( L^\infty(\Omega) \) for every \( t \in [0, T] \) since

\[
\| e^{A(x)t} \|_\infty \leq e^{\| A \|_\infty t}.
\]

\( \| A \|_\infty \) denotes the operator norm of \( A \). The operator \( \Lambda(\cdot, x) : [0, T] \to L^\infty(\Omega)^{3 \times 3 \times 3 \times 3} \) is continuously differentiable (cf.,[61, p. 367, Satz VII.4.7]) and \( \Lambda \in C^1([0, T], L^\infty(\Omega)^{3 \times 3 \times 3 \times 3}) \). Differentiation with respect to \( t \) in (3.2.21) is admissible in terms of the Bochner integral,

\[
\partial_t \varepsilon(t, x) = \Lambda(0, x) : \sigma(t, x) + \int_0^t \partial_t \Lambda(t-s, x) : \sigma(s, x) \, ds.
\]

In the later proof for an existence of a unique solution, we will need that \( \Lambda(0, x) \) is a positive tensor for almost all \( x \in \Omega \) and bounded below by a constant. We motivate it by the following remark.

**Remark 3.2.4.** If for the initial condition \( \sigma(0, x) \neq 0 \) holds, it yields by the equations (3.2.2), (3.2.8c), (3.2.9c) and (3.2.23) at \( t = 0 \)

\[
\Lambda(0, x) : \sigma(0, x) = \partial_t \varepsilon(0, x) = \sum_{j \in \mathbb{N}} \partial_t \varepsilon_j(0, x) = \left( \sum_{j \in \mathbb{N}} V_j^{-1}(x) \right) : \sigma(0, x)
\]

and thus for arbitrary initial condition of \( \sigma \)

\[
\Lambda(0, x) = \sum_{j \in \mathbb{N}} V_j^{-1}(x).
\]

Due to 3.2.12f there exists a \( c > 0 \) such that

\[
f.a. \ i, j, k, l \in \{1, 2, 3\} f.a.a. \ x \in \Omega : \Lambda_{ijkl}(0, x) > c.
\]

Problem 3.2.1 will be rewritten in a displacement-stress based on the Remarks 3.2.3 and 3.2.4 form before we conduct its weak analysis.

**Problem 3.2.5.** Let \( S := [0, T], T > 0 \) be a time interval, \( \Omega \subset \mathbb{R}^3 \) a bounded open \( C^{0,1} \)-domain with closed \( \Gamma_u \subset \partial\Omega \), where \( \Gamma_u \) has positive surface measure, and \( \Gamma_\sigma := \partial\Omega \setminus \Gamma_u \). Find functions

\[
u(t, x) : S \times \Omega \to \mathbb{R}^3,\\
\sigma(t, x) : S \times \Omega \to \mathbb{R}^{3 \times 3},
\]

such that for given \( \varrho \) and tensor function \( \Lambda : [0, T] \times \Omega \to \mathbb{R}^{3 \times 3 \times 3 \times 3} \), continuous and differentiable for all \( t \in [0, T] \) with \( \Lambda(0, x) \) positive definite

\[
\varrho \partial_t u - \text{Div} \sigma = f, \quad \text{in } S \times \Omega,
\]

\[
\frac{1}{2} (\nabla u + \nabla u^T) = \int_0^t \Lambda(t-s, x) : \sigma(s, x) \, ds, \quad \text{in } S \times \Omega,
\]

is fulfilled for initial conditions with given \( u^{(1)} \),

\[
u(0, x) = 0, \quad \text{in } \Omega,
\]

\[
\partial_t u(0, x) = u^{(1)}(x), \quad \text{in } \Omega,
\]


and boundary conditions ($\nu$ unit normal vector on $\partial\Omega$)

\begin{align*}
(3.2.30a) \quad & u(t, x) = 0, \quad \text{on } S \times \Gamma_u, \\
(3.2.30b) \quad & \sigma(t, x)\nu(x) = \gamma_\sigma(t, x), \quad \text{on } S \times \Gamma_\sigma.
\end{align*}

We modify Setting 3.2.2.

**Setting 3.2.6.**

*Time & Space:*

\begin{align*}
(3.2.31a) \quad & S := [0, T] \subset \mathbb{R}, \; T > 0 \quad \text{(time)} \\
(3.2.31b) \quad & \Omega \subset \mathbb{R}^3, \text{bounded, open } C^{0,1}\text{-domain} \quad \text{(space)} \\
(3.2.31c) \quad & \Gamma_u \subset \partial\Omega \text{ closed with positive surface measure}, \; \Gamma_\sigma := \partial\Omega \setminus \Gamma_u \quad \text{(boundary)}
\end{align*}

*Parameter (-functions):*

\begin{align*}
(3.2.32a) \quad & \varrho \in L^\infty(\Omega), \; \exists \varrho_0 > 0 \text{ f.a.a. } x \in \Omega : \varrho(x) > \varrho_0 \quad \text{(mass density)} \\
(3.2.32b) \quad & \Lambda \in C^1([0, T], L^\infty(\Omega)^{3\times3}) \\
(3.2.32c) \quad & \text{f.a.a. } x \in \Omega : \Lambda(0, x) \text{ positive definite} \\
(3.2.32d) \quad & \exists c > 0 \text{ f.a.a. } i, j, k, l \in \{1, 2, 3\} \text{ f.a.a. } x \in \Omega : \Lambda_{ijkl}(0, x) > c
\end{align*}

*Function spaces:*

\begin{align*}
(3.2.33a) \quad & V := W^{1,2}(\Omega), \; V_0 := \{v \in W^{1,2}(\Omega) | v(x)|_{\Gamma_0} = 0\}, \; H := L^2(\Omega) \\
(3.2.33b) \quad & V := L^2(S, V), \; V_0 := L^2(S, V_0), \; \mathcal{H} := L^2(S, H) \\
(3.2.33c) \quad & \mathcal{H}^{3\times3} := \left\{ \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{pmatrix} | h_{ij} \in \mathcal{H} \right\} \\
(3.2.33d) \quad & \mathcal{H}_{sym}^{3\times3} := \{ h \in \mathcal{H}^{3\times3} | h_{ij} = h_{ji}, \; i, j = 1, \ldots, 3\}
\end{align*}

*Data:*

\begin{equation}
(3.2.34) \quad f \in \mathcal{H}, \quad u^{(0)} \in V_0^3, \quad u^{(1)} \in H^3, \quad \gamma_\sigma \in [L^2(S, L^2(\Gamma_1))]^3
\end{equation}

**Definition 3.2.7.** Consider Setting 3.2.6. $(u, \sigma) \in V_0^3 \times \mathcal{H}_{sym}^{3\times3}$ with $\partial_t u \in \mathcal{H}$ is called a weak solution of Problem 3.2.5 if and only if for all $\varphi \in V_0^3$ with $\partial_t \varphi \in \mathcal{H}^3$, $\varphi(T) = 0$ and $\psi \in \mathcal{H}_{sym}^{3\times3}$

\begin{align*}
(3.2.35a) \quad & -\int_0^T \left( \varrho(x) \partial_t u(t, x) | \partial_t \varphi(t, x) \right)_{H^3} dt + \int_0^T \left( \sigma(t, x) | \nabla \varphi(t, x) \right)_{H^{3\times3}} dt \\
& = \int_0^T \left( f(t, x) | \varphi(t, x) \right)_{H^3} dt + \left( \varrho(x) u^{(1)}(x) | \varphi(0, x) \right)_{H^3} + \int_{\Gamma_\sigma} \gamma_\sigma(t, x) : \varphi(t, x) \, d\sigma_x dt,
\end{align*}

\begin{equation}
(3.2.35b) \quad \int_0^T \left( \nabla u(t, x) | \psi(t, x) \right)_{H^{3\times3}} dt = \int_0^T \left( \int_0^t \Lambda(t-s, x) : \sigma(s, x) \, ds | \psi(t, x) \right)_{H^{3\times3}} dt
\end{equation}

and $u(0, x) = u^{(0)}$ for almost all $x \in \Omega$. 

47
3.2. Isothermal Viscoelasticity (3d)

**Theorem 3.2.8.** Under the assumptions of Setting 3.2.6, there exists a unique weak solution \((u, \sigma) \in [L^\infty(S, V_0)]^3 \times H^3_{sym}\) with \(\partial_t u \in \mathcal{H}^3\) of the Problem 3.2.5. Moreover, the map

\[
S : (u^{(0)}, u^{(1)}, \gamma, f) \mapsto (u, \partial_t u, \sigma)
\]

with

\[
S : V^3_0 \times H^3 \times [L^2(S, L^2(\Gamma^*))]^3 \times \mathcal{H}^3 \rightarrow [L^\infty(S, V_0)]^3 \times V^3 \times \mathcal{H}^3_{sym}
\]

is linear and continuous.

**Proof.** The proof is based on Galerkin's method.

**Step 1 – Galerkin approximation**

Let \((w_q)_{q \in \mathbb{N}}\) be a Galerkin basis of \(V^3_0\) and \((v_q)_{q \in \mathbb{N}}\) a Galerkin basis of \(H^3_{sym}\) as well as \(V^3_{0,n} := \text{span}\{w_1, \ldots, w_n\}\) and \(H^3_{sym,n} := \text{span}\{v_1, \ldots, v_n\}\) the related Galerkin schemes. Whereas the basis of \(H^3_{sym}\) is chosen such that \(\nabla w_q + (\nabla w_q)^T \in H^3_{sym} \) for all \(q = 1, \ldots, n\). Moreover, let \((u^{[n],(0)}), (u^{[n],(1)})\) be sequences in \(V^3_0\) with

\[
\begin{align*}
(u^{[n],(0)}) & \in V^3_{0,n}, \quad (u^{[n],(1)}) \in H^3_{0,n}, \quad \forall n \in \mathbb{N}, \\
(u^{[n],(0)}) & \rightarrow u^{(0)} \text{ in } V^3_0, \\
(u^{[n],(1)}) & \rightarrow u^{(1)} \text{ in } H^3.
\end{align*}
\]

Every element \(u^{[n]}\) of \([V^3_{0,n}] = [L^2(S, V^3_{0,n})]^3\) and \(\sigma^{[n]}\) of \(H^3_{sym,n} = [L^2(S, H^3_{sym,n})]^3\) can be represented with respect to the Galerkin basis of \(V^3_{0,n}\) and \(H^3_{sym,n}\) respectively for all \(t \in S\) as

\[
\begin{align*}
(u^{[n],(0)}) & = \sum_{j=1}^{n} \phi^{[n],(0)}_j w_j, \\
(u^{[n],(1)}) & = \sum_{j=1}^{n} \phi^{[n],(1)}_j w_j,
\end{align*}
\]

where \(\phi^{[n]}_j : S \rightarrow \mathbb{R}\) and \(h^{[n]}_j : S \rightarrow \mathbb{R}\) are coefficient functions which have to be determined. Due to \((u^{[n],(0)}), (u^{[n],(1)}) \in [V^3_{0,n}]^3\), the initial conditions are represented by

\[
\begin{align*}
(u^{[n],(0)}) & = \sum_{j=1}^{n} \phi^{[n],(0)}_j w_j, \quad \text{and} \quad (u^{[n],(1)}) = \sum_{j=1}^{n} \phi^{[n],(1)}_j w_j.
\end{align*}
\]

We define the vectors

\[
\begin{align*}
g^{[n]} := (g^{[n]}_1, \ldots, g^{[n]}_n)^T \quad \text{and} \\
h^{[n]} := (h^{[n]}_1, \ldots, h^{[n]}_n)^T.
\end{align*}
\]

Now, the problem is to find a solution \((g^{[n]}, h^{[n]})\) to the initial conditions

\[
\begin{align*}
g^{[n]}(0) & = (\alpha_1, \ldots, \alpha_n)^T =: g^{[n]}_0, \\
(\partial_t g^{[n]}(0)) & = (\beta_1, \ldots, \beta_n)^T =: g^{[n]}_1,
\end{align*}
\]

48
of the Galerkin equation system related to Problem 3.2.5

\[(3.2.43a) \quad \int_{\Omega} \varrho \partial_t u^{[n]} : \varphi \, dx + \int_{\Omega} \sigma^{[n]} : \nabla \varphi \, dx = \int_{\Omega} f : \varphi \, dx + \int_{\Gamma_d} \gamma : \varphi \, ds,\]

\[(3.2.43b) \quad \frac{1}{2} \int_{\Omega} (\nabla u^{[n]} + (\nabla u^{[n]})^T) : \psi \, dx = \int_{\Omega} \left( \int_0^t \Lambda(t - s) : \sigma^{[n]}(s) \, ds \right) : \psi \, dx,\]

for \((\varphi, \psi) \in V_0 \times H^{3 \times 3}_{sym,n}.\) Testing with \(\varphi = w_q\) and \(\psi = v_q\) leads to

\[(3.2.44a) \quad \int_{\Omega} \varrho \partial_t u^{[n]} : w_q \, dx + \int_{\Omega} \sigma^{[n]} : \nabla w_q \, dx = \int_{\Omega} f : w_q \, dx + \int_{\Gamma_d} \gamma : w_q \, ds,\]

\[(3.2.44b) \quad \int_{\Omega} \nabla u^{[n]} : v_q \, dx = \int_{\Omega} \left( \int_0^t \Lambda(t - s) : \sigma^{[n]}(s) \, ds \right) : v_q \, dx,\]

for \(q = 1, \ldots, n.\) Differentiation in (3.2.44b) leads to

\[(3.2.44c) \quad \int_{\Omega} \partial_t \nabla u^{[n]} : v_q \, dx = \int_{\Omega} \Lambda(0) : \sigma^{[n]} : v_q \, dx + \int_{\Omega} \left( \int_0^t \partial_t \Lambda(t - s) : \sigma^{[n]}(s) \, ds \right) : v_q \, dx,\]

for \(q = 1, \ldots, n.\) We sort the equations, such that with (3.2.39a) and (3.2.39b)

\[(3.2.45a) \quad \sum_{p=1}^n \left( \int_{\Omega} \varrho w_p : w_q \, dx \right) \partial_t g^{[n]}_p + \sum_{p=1}^n \left( \int_{\Omega} \nabla w_p : \nabla w_q \, dx \right) h^{[n]}_p = \int_{\Omega} f : w_q \, dx + \int_{\Gamma_d} \gamma : w_q \, ds,\]

\[(3.2.45b) \quad \sum_{p=1}^n \left( \int_{\Omega} \nabla w_p : v_q \, dx \right) g^{[n]}_p = \sum_{p=1}^n \int_{\Omega} \left( \int_0^t \Lambda(t - s) : v_p : v_q \, dx \right) h^{[n]}_p(s) \, ds,\]

\[(3.2.45c) \quad \sum_{p=1}^n \left( \int_{\Omega} \nabla w_p : v_q \, dx \right) \partial_t g^{[n]}_p = \sum_{p=1}^n \left( \int_{\Omega} \Lambda(0) : v_p : v_q \, dx \right) h^{[n]}_p + \sum_{p=1}^n \int_{\Omega} \left( \int_0^t \partial_t \Lambda_p(t - s) : v_p : v_q \, dx \right) h^{[n]}_p(s) \, ds.\]

In order to find a solution of the Galerkin equation system (3.2.44), we consider (3.2.45a) and (3.2.45c) for \(q = 1, \ldots, n.\) We define the following matrices componentwise, for \(p, q = 1, \ldots, n,\)

\[(3.2.46a) \quad (G^V)_pq := \int_{\Omega} \varrho w_p : w_q \, dx,\]

\[(3.2.46b) \quad (M^1)_pq := \int_{\Omega} v_p : \nabla w_q \, dx,\]
3.2. Isothermal Viscoelasticity (3d)

(3.2.46c) \( (M_2)_{pq} := \int_\Omega \nabla w_p : v_q \, dx, \)

(3.2.46d) \( (G^H)_{pq} := \int_\Omega \Lambda(0) : v_p : v_q \, dx, \)

(3.2.46e) \( (M_\Lambda)_{pq}(t-s) := \int_\Omega \partial_t \Lambda_p(t-s) : v_q \, dx. \)

Note that \( G^V \) and \( G^H \) are Gramian matrices related to a weighted scalar product, and therefore invertible. Moreover, we define

(3.2.47) \( (F)_q := \int_\Omega f : w_q \, dx + \int_{\Gamma_\sigma} \gamma_\sigma : w_q \, ds. \)

We are able to rewrite (3.2.45a) and (3.2.45c), for \( q = 1, \ldots, n \), in a compact form:

(3.2.48a) \( G^V \partial_t g^{[n]} + M_1 h^{[n]} = F, \)

(3.2.48b) \( M_2 \partial_t g^{[n]} = G^H \Lambda h^{[n]} + \int_0^t M_\Lambda(t-s)h^{[n]}(s) \, ds. \)

By introducing \( k^{[n]} := \partial_t g^{[n]} \), we transform the system into

(3.2.49a) \( \partial_t g^{[n]} = k^{[n]}, \)

(3.2.49b) \( \partial_t k^{[n]} = -(G^V)^{-1} M_1 h^{[n]} + (G^V)^{-1} F, \)

(3.2.49c) \( h^{[n]} = (G^H)^{-1} M_2 k^{[n]} - (G^H)^{-1} \int_0^t M_\Lambda(t-s)h^{[n]}(s) \, ds. \)

This is equivalent to

(3.2.50a) \( g^{[n]} = g_0^{[n]} + \int_0^t k^{[n]}(s) \, ds, \)

(3.2.50b) \( k^{[n]} = g_1^{[n]} - (G^V)^{-1} M_1 \int_0^t h^{[n]}(s) \, ds + (G^V)^{-1} \int_0^t F(s) \, ds, \)

(3.2.50c) \( h^{[n]} = (G^H)^{-1} M_2 k^{[n]} - (G^H)^{-1} \int_0^t M_\Lambda(t-s)h^{[n]}(s) \, ds \)

and therefore to the system

(3.2.51) \[
\begin{pmatrix}
g^{[n]} \\
k^{[n]} \\
h^{[n]}
\end{pmatrix} = \begin{pmatrix}
g_0^{[n]} \\
g_1^{[n]} \\
0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & (G^H)^{-1} M_2 & 0
\end{pmatrix} \begin{pmatrix}
g^{[n]} \\
k^{[n]} \\
h^{[n]}
\end{pmatrix} + \int_0^t \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & (G^H)^{-1} M_2 & 0
\end{pmatrix} \begin{pmatrix}
g^{[n]} \\
k^{[n]} \\
h^{[n]}
\end{pmatrix} \]
3. Viscoelasticity within the Framework of MM Models

\[ + \int_0^t \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -(G^V_\phi)^{-1}M_1 \end{pmatrix} \begin{pmatrix} g^{[n]}(s) \\ k^{[n]}(s) \end{pmatrix} ds + \int_0^t \begin{pmatrix} 0 \\ -(G^H_\Lambda)^{-1}M_1(t-s) \end{pmatrix} \begin{pmatrix} h^{[n]}(s) \end{pmatrix} ds \]

and further

\[
(3.2.52) \quad \begin{pmatrix} g^{[n]} \\ k^{[n]} \\ h^{[n]} \end{pmatrix} = \begin{pmatrix} g^{[n]}_0 \\ g^{[n]}_1 \\ (G^H_\Lambda)^{-1}M_2g^{[n]}_1 \end{pmatrix} + \int_0^t \begin{pmatrix} 0 \\ (G^V_\phi)^{-1}F(s) \\ (G^H_\Lambda)^{-1}M_2(G^V_\phi)^{-1}F(s) \end{pmatrix} ds
\]

\[
+ \int_0^t \begin{pmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -(G^V_\phi)^{-1}M_1 \\ -(G^H_\Lambda)^{-1}M_1(t-s) \end{pmatrix} \begin{pmatrix} g^{[n]}(s) \\ k^{[n]}(s) \\ h^{[n]}(s) \end{pmatrix} ds.
\]

The system (3.2.52) is a system of Volterra-integral equations of second kind,

\[
(3.2.53) \quad \zeta = \zeta_0 + \int_0^t M(t-s)\zeta(s) ds + \int_0^t \tilde{F}(s) ds \quad \equiv T(t,\zeta)
\]

with the notations

\[
(3.2.54a) \quad \zeta := \begin{pmatrix} g^{[n]} \\ k^{[n]} \\ h^{[n]} \end{pmatrix},
\]

\[
(3.2.54b) \quad \zeta_0 := \begin{pmatrix} g^{[n]}_0 \\ g^{[n]}_1 \\ (G^H_\Lambda)^{-1}M_2g^{[n]}_1 \end{pmatrix},
\]

\[
(3.2.54c) \quad \tilde{F}(s) := \begin{pmatrix} 0 \\ (G^V_\phi)^{-1}F(s) \\ (G^H_\Lambda)^{-1}M_2(G^V_\phi)^{-1}F(s) \end{pmatrix},
\]

\[
(3.2.54d) \quad M := \begin{pmatrix} 0 & I \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -(G^V_\phi)^{-1}M_1 \\ -(G^H_\Lambda)^{-1}M_1(t-s) \end{pmatrix}.
\]

It holds using the Bielicki-norm\(^6\) for \(\zeta, \eta \in C(S, \mathbb{R})\)

\[
(3.2.55) \quad |T(t,\zeta) - T(t,\eta)| \leq \max_{t \in S} |M(t)||S| \int_0^t e^{\lambda s} e^{-\lambda s} |\zeta(s) - \eta(s)| ds
\]

\[\leq \max_{t \in S} |M(t)||S| \int_0^t e^{\lambda s} e^{-\lambda s} |\zeta(s) - \eta(s)| ds \quad \equiv L\]

\(^6\)We define \(\|u\|_\Lambda := \max_{t \in [a,b]} \{e^{-\lambda(t-s)}|u(t)|\} \) with \(\lambda > 0\).
3.2. Isothermal Viscoelasticity (3d)

\[
\leq L \int_0^t e^{\lambda s} ds \cdot ||\zeta - \eta||_\lambda \\
\leq \frac{L}{\lambda} e^{\lambda t} ||\zeta - \eta||_\lambda.
\]

We choose \( \lambda \) such that \( c_\lambda := \frac{L}{\lambda} < 1 \) and

\[
(T(t, \zeta) - T(t, \eta))|_\lambda \leq c_\lambda ||\zeta - \eta||_\lambda.
\]

(3.2.56)

The Banach fixed-point theorem (e.g. [81, p. 17]) yields that (3.2.52) has a unique global solution \( \zeta \in C(S, \mathbb{R}^{3n}) \) and, therefore, (3.2.44) has a unique global solution \((u^{[n]}, \sigma^{[n]}) \in C^1(S, V^3_n) \times C(S, H^{3 \times 3}_{sym,n})\).

**Step 2 - A priori estimate**

In order to obtain suitable a priori estimates, we multiply (3.2.44a) by \( \partial_t g^{[n]} \), (3.2.44c) by \( h^{[n]} \), sum up each equation by \( q = 1, \ldots, n \) and integrate over \([0, \tau]\), \( \tau \leq T \), such that

\[
\int_0^\tau \left( \partial_t g^{[n]} | \partial_t u^{[n]} \right)_{H^3} dt + \int_0^\tau \left( \sigma^{[n]} | \nabla (\partial_t u^{[n]}) \right)_{H^{3 \times 3}} dt \\
= \int_0^\tau \left( f | \partial_t u^{[n]} \right)_{H^3} dt + \int_0^\tau \gamma : \partial_t u^{[n]} d\sigma \, dt,
\]

(3.2.57a)

\[
\int_0^\tau \left( \nabla (\partial_t u^{[n]}) | \sigma^{[n]} \right)_{H^{3 \times 3}} dt = \int_0^\tau \left( \Lambda(0) : \sigma^{[n]} | \sigma^{[n]} \right)_{H^{3 \times 3}} dt \\
+ \int_0^\tau \left( \int_0^t (\partial_t \Lambda(t-s) : \sigma^{[n]}(s) ds | \sigma^{[n]} \right)_{H^{3 \times 3}} dt.
\]

(3.2.57b)

Inserting (3.2.57b) into (3.2.57a) (note that the scalar product is symmetric and \( \Lambda(0) \) positive definite), we obtain for \( c_{\Lambda(0)} > 0 \)

\[
\|\partial_t u^{[n]}\|_{H^3}^2 + c_{\Lambda(0)} \int_0^\tau \|\sigma^{[n]}\|_{H^{3 \times 3}}^2 dt + \int_0^\tau \left( \int_0^t (\partial_t \Lambda(t-s) : \sigma^{[n]}(s) ds | \sigma^{[n]} \right)_{H^{3 \times 3}} dt \\
\leq \|u^{(1),[n]}\|_{H^3}^2 + \int_0^\tau \left( f | \partial_t u^{[n]} \right)_{H^3} dt + \int_0^\tau \gamma : \partial_t u^{[n]} d\sigma \, dt.
\]

(3.2.58)
Due to Young’s inequality and the trace theorem, it follows
\[
\left| \int_0^\tau \int_{\Gamma_t} \gamma_{\sigma} : \partial_t u^{[n]} d\sigma_t dt \right| \leq \frac{1}{4\delta} \int_0^\tau \int_{\Gamma_t} |\gamma_{\sigma}|^2 d\sigma_t dt + \delta \int_0^\tau \int_{\Gamma_t} |\partial_t u^{[n]}|^2 d\sigma_t dt
\]
(3.2.59)
\[
\leq \frac{1}{4\delta} \| \gamma_{\sigma} \|^2_{L^2(\Gamma_t)} dt + \delta \int_0^\tau \| \partial_t u^{[n]} \|^2_{L^2(\Gamma_t)} dt
\]
\[
\leq \frac{1}{4\delta} \| \gamma_{\sigma} \|^2_{L^2(S,L^2(\Gamma_t))} + \delta \int_0^\tau \| \partial_t u^{[n]} \|^2_{V} dt.
\]

Further, testing (3.2.43b) by \( \nabla w_q + (\nabla w_q)^T (\in H^{3,3}_{sym,n}) \), multiplying by \( \partial_t g^{[n]}_q \), summing over \( q = 1, \ldots, n \) and integrating over \( [0, \tau], \tau \leq T \), we obtain

\[
\int_0^\tau \left( \nabla \partial_t u^{[n]} + \nabla \partial_t u^{[n]} T \right) \left( \nabla \partial_t u^{[n]} + \nabla \partial_t u^{[n]} T \right)_H^{3,3} dt
\]
(3.2.60)
\[
= \int_0^\tau \left( 2\Lambda(0) : \sigma^{[n]} \left| \nabla \partial_t u^{[n]} + \nabla \partial_t u^{[n]} T \right| \right)_H^{3,3} dt
\]
\[
+ \int_0^\tau \left( 2 \int_0^t \partial_t \Lambda(t-s) : \sigma^{[n]}(s) ds \left| \nabla \partial_t u^{[n]} + \nabla \partial_t u^{[n]} T \right| \right)_H^{3,3} dt.
\]

By the inequalities of Korn and Young, it follows for \( C > 0 \) and \( \varepsilon > 0 \) the estimate

\[
\int_0^\tau \| \nabla \partial_t u^{[n]} \|^2_{H^{3,3}} dt \leq \frac{C}{\varepsilon} \left( \| \sigma^{[n]} \|^2_{H^{3,3}} + \int_0^\tau \| \sigma^{[n]} \|^2_{H^{3,3}} dt \right).
\]
(3.2.61)

By suitable choice of \( \varepsilon \) and \( \delta \), there exist positive constants \( c_1, c_2, C_3, C_4, C_5, C_6 \) and \( C_7 \) such that

\[
c_1 \| \partial_t u^{[n]} \|^2_{H^3} + c_2 \| \sigma^{[n]} \|^2_{H^{3,3}} \leq C_3 \| f \|^2_{H^3} + C_4 \int_s^\tau \| \partial_t u^{[n]} \|^2_{H^3} dt + C_5 \int_s^\tau \| \sigma^{[n]} \|^2_{H^{3,3}} dt
\]
(3.2.62)
\[
+ C_6 \| u^{(1),[n]} \|^2_{L^2(\Gamma)} + C_7 \| \gamma_{\sigma} \|^2_{L^2(S,L^2(\Gamma_t))}.
\]

Gronwall’s lemma ensures the desired estimate for a \( C > 0 \) independent of \( \partial_t u^{[n]} \) and \( \sigma^{[n]} \)

\[
\| \partial_t u^{[n]} \|^2_{H^3} + \| \sigma^{[n]} \|^2_{H^{3,3}} \leq C \left( \| f \|^2_{H^3} + \| u^{(1),[n]} \|^2_{H^3} + \| \gamma_{\sigma} \|^2_{L^2(S,L^2(\Gamma_t))} \right).
\]
(3.2.63)

In order to obtain an estimate for \( u^{[n]} \), we test (3.2.43b) by \( \nabla w_q + (\nabla w_q)^T \), multiply by \( g^{[n]}_q \) and sum up over \( q = 1, \ldots, n \). Then, by Korn’s inequality and Young’s inequality, it holds for \( C > 0 \) and \( \varepsilon > 0 \)

\[
\| u^{[n]} \|^2_{V^3} \leq \frac{C}{\varepsilon} \int_0^\tau \| \sigma^{[n]} \|^2_{H^{3,3}} ds.
\]
(3.2.64)
3.2. Isothermal Viscoelasticity (3d)

Due to the above estimate (3.2.62) for $\sigma^{[n]}$, it exists a $C > 0$ with

\[ (3.2.65) \quad \|u^{[n]}\|_{H^3}^2 \leq C \left( \|f\|_{H^3}^2 + \|u^{(1),[n]}\|_{H^3}^2 + \|\gamma_\sigma\|_{L^2(S,L^2(\Gamma_\sigma))}^2 \right). \]

These estimates hold for all $\tau \in [0, T]$ and it follows

\[ (3.2.66) \quad \|u^{[n]}\|_{L^\infty(S,V^3)}^2 \leq C \left( \|f\|_{H^3}^2 + \|u^{(1),[n]}\|_{H^3}^2 + \|\gamma_\sigma\|_{L^2(S,L^2(\Gamma_\sigma))}^2 \right) < \infty, \]
\[ (3.2.67) \quad \|\partial_t u^{[n]}\|_{L^\infty(S,H^3)}^2 \leq C \left( \|f\|_{H^3}^2 + \|u^{(1),[n]}\|_{H^3}^2 + \|\gamma_\sigma\|_{L^2(S,L^2(\Gamma_\sigma))}^2 \right) < \infty, \]
\[ (3.2.68) \quad \|\sigma^{[n]}\|_{\widetilde{H}^{3 \times 3}}^2 \leq C \left( \|f\|_{H^3}^2 + \|u^{(1),[n]}\|_{H^3}^2 + \|\gamma_\sigma\|_{L^2(S,L^2(\Gamma_\sigma))}^2 \right) < \infty \]

and, therefore,

\[ (3.2.69) \quad u^{[n]} \in L^\infty(S,V^3), \quad \partial_t u^{[n]} \in L^\infty(S,H^3), \quad \text{and} \quad \sigma^{[n]} \in \mathcal{H}^{3 \times 3}. \]

**Step 3 – Limiting process**

The a priori estimates ensure the existence of the subsequences $u^{[n_k]}$ of $u^{[n]}$ and $\sigma^{[n_k]}$ of $\sigma^{[n]}$ with

\[ (3.2.70) \quad u^{[n_k]} \xrightarrow{*} u, \quad \text{in} \quad L^\infty(S,V^3), \]
\[ (3.2.71) \quad \sigma^{[n_k]} \xrightarrow{*} \sigma, \quad \text{in} \quad L^2(S,H^{3 \times 3}), \]
\[ (3.2.72) \quad \partial_t u^{[n_k]} \xrightarrow{*} \partial_t u, \quad \text{in} \quad L^\infty(S,H^3), \]

for $n_k \to \infty$, for (3.2.72) cf. e.g. [80, Proposition (23.19)].

Let $\varphi \in C^1(S)$ with $\varphi(T) = 0$ and $\psi \in C^1(S)$. We multiply (3.2.44a) by $\varphi$, integrate over $S$ and integrate by parts in the first term. Moreover, we multiply (3.2.44b) with $\psi$ and integrate over $S$, then it yields

\[ (3.2.73a) \quad - \int_S \left( \varrho \partial_t u^{[n_k]} | \partial_t \varphi w_p \right)_{H^3} dt + \int_S \left( \sigma^{[n_k]} | \nabla \varphi w_p \right)_{H^{3 \times 3}} dt \]
\[ = \int_S \left( f | \varphi w_p \right)_{H^3} dt + \int_S \left( \varrho u^{[n_k],(1)}(\varphi(0)) w_p \right)_{H^3} + \int_{\Gamma_\sigma} \gamma_\sigma : \varphi w_p d\sigma_x dt, \]

\[ (3.2.73b) \quad \int_S \left( \nabla u^{[n_k]} | \psi w_p \right)_{H^{3 \times 3}} dt = \int_S \left( \int_0^t \Lambda(t-s) : \sigma^{[n_k]}(s) ds | \psi w_p \right)_{H^{3 \times 3}} dt. \]

The limiting process is correct for all terms and, therefore,

\[ (3.2.74a) \quad - \int_S \left( \varrho u | \psi w_p \right)_{H^3} dt + \int_S \left( \sigma | \nabla \varphi w_p \right)_{H^{3 \times 3}} dt \]
\[ = \int_S \left( f | \varphi w_p \right)_{H^3} dt + \left( \varrho u^{(1)}(\varphi(0)) w_p \right)_{H^3} + \int_{\Gamma_\sigma} \gamma_\sigma : \varphi d\sigma_x dt, \]
3. Viscoelasticity within the Framework of MM Models

\[
\int_S \left( \nabla u_3 \psi v^3 \right)_{H^{3	imes3}} dt = \int_S \left( \int_0^t \Lambda(t-s) : \sigma(s) ds \psi v^3 \right)_{H^{3	imes3}} dt.
\]

The linear hull of the functions \( \varphi w_p(j \in \mathbb{N}) \) is dense in \( V^3 \) as well as the linear hull of the functions \( \psi v^3_p(j \in \mathbb{N}) \) is dense in \( H^{3	imes3} \). It remains to prove

\[
\mathbf{u}(0, x) = \mathbf{u}^{(0)}(x).
\]

It holds for all \( p \in \{1, \ldots, n\} \)

\[
\int_S \left( \partial_t u_j^{[n_k]} | \varphi w_p \right)_{H^3} dt = - \int_S \left( u_j^{[n_k]} | \partial_t \varphi w_p \right)_{H^3} dt - \left( u_j^{[n_k],(0)}(x) | \varphi(0) w_p \right)_{H^3}
\]

and all \( p \in \mathbb{N} \)

\[
\int_S \left( \partial_t u_j | \varphi w_p \right)_{H^3} dt = - \int_S \left( u_j | \partial_t \varphi w_p \right)_{H^3} dt - \left( u_j^{(0)}(x) | \varphi(0) w_p \right)_{H^3}.
\]

Combined, it yields

\[
\left( u_j^{[n_k],(0)}(x) | \varphi(0) w_p \right)_{H^3} \rightarrow \left( u_j^{(0)}(x) | \varphi(0) w_p \right)_{H^3}
\]

for \( n_k \to \infty \), for all \( p \in \mathbb{N} \), and thus

\[
\mathbf{u}(0, x) = \mathbf{u}^{(0)}(x).
\]

**Step 4 – Uniqueness**

Let \((\mathbf{u}_i, \sigma_i), i = 1, 2, \) be solutions of Problem 3.2.5 and \( \varphi \in H^3, \psi \in H^{3	imes3}_{\text{sym}} \). We define

\[
\mathbf{\hat{u}} := \mathbf{u}_1 - \mathbf{u}_2, \quad \mathbf{\hat{\sigma}} := \sigma_1 - \sigma_2,
\]

then (3.2.35a) yields

\[
- \int_S \left( \varphi \partial_t \mathbf{\hat{u}} | \partial_t \varphi \right)_{H^3} dt + \int_S \left( \mathbf{\hat{\sigma}} | \nabla \varphi \right)_{H^{3	imes3}} dt = 0,
\]

\[
\int_S \left( \frac{1}{2} (\nabla \mathbf{\hat{u}} + \nabla \mathbf{\hat{u}}^T) | \psi \right)_{H^{3	imes3}} = \int_S \left( \int_0^t \Lambda(t-s) : \mathbf{\hat{\sigma}}(s) ds | \psi \right)_{H^{3	imes3}} dt.
\]

We define with \( \chi_r(t) \), the characteristic function of the interval \([0, \tau], 0 \leq \tau \leq T\),

\[
\varphi(t) := \chi_r(t) \mathbf{\hat{u}}(t),
\]

\[
\psi(t) := \int_0^t \chi_r(s) \mathbf{\hat{\sigma}}(s) ds + c,
\]

where \( c \) is chosen such that \( \psi(T) = 0 \) holds. Moreover, we obtain

\[
\psi(t) = 0 \quad \text{for } \tau \leq t \leq T,
\]

\[
\partial_t \psi(t) = \chi_r(t) \mathbf{\hat{\sigma}}(t).
\]
Due to the chosen test functions $\varphi$, it holds

$$
(3.2.84) \quad - \int_\mathcal{S} \left( \varphi \partial_t \hat{u} \right) \partial_t \chi \hat{u} \, H^3 + \int_\mathcal{S} \left( \sigma \nabla \chi \hat{u} \right) \, H^{3 \times 3} \, dt = -\|\partial_t \chi \hat{u}\|_{L^2 H^3}^2 + \int_\mathcal{S} \left( \chi \sigma \nabla \hat{u} \right) \, H^{3 \times 3} \, dt.
$$

Testing (3.2.81b) by $\chi \sigma$, we conclude

$$
(3.2.85) \quad \int_\mathcal{S} \left( \nabla \hat{u} | \chi \sigma \right) \, H^{3 \times 3} \, dt = \int_0^t \Lambda (t - s) : \sigma (s) \, ds \chi \sigma \, H^{3 \times 3} \, dt
$$

$$
= - \int_0^T \left( \int_0^t \Lambda : (t - s) \sigma (s) \, ds \right) \int_0^T \chi \sigma (s) \, ds + c \, H^{3 \times 3} \, dt
$$

$$
= - \int_0^T \Lambda (t - s) : \sigma (s) \, ds \chi \sigma (s) \, ds + c \, H^{3 \times 3} \, dt
$$

$$
= 0.
$$

Then, we obtain

$$
(3.2.86) \quad \|\partial_t \chi \hat{u}\|_{L^2 H^3}^2 = 0,
$$

such that $\partial_t \hat{u} = 0$ and due to the initial condition $\hat{u}(0) = 0$, it holds $\hat{u} = 0$. Furthermore, it follows $\nabla \hat{u} = 0$ and, therefore,

$$
(3.2.87) \quad 0 = \nabla \hat{u} + \nabla \hat{u}^T = \int_0^t \Lambda (t - s) : \sigma (s) \, ds,
$$

which implies $\sigma = 0$, since the equation is a homogeneous Volterra integral equation of first kind and possesses only the trivial solution.

Due to the assumption that we also consider $\Lambda (0, x)$ as positive definite for situations with $\sigma (0, x) = 0$, see Remark 3.2.4, the Problem 3.2.5 is an auxiliary problem of Problem 3.2.1 but also generalizes the original problem, since we do not use the special structure of $\Lambda$, cf. Remark 3.2.3.

---

\[7\int_\mathcal{S} \left( \varphi | \partial_t \psi \right) \, dt = - \int_\mathcal{S} \left( \partial_t \varphi | \psi \right) \, dt + \left( \varphi (T) | \psi (T) \right) - \left( \varphi (0) | \psi (0) \right) - \int_\mathcal{S} \frac{d}{dt} \left( \varphi | \psi \right) \, dt + \left( \varphi (T) | \psi (T) \right) - \left( \varphi (0) | \psi (0) \right)\]
3.2.3. Numerical Simulations with COMSOL 4.3a – 3d Narrow Rod

We recall the implementation and discussion of [36]. In order to examine the behavior of the viscoelastic MM model with $N$ coupled Kelvin-Voigt elements, we conduct a numerical simulation using a dimensionless approach (cf. Section 3.3.5 and [35] for the 1d case). Our model problem is a narrow 3d rod. For our numerical simulations, we consider a stress driven test with homogeneous Dirichlet boundary conditions at the left edge ($x = 0$) and inhomogeneous Neumann boundary at the right edge ($x = 1$) for a cyclic boundary load. The geometry is chosen comparable to the 1d case of Section 3.3.5. Other geometries can be treated as well without further effort. For simplicity, we restrict ourselves for the simulations to $N = 2$, cf. Fig. 3.2, the isotropic case and the Poisson’s ratio $\nu$ is assumed to be equal for every material tensor. In the isotropic case, the material matrices have a special shape, see [4], and are determined by two constants (resp. scalar material functions). Therefore, we define

\[
M_\nu := \begin{pmatrix}
\frac{1-\nu}{(1-2\nu)(1+\nu)} & \frac{\nu}{(1-2\nu)(1+\nu)} & \frac{\nu}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu}{(1-2\nu)(1+\nu)} & \frac{1-\nu}{(1-2\nu)(1+\nu)} & \frac{\nu}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu}{(1-2\nu)(1+\nu)} & \frac{\nu}{(1-2\nu)(1+\nu)} & \frac{1}{2(1+\nu)} & \frac{1}{2(1+\nu)} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2(1+\nu)} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2(1+\nu)} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2(1+\nu)} 
\end{pmatrix}.
\]

Figure 3.2.: Series connection of two coupled Kelvin-Voigt elements
In the isotropic case, it holds for all material matrices $M_T$

\[(3.2.89)\]

$$M_T = C_T M_\nu,$$

where $C_T$ is a scalar (resp. scalar material function). Here, $C_T$ will be the elastic moduli $E_1$ and $E_2$ and their coupling $E_{12}$. We identify the space variable $x$ with $(x,y,z)^T$ and since we use the matrix-vector notations, note that

\[(3.2.90)\]

$$\text{Div } \sigma = \tilde{\text{div }} \tilde{\sigma}$$

with

\[(3.2.91)\]

$$\tilde{\text{div }} := \begin{pmatrix}
\partial_x & 0 & 0 & \partial_z & \partial_y \\
0 & \partial_y & \partial_z & 0 & \partial_x \\
0 & \partial_z & \partial_y & \partial_x & 0
\end{pmatrix}.$$

With the above notation, we formulate Problem 3.2.9 in differential form of the model (cf. Problem 3.2.1).

**Problem 3.2.9.** Let $S$ be the interval $S := [0,80]$ and $\Omega$ the narrow rod $\Omega := ]0,1[ \times ]0,0.1[ \times ]0,0.1[$. Find $(u, \sigma, \varepsilon_1, \varepsilon_2) : S \times \Omega \rightarrow \mathbb{R}^3 \times \mathbb{R}^6 \times \mathbb{R}^6 \times \mathbb{R}^6$ such that for given $\gamma_\sigma$

\[(3.2.92a)\]

$$\rho \partial_t u - \tilde{\text{div }} \tilde{\sigma} = 0,$$

\[(3.2.92b)\]

$$2\tilde{\sigma} = V_1 M_\nu \partial_t \varepsilon_1 + V_2 M_\nu \partial_t \varepsilon_2 + (E_1 + E_{12}) M_\nu \varepsilon_1 + (E_2 + E_{12}) M_\nu \varepsilon_2,$$

\[(3.2.92c)\]

$$V_1 M_\nu \partial_t \varepsilon_1 + E_1 M_\nu \varepsilon_1 = \tilde{\sigma} - E_{12} M_\nu \varepsilon_2,$$

\[(3.2.92d)\]

$$\varepsilon_2 = \frac{1}{2} \left( \nabla u + \nabla u^T \right) - \varepsilon_1,$$

is fulfilled for initial conditions

\[(3.2.92e)\]

$$u = 0, \ \partial_t u = 0, \ \varepsilon_1 = 0, \ \varepsilon_2 = 0,$$

in $\Omega$,

and mixed boundary conditions – homogeneous Dirichlet boundary conditions at the left edge $(x = 0)$

\[(3.2.92f)\]

$$u = 0,$$

in $S \times \{0\} \times ]0,0.1[^2,$

and inhomogeneous Neumann boundary conditions at the right edge $(x = 1)$ – $\nu$ unity normal vector

\[(3.2.92g)\]

$$\begin{pmatrix}
\sigma_1 + \sigma_5 + \sigma_6 \\
\sigma_6 + \sigma_2 + \sigma_4 \\
\sigma_5 + \sigma_4 + \sigma_3
\end{pmatrix} \cdot \nu = \gamma_\sigma,$$

in $S \times \{1\} \times ]0,0.1[^2$.

### 3.2.3.1. Cyclic Tension Test with Relaxation

In order to conduct dimensionless simulations of Problem 3.2.9, we state the settings used in COMSOL Multiphysics® 4.3a. The unit system is set to None. Since we consider the isotropic case, the restriction for $E_{12}$ (3.2.12c) reduces here to

\[(3.2.93)\]

$$p^2 := \frac{E_{12}^2}{E_1 E_2} \leq 1.$$
We are interested in the qualitative behavior of the model and, therefore, we choose three values for $p$ to simulate the cases of no coupling ($p = 0$, resp. $E_{12} = 0$), strong regular coupling ($p = 0.98$ with $E_{12}^2 < E_1 E_2$) and singular coupling ($p = 1$ with $E_{12}^2 = E_1 E_2$). We define in the Global Definitions node the material parameters

$$
\begin{align*}
\rho &:= 60, & E_1 &:= 1, & E_2 &:= 2, & E_{12} &:= p \sqrt{E_1 E_2}, \\
p &:= \{0, 0.98, 1\}, & V_1 &:= 0.1, & V_2 &:= 1, & \nu &:= 0.3144,
\end{align*}
$$

as well as the parameters and functions for the cyclic boundary load $\gamma_\sigma$. The simulation computes the behavior of the rod under load by an asymmetric, cyclic boundary force $\gamma_\sigma$ and after the load has been removed. For this, we define $\alpha := 0.2, \omega := 1, c := 0.2, T_R := 16 \cdot \pi \approx 50.27$ and

$$\gamma_\sigma(t) := \begin{pmatrix} \gamma_{1x}(t) \\ 0 \\ 0 \end{pmatrix}, \quad \gamma_{1x}(t) := \begin{cases} 
\alpha (\sin(\omega t - \arcsin(c)) + c), & 0 \leq t < T_R, \\
0, & t \geq T_R.
\end{cases}$$

(3.2.94)

As geometry, we define a squared rod (Block) of length $l = 1$ ($x$ direction) and thickness $h = 0.1$

![Figure 3.3: Responses of a 3d rod in x direction at the rod’s center point of the right edge as (2)-coupled KV model to an asymmetric cyclic body load for various coupling parameters](image)

(a) Temporal strain curve for $p = 0$ (---), $p = 0.98$ (···), $p = 1$ (---) and temporal stress curve (−)

(b) No coupling for $p = 0$

(c) Regular coupling close to singular coupling for $p = 0.98$

(d) Singular coupling for $p = 1$

The parameter $p$ is altered by three studies instead of one parameter sweep with three values of $p$.  

---

Footnote: The parameter $p$ is altered by three studies instead of one parameter sweep with three values of $p$.  

---

59
(y and z direction). The equations of Problem 3.2.9 are implemented in four 3d Coefficient Form PDEs of the *Mathematica* toolbox for \(u, \sigma, \varepsilon_1\) and \(\varepsilon_2\). The boundary conditions are realized in the PDE system (3.2.9a). On the left edge \((x = 0)\), we prescribe the values of the deformation with the Dirichlet conditions \(u = 0\), while on the right edge \((x = 1)\) a cyclic boundary force \(\gamma_\sigma\) (see (3.2.94)) is applied as Flux/Source. For the PDE systems (3.2.9b) to (3.2.9d), we choose the default setting Zero Flux.

The PDE system (3.2.92) is solved by the implemented time dependent default solver MUMPS as fully coupled system up to \(T = 80\). The extremely fine mesh for the rod contains 19454 tetrahedral elements. For the PDE system (3.2.92a), we use Lagrange elements of cubic order while for the PDE systems (3.2.92b) - (3.2.92d) Lagrange elements of linear order are chosen. Combined, this leads to a number of degrees of freedom of 354027. The calculations were performed on a Intel i7-2600 (3.4 GHz) processor with 16 GB RAM and last less than 50 minutes for each value of \(p\).

![Figure 3.4: Von-Mises stress response of a 3d rod at the midsection \((z = 0.05)\) as \((2)\)-coupled KV model to a asymmetric cyclic body load for \(p = 1\)](image)

**3.2.3.2. Results**

The results are illustrated in time-strain and stress-strain graphs (Fig 3.3) for the rod’s center point of the right edge \((x = 1, y = 0.05, z = 0.05)\) with strains \((\varepsilon_{11})\) and stresses \((\sigma_{11})\) in \(x\)-direction for different coupling parameters \(p\).

As expected, for the case

- without coupling \((p = 0, \text{Fig. 3.3(b)})\),

the simulation shows the classical behavior of two Kelvin-Voigt models in series. After very few stress cycles the stress-strain curve converges to a constant orbit. When the applied boundary force tends to zero, the strain does too. For

- strong coupling near the singular case \((p = 0.98, \text{Fig. 3.3(c)})\),

the stress-strain curve shows the behavior of a shakedown (cf. [42]) and recovers slowly towards a strainless state after the stress is released. The most interesting case occurs for the

- singular case \((p = 1, \text{Fig. 3.3(d)})\).

This dissolves the usual structure of the stress-strain curve and there is no stabilizing of an orbit. Instead, the strain increases with increasing number of cycles. A *ratcheting effect* occurs. Furthermore, a constant strain level remains in the singular case, while the applied boundary
3. Viscoelasticity within the Framework of MM Models

force tends to zero. Similar results (for singular coupling) are obtained for plastic/viscoplastic material models with a multi-mechanism approach, cf. [17].

Contrary to the 1d case (cf. [35]), the strain still oscillates after the stress is released; presumable due to numerical reasons since the oscillation decreases with increasing mesh elements. However, the 3d simulations match well with the qualitative behavior of the 1d model on the boundary. In order to point out the stress evolution inside the rod, contour plots of the von-Mises stress at the mid section of the rod are presented in Fig. 3.4. The applied boundary force leads to a stress which evolves similar to a wave inside the rod, exemplarily illustrated for $p = 1$. After the boundary load is released, the stress tends to zero. Hereby, the decrease of the stress for material points inside the rod is delayed. The strain evolves similarly in the midpoint of the rod as on the boundary, cf. Fig. 3.5 and tends to stabilize. Although, interference effects take place as well.

![Figure 3.5: Strain response ($\varepsilon_{11}$) of a 3d rod in x direction at the rod’s midpoint ($x = 0.5, y = z = 0.05$) as (2)-coupled KV model to an asymmetric cyclic body, $p = 0 \ (- \cdot -)$, $p = 0.98 \ (\cdots)$, $p = 1 \ (- \cdot -)$](image)

3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

In this section, we deal with the derivation of a system of partial differential equations for the description of viscoelastic behavior of a 1d rod based on Section 2.3.3.2 – simplified to (2)-coupled Kelvin-Voigt elements in series (cf. [35, 37]), and attend to appropriate initial and boundary conditions. By simplifications of the 1d case, new problems arise in form of a singular mass matrix and inconsistent Dirichlet boundary conditions. We give a detailed derivation and motivation of the model and treat several boundary value problems – suitable to describe displacement and stress driven test situations – concerning their weak solvability. A nondimensionalization and scaling of the mixed boundary value problem is used to conduct qualitative numerical simulations with COMSOL Multiphysics® 4.3a to show the quality of the model.

3.3.1. Modeling – (2)-coupled-KV Model

The considered rod has the length $l > 0$ and the cross section $Q$. Its axis extends parallel in $x$-direction. We examine a small segment of the rod of length $h$ from $x$ to $x + h$, see Fig. 3.6. Let $u(t, x)$ be the displacement in $x$-direction in point $x$ at time $t$ of the time interval $S := [0, T]$, $T > 0$ and for the space interval $\Omega := [0, l]$. We assume differentiability of the displacement, so
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

that

$$\partial_x u(t, x) \approx \frac{u(t, x + h) - u(t, x)}{h}$$

holds approximately. All forces are intended to act exclusively in the direction of the x-axis. The response of the rod to external forces is determined by the model approach of (2)-coupled Kelvin-Voigt elements in series. Additionally, inertial effects are taken into account. D’Alembert’s principle provides the following relationship between the inertial forces $F_T$, the external volume forces $F_V$ and the internal forces $F_{in}$ as response to external forces in $S \times \Omega$:

$$F_T(t, x) - F_{in}(t, x) = F_V(t, x)$$

based on the segment $Q \times [x, x + h)$. The inertia is as usual determined by

$$F_T(t, x) = \rho(x) \partial_t u(t, x) |Q|h,$$

where $\rho(x)$ denotes the mass density in $x$. If the rod is homogeneous, $\rho$ is constant. The external volume forces are defined by

$$F_V(t, x) = \rho(x)f_V(t, x)|Q|h.$$  

Hereby, $f_V$ denotes the mass density of the external forces, e.g. gravity in $x$-direction.

The modeling of the response of the internal forces is illustrated in the following. Let $F_1$ and $F_2$ be contact forces acting in $x$ at time $t$ in longitudinal direction related to their respective Kelvin-Voigt element as response to external forces. Due to the series connection of the Kelvin-Voigt elements (cf. Section 2.3.3), it holds

$$\partial_x u(t, x) = \partial_x u_1(t, x) + \partial_x u_2(t, x)$$

and

$$F_{in}(t, x) = F_1(t, x) = F_2(t, x).$$

Moreover, we assume an additive split of the displacement itself into two partial displacements,

$$u(t, x) = u_1(t, x) + u_2(t, x).$$
This assumption is reasonable for the 1d case - especially, when we pose homogeneous Dirichlet boundary conditions at one end of the rod (cf. Section 3.3.3). In the material point \( x \), it holds with the elasticities \( E_i \), the viscosities \( V_i \), \( i = 1, 2 \), and the coefficient for interaction \( E_{12} \)

(3.3.8a) \[ \frac{F_1(t, x)}{|Q|} = E_1(x) \partial_x u_1(t, x) + E_{12}(x) \partial_x u_2(t, x) + V_1(x) \partial_x u_1(t, x), \]

(3.3.8b) \[ \frac{F_2(t, x)}{|Q|} = E_{12}(x) \partial_x u_1(t, x) + E_2(x) \partial_x u_2(t, x) + V_2(x) \partial_x u_2(t, x). \]

Then, the (contact) forces

\[
F_1(t, x + h) - F_1(t, x) = |Q| ( (E_1(x + h) \partial_x u_1(t, x + h) - E_1(x) \partial_x u_1(t, x))
\]

(3.3.9a) \[ + (E_{12}(x + h) \partial_x u_2(t, x + h) - E_{12}(x) \partial_x u_2(t, x))
\]

\[
+ (V_1(x + h) \partial_x u_1(t, x + h) - V_1(x) \partial_x u_1(t, x))
\]

\[ \approx \partial_x (E_1(x) \partial_x u_1(t, x) + E_{12}(x) \partial_x u_2(t, x)) |Q|h, \]

(3.3.9b) \[ F_2(t, x + h) - F_2(t, x) = |Q| ((E_{12}(x + h) \partial_x u_1(t, x + h) - E_{12}(x) \partial_x u_1(t, x))
\]

\[
+ (E_2(x + h) \partial_x u_2(t, x + h) - E_2(x) \partial_x u_2(t, x))
\]

\[
+ (V_2(x + h) \partial_x u_2(t, x + h) - V_2(x) \partial_x u_2(t, x))
\]

\[ \approx \partial_x (E_{12}(x) \partial_x u_1(t, x) + E_2(x) \partial_x u_2(t, x) + V_2(x) \partial_x u_2(t, x)) |Q|h, \]

are acting on the segment \( Q \times [x, x+h] \). Thus, under considerations of the fact that unidirectional forces \( F \) do not result in relative length changes, d’Alembert’s principle yields

(3.3.10a) \[ F_T - (F_1(t, x + h) - F_1(t, x)) = F_V, \]

(3.3.10b) \[ F_T - (F_2(t, x + h) - F_2(t, x)) = F_V. \]

The equation of momentum leads to the following system of partial differential equation of third order

(3.3.11a) \[ \varrho \partial_t(u_1 + u_2) |Q|h - \partial_x (E_1 \partial_x u_1 + E_{12} \partial_x u_2 + V_1 \partial_x u_1) |S|h = \rho f_V |S|h, \]

(3.3.11b) \[ \varrho \partial_t(u_1 + u_2) |Q|h - \partial_x (E_{12} \partial_x u_1 + E_2 \partial_x u_2 + V_2 \partial_x u_2) |S|h = \rho f_V |S|h \]

and, therefore,

(3.3.12a) \[ \varrho \partial_t(u_1 + u_2) - \partial_x (E_1 \partial_x u_1 + E_{12} \partial_x u_2 + V_1 \partial_x u_1) = \rho f_V, \]

(3.3.12b) \[ \varrho \partial_t(u_1 + u_2) - \partial_x (E_{12} \partial_x u_1 + E_2 \partial_x u_2 + V_2 \partial_x u_2) = \rho f_V. \]

The system (3.3.12) can be written in a compact form while introducing the notations

(3.3.13a) \[ u(t, x) := \begin{pmatrix} u_1(t, x) \\ u_2(t, x) \end{pmatrix}, \quad f(t, x) := \begin{pmatrix} f_1(t, x) \\ f_2(t, x) \end{pmatrix} := \begin{pmatrix} f_V(t, x) \\ f_V(t, x) \end{pmatrix}, \]

Integrating (3.3.5) from 0 to \( y, y \leq l \), and posing homogeneous Dirichlet boundary conditions at \( x = 0 \), we obtain

\[ u(t, y) = \int_0^y \partial_x u(x, t) \, dx = \int_0^y \partial_x u_1(x, t) \, dx + \int_0^y \partial_x u_2(x, t) \, dx = u_1(t, y) + u_2(t, y). \]
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

\[ E := \begin{pmatrix} E_1(x) & E_{12}(x) \\ E_{12}(x) & E_2(x) \end{pmatrix}, \quad V := \begin{pmatrix} V_1(x) & 0 \\ 0 & V_2(x) \end{pmatrix}, \]

and

\[ P(\cdot) := [E \partial_x(\cdot) + V \partial_{tx}(\cdot)]. \]

The system (3.3.12) now reads as

\[ \rho(x) \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \partial_{tt} u(t,x) - \partial_x P(t,x) = \rho(x) f(t,x). \]  

(3.3.14)

### 3.3.2. Initial and Boundary Conditions

#### 3.3.2.1. Initial Conditions

The initial conditions describe the initial state of the rod at the time \( t = 0 \)

\[ u_1(0,x) = u_1^{(0)}(x), \quad \partial_t u_1(0,x) = u_1^{(1)}(x), \]

\[ u_2(0,x) = u_2^{(0)}(x), \quad \partial_t u_2(0,x) = u_2^{(1)}(x). \]  

(3.3.15)  

(3.3.16)

In compact notation the initial conditions are given by

\[ u(0,x) = u^{(0)}(x) := \begin{pmatrix} u_1^{(0)}(x) \\ u_2^{(0)}(x) \end{pmatrix}, \quad \partial_t u(0,x) = u^{(1)}(x) := \begin{pmatrix} u_1^{(1)}(x) \\ u_2^{(1)}(x) \end{pmatrix}. \]  

(3.3.17)

#### 3.3.2.2. Boundary Conditions

In this section, we motivate and discuss reasonable boundary conditions for a rod. The boundary condition are valid for all \( t \geq 0 \).

**Dirichlet Boundary Conditions** If the rod is fixed on both edges, homogeneous Dirichlet boundary conditions have to be chosen,

\[ u(t,0) = u(t,l) = 0. \]  

(3.3.18)

Inhomogeneous Dirichlet boundary conditions occur if the displacement is given as a function of time on the boundary. This case occurs for example during displacement controlled experiments.

Exemplarily, let the situation of Fig. 3.7(b) be given. The left edge of the rod is fixed for all times at a fixed place, so that homogeneous Dirichlet boundary conditions hold on the left edge.
At the right edge \((x = l)\) a known displacement \(u^R\) is given,

\[
(3.3.19) \quad u(t, l) = u_1(t, l) + u_2(t, l) = u^R(t).
\]

If a displacement \(u^L\) is also given for the left edge \((x = 0)\), similarly

\[
(3.3.20) \quad u(t, 0) = u_1(t, 0) + u_2(t, 0) = u^L(t)
\]

applies. The displacement of the edge \(u^R\) has to be split to both partial displacements \(u_1(t, l)\) and \(u_2(t, l)\) on the edge. A reasonable ansatz is to choose \(u_1(t, l) = q(t)u^R(t)\) and \(u_2(t, l) = (1 - q(t))u^R(t)\) and determine \(q(t)\) by the material law. The same applies to \(u^L\) (see Section 3.3.3.4 for details).

**Neumann Boundary Conditions** If we assume free edges, Neumann boundary conditions have to be chosen. Homogeneous conditions hold if the rod lies in space without any acting contact on the edges. This might apply e.g. for Zero-G-states like in outer space or during zero gravity maneuvers of airplanes.

Considering d’Alembert’s principle on both edges, in \([0, h]\) (resp. in \([l - h, l]\), cf. Fig. 3.6). We obtain for \([0, h]\)

\[
(3.3.21) \quad F_T(t, 0) = F_V(t, 0) + F_i(t, h), \ i \in \{1, 2\},
\]

and, hence,

\[
(3.3.22) \quad g(0)\partial_h u(t, 0)Qh - g(0)f_V(t, 0)Qh = F_i(t, h), \ i \in \{1, 2\}.
\]

The limiting process \(h \to 0\) leads to

\[
(3.3.23) \quad \lim_{h \to 0} g(0)\partial_h u(t, 0)Qh - g(0)f_V(t, 0)Qh = F_i(t, h), \text{ and, therefore}, \ F_i(t, 0) = 0, \ i \in \{1, 2\}.
\]

We obtain

\[
(3.3.24a) \quad E_1(0)\partial_x u_1(t, 0) + E_{12}(0)\partial_x u_2(t, 0) + V_1(0)\partial_x u_1(t, 0) = 0,
\]

\[
(3.3.24b) \quad E_{12}(0)\partial_x u_1(t, 0) + E_2(0)\partial_x u_2(t, 0) + V_2(0)\partial_x u_2(t, 0) = 0.
\]

Analogous conclusions hold for the right edge \((x = l)\).

If a time dependent stress \(\sigma^R\) is applied to the cross section \(Q\) of the right edge of the rod, we obtain inhomogeneous Neumann boundary conditions on this edge. Considering d’Alembert’s principle on the right edge and noting that \(|Q| \cdot \sigma^R\) is a force, it follows

\[
(3.3.25) \quad F_T(t, l) = F_V(t, l) - F_i(t, l - h) + |Q|\sigma^R(t), \ i \in \{1, 2\}.
\]

Inserting and transforming the equation leads to

\[
(3.3.26) \quad g(l)\partial_h u(t, l)|Q|h - g(l)f_V(t, l)|Q|h = -F_i(t, l - h) + |Q|\sigma^R(t), \ i \in \{1, 2\}.
\]

Passing the limit \(h \to 0\), we obtain

\[
(3.3.27) \quad F_i(t, l) = |Q|\sigma^R(t)
\]
and, therefore,
\begin{align}
(3.3.28a) \quad & E_1(l) \partial_x u_1(t, l) + E_{12}(l) \partial_x u_2(t, l) + V_1(l) \partial_{tx} u_1(t, l) = \sigma^R(t), \\
(3.3.28b) \quad & E_{12}(l) \partial_x u_1(t, l) + E_2(l) \partial_x u_2(t, l) + V_2(l) \partial_{tx} u_2(t, l) = \sigma^R(t),
\end{align}
respectively in compact notation
\begin{equation}
(3.3.29) \quad \left( \begin{array}{c}
\sigma^R(t) \\
\sigma^R(t)
\end{array} \right).
\end{equation}
Combined with a homogeneous Dirichlet boundary condition on the left edge, we gain the basis to describe a stress controlled experiment.

Analogously, we can derive inhomogeneous Neumann boundary conditions for the left edge.

**Robin Boundary Conditions** If the rod is clamped between two additional bodies \(K_1\) and \(K_2\), Robin boundary conditions apply (cf. Fig. 3.8). Let \(K_1\) and \(K_2\) be two elastic bodies. Then, it holds at \(x = 0\)
\begin{align}
(3.3.30a) \quad & E^L (u_1(t, 0) + u_2(t, 0)) = E_1 \partial_x u_1(t, 0) + E_{12} \partial_x u_2(t, 0) + V_1 \partial_{tx} u_1(t, 0), \\
(3.3.30b) \quad & E^L (u_1(t, 0) + u_2(t, 0)) = E_{12} \partial_x u_1(t, 0) + E_2 \partial_x u_2(t, 0) + V_2 \partial_{tx} u_2(t, 0),
\end{align}
and at \(x = l\)
\begin{align}
(3.3.31a) \quad & -E^R (u_1(t, l) + u_2(t, l)) = E_1 \partial_x u_1(t, l) + E_{12} \partial_x u_2(t, l) + V_1 \partial_{tx} u_1(t, l), \\
(3.3.31b) \quad & -E^R (u_1(t, l) + u_2(t, l)) = E_{12} \partial_x u_1(t, l) + E_2 \partial_x u_2(t, l) + V_2 \partial_{tx} u_2(t, l).
\end{align}
Hereby, \(E^L\) and \(E^R\) are the elasticities of the left and right adjoint elastic bodies. If \(K_1\) and \(K_2\) are viscoelastic bodies, let additionally \(V^L\) and \(V^R\) be the viscosities of each adjoint body. The boundary conditions are then posed as
\begin{align}
(3.3.32a) \quad & E^L (u_1(t, 0) + u_2(t, 0)) + V^L \partial_t (u_1(t, 0) + u_2(t, 0)) \\
& \quad = E_1(0) \partial_x u_1(t, 0) + E_{12}(0) \partial_x u_2(t, 0) + V_1(0) \partial_{tx} u_1(t, 0), \\
(3.3.32b) \quad & E^L (u_1(t, 0) + u_2(t, 0)) + V^L \partial_t (u_1(t, 0) + u_2(t, 0)) \\
& \quad = E_{12}(0) \partial_x u_1(t, 0) + E_2(0) \partial_x u_2(t, 0) + V_2(0) \partial_{tx} u_2(t, 0)
\end{align}
and
\begin{align}
(3.3.33a) \quad & -E^R (u_1(t, l) + u_2(t, l)) - V^R \partial_t (u_1(t, l) + u_2(t, l)) \\
& \quad = E_1(l) \partial_x u_1(t, l) + E_{12}(l) \partial_x u_2(t, l) + V_1(l) \partial_{tx} u_1(t, l),
\end{align}
3.1. Viscoelasticity within the Framework of MM Models

\[(3.3.33b) \quad -E^R(u_1(t,l) + u_2(t,l)) - V^R \partial_t (u_1(t,l) + u_2(t,l))
\]
\[= E_{12}(l) \partial_x u_1(t,l) + E_2(l) \partial_x u_2(t,l) + V_2(l) \partial_{xx} u_2(t,l).\]

We define the compact notation

\[(3.3.34) \quad E^L := \begin{pmatrix} E^L & E^L \\ E^L & E^L \end{pmatrix}, \quad E^R := \begin{pmatrix} E^R & E^R \\ E^R & E^R \end{pmatrix}.\]

\(E^L, E^R\) are the elasticities of the left \((x = 0)\) and right \((x = l)\) adjoint body. Analogously, we define the viscosities

\[(3.3.35) \quad V^L := \begin{pmatrix} V^L \\ V^L \end{pmatrix}, \quad V^R := \begin{pmatrix} V^R \\ V^R \end{pmatrix}.\]

The compact form of the Robin boundary conditions is

\[(3.3.36) \quad P u(t,0) = E^L u(t,0) + V^L \partial_t u(t,0), \quad P u(t,l) = -E^R u(t,l) - V^R \partial_t u(t,l).\]

If we consider an ideal elastic or ideal viscous attached body, we set the elastic respectively with viscous coefficient to 0.

Again, also inhomogeneous Robin boundary conditions are applicable. In this case, we add the boundary stresses \(\sigma^L\) and \(\sigma^R\),

\[(3.3.37) \quad \sigma^L(t) = \begin{pmatrix} \sigma^L(t) \\ \sigma^L(t) \end{pmatrix}, \quad \sigma^R(t) = \begin{pmatrix} \sigma^R(t) \\ \sigma^R(t) \end{pmatrix},\]

on the related edge, such that

\[(3.3.38a) \quad P u(t,0) = E^L u(t,0) + V^L \partial_t u(t,0) + \sigma^L(t), \quad P u(t,l) = -E^R u(t,l) - V^R \partial_t u(t,l) + \sigma^R(t).\]

Clearly, mixed boundary conditions are reasonable as well.

Figure 3.9.: Inhomogeneous Robin boundary at the right edge

3.3.3. Weak Solvability of the Transient Problem

In the following, we deal with the weak solvability of the system (3.3.12). Formulating the mathematical problems

- for homogeneous Dirichlet boundary conditions,
- for inhomogeneous Dirichlet boundary conditions, and
- for mixed boundary conditions,
we make statements about existence and uniqueness of weak solutions of each problem. General mathematical references concerning this topic are e.g. [25], [62] or [80].

3.3.3.1. Homogeneous Dirichlet Boundary Value Problem

We pose the problem for homogeneous Dirichlet boundary as

**Problem 3.3.1.** Let \( S := [0, T], \Omega := [0, l], P \) and \( f \) be as in (3.3.13). Find \( u : S \times \Omega \to \mathbb{R}^2 \) such that for given \( \varrho, E_1, E_2, E_{12}, V_1, V_2 \) and \( f \)

\[
(3.3.39) \quad \varrho(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_t u(t, x) - \partial_x P u(t, x) = \varrho(x) f(t, x), \quad \text{in} \ S \times \Omega,
\]

is fulfilled for given initial condition \( u(0) \) and \( u(1) \)

\[
(3.3.40a) \quad u(0, x) = u^{(0)}(x), \quad \partial_t u(0, x) = u^{(1)}(x), \quad \text{in} \ \Omega,
\]

and given homogeneous boundary conditions

\[
(3.3.40b) \quad u(t, x) = 0, \quad \text{on} \ S \times \partial \Omega.
\]

We search a weak solution of Problem 3.3.1 in the following setting.

**Setting 3.3.2.**

**Time & Space:**

\[
(3.3.41) \quad S := [0, T] \subset \mathbb{R}, \ T > 0 \ (\text{time}), \quad \Omega := [0, l] \subset \mathbb{R}, \ l > 0 \ (\text{space})
\]

**Parameter (functions):**

\[
(3.3.42a) \quad \varrho \in L^\infty(\Omega), \quad \exists \varrho_0 \ \text{f.a.a.} \ x \in \Omega : \ \varrho(x) > \varrho_0 \quad \text{(mass density)}
\]

\[
(3.3.42b) \quad E_i \in L^\infty(\Omega), \quad \exists E_{0,i} \ \text{f.a.a.} \ x \in \Omega : \ E_i(x) > E_{0,i}, \ i = 1, 2 \quad \text{(elasticities)}
\]

\[
(3.3.42c) \quad V_i \in L^\infty(\Omega), \quad \exists V_{0,i} \ \text{f.a.a.} \ x \in \Omega : \ V_i(x) > V_{0,i}, \ i = 1, 2 \quad \text{(viscosities)}
\]

\[
(3.3.42d) \quad E_{12} \in L^\infty(\Omega) \quad \text{(coupling coefficient)}
\]

**Function spaces:**

\[
(3.3.43a) \quad V := W^{1,2}(\Omega), \ V_0 := W^{1,2}_0(\Omega), \ H := L^2(\Omega)
\]

\[
(3.3.43b) \quad V := L^2(S, V), \ V_0 := L^2(S, V_0), \ H := L^2(S, H)
\]

**Data:**

\[
(3.3.44) \quad f \in H^2, \ u^{(0)} \in V_0^2, \ u^{(1)} \in H^2
\]

**Definition 3.3.3.** Consider Setting 3.3.2. \( u \in V_0^2 \) is a weak solution of Problem 3.3.1 if and only if \( \partial_t u \in V_0^2 \) and for all \( \varphi \in V_0^2 \) with \( \partial_t \varphi \in H^2 \) and \( \varphi(T) = 0 \),

\[
(3.3.45) \quad - \int_S \int_\Omega \left( \varrho(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \partial_t u(t, x) \right) \cdot \partial_t \varphi(t, x) \, dx \, dt + \int_S \int_\Omega P u(t, x) \cdot \partial_x \varphi(t, x) \, dx \, dt
\]

\[
= \int_S \int_\Omega \varrho(x) f \cdot \varphi(t, x) \, dx \, dt + \int_\Omega \left( \varrho(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u^{(1)}(x) \right) \cdot \varphi(0, x) \, dx
\]
Theorem 3.3.4. Under the assumptions of Setting 3.3.2 there exists a unique weak solution \( u \) of (3.3.1) in \( W^{1,2}(S, V_0) \) and \( \partial_t u \in [L^2(S, V_0)]^2 \) of Problem 3.3.1. Moreover, the map

\[
S : \left( u^{(0)}, u^{(1)}, f \right) \mapsto (u, \partial_t u)
\]

is linear and continuous. In particular, there exists a constant \( C \) independent of \( u, u^{(0)}, u^{(1)} \) and \( f \), such that

\[
\|u\|_{L^\infty(S, V_0)} + \|\partial_t u\|_{V^2} \leq C \left( \|u^{(0)}\|_{V^2} + \|u^{(1)}\|_{H^2} + \|f\|_{V^2} \right).
\]

Extra: If the constraint for strict convexity of the free energy is fulfilled,

\[
E_2^1 < E_1^2 \text{ f.a.a. } x \in \Omega,
\]

the constant \( C \) in (3.3.48) is independent of the time interval \( S \).

The constraint due to thermomechanical consistency \( E_2^1 \leq E_1^2 \) is not crucial for the proof of the theorem, although it is physically problematic.

Proof. The existence and uniqueness of the solution of the transient Problem 3.3.1 will be proven via Galerkin’s method, see [37].

Step 1 – Galerkin approximation

Let \( (w_j)_{j \in \mathbb{N}} \) be a Galerkin basis in the separable Banach space \( V_0 \) and \( (V_0, n)_{n \in \mathbb{N}} \) with \( V_0, n := \text{span}\{w_1, \ldots, w_n\} \) the corresponding Galerkin scheme. Furthermore, let \( (u^{[n], (0)}), (u^{[n], (1)}))_{n \in \mathbb{N}} \) be sequences in \( [V_0, n]^2 \) with

\begin{align*}
\text{(3.3.50a)} & \quad u^{[n], (0)}, u^{[n], (1)} \in [V_0, n]^2, \\
\text{(3.3.50b)} & \quad u^{[n], (0)} \rightarrow u^{(0)} \quad \text{in } V_0^2 \text{ for } n \rightarrow \infty, \\
\text{(3.3.50c)} & \quad u^{[n], (1)} \rightarrow u^{(1)} \quad \text{in } H^2 \text{ for } n \rightarrow \infty.
\end{align*}

The Galerkin method projects for every time \( t \) the solution \( u(t, \cdot) \in V^2 \) onto \( u^{[n]}(t, \cdot) \in [V_0, n]^2 \). Every element \( u^{[n]} \in [L^2(S, V_0)]^2 \) is represented via the Galerkin basis of \( V_0, n \) by

\[
\text{(3.3.51)} \quad u^{[n]}(t) = \sum_{j=1}^{n} g_j^{[n]}(t) w_j, \quad \forall t \in S \text{ with } g_j^{[n]}(t) = \begin{pmatrix} g_j^{[n], (1)}(t) \\ g_j^{[n], (2)}(t) \end{pmatrix},
\]

where \( g_j^{[n]}(t) : S \rightarrow \mathbb{R} \) are coefficient functions, which are to be determined. Due to \( u^{[n], (0)}, u^{[n], (1)} \in [V_0, n]^2 \), there exist representations

\begin{align*}
\text{(3.3.52)} & \quad u^{[n], (0)} = \sum_{j=1}^{n} \alpha_j^{[n]} w_j, \\
& \quad u^{[n], (1)} = \sum_{j=1}^{n} \beta_j^{[n]} w_j,
\end{align*}

as well as \( u(0, x) = u^{(0)}(x) \) hold, (cf.[79, Section 24]).
with

(3.3.53) \[
\alpha_j^{[n]} = \left( \frac{\alpha_{j,1}^{[n]}}{\alpha_{j,2}^{[n]}} \right), \quad \beta_j^{[n]} = \left( \frac{\beta_{j,1}^{[n]}}{\beta_{j,2}^{[n]}} \right).
\]

The matrix in front of the \( \partial_t u^{[n]} \)-term is singular, therefore, we use for the Galerkin formulation of the Problem (3.3.1) a regularized form of the equation:

Find \( u^{[n]} \in [L^2(S, V_{0,n})]^2 \), such that for all \( \varphi \in V_{0,n}^2 \) almost everywhere in \( S \)

(3.3.54a) \[
\int_\Omega \left( \varphi(x) \left( \frac{1}{1 - \frac{1}{n}} \frac{1 - \frac{1}{n}}{1} \right) \partial_t u^{[n]}(t, x) \right) \cdot \varphi \, dx + \int_\Omega P u^{[n]}(t, x) \cdot \partial_x \varphi \, dx = \int_\Omega \varphi(x) f \cdot \varphi \, dx,
\]

(3.3.54b) \[
u^{[n]}(0, x) = u^{[n],(0)}(x), \quad \partial_t u^{[n]}(0, x) = u^{[n],(1)}(x)
\]
holds.

Testing the Galerkin equation with \( \varphi = (w_j, w_j)^T, \ j = 1, \ldots, n \), we obtain

(3.3.55) \[
\int_\Omega \left( \varphi(x) \partial_t \left( \frac{1}{1 - \frac{1}{n}} \frac{1 - \frac{1}{n}}{1} \right) u^{[n]}(t, x) \right) \cdot \left( \begin{array}{c} w_j \\ w_j \end{array} \right) \, dx + \int_\Omega P u^{[n]}(t, x) \cdot \partial_x \left( \begin{array}{c} w_j \\ w_j \end{array} \right) \, dx = \int_\Omega \varphi(x) f \cdot \left( \begin{array}{c} w_j \\ w_j \end{array} \right) \, dx.
\]

Inserting the basis representation of \( u^{[n]} \) in (3.3.55), we obtain for the first term of the left side of the equation

(3.3.56) \[
\int_\Omega \left( \begin{array}{c} 1 \\ 1 - \frac{1}{n} \end{array} \right) \left( \begin{array}{c} \partial_t \sum_{k=1}^n g_{k,1}^{[n]} w_k \\ \partial_t \sum_{k=1}^n g_{k,2}^{[n]} w_k \end{array} \right) \cdot \left( \begin{array}{c} w_j \\ w_j \end{array} \right) \, dx = \left( \begin{array}{c} 1 \\ 1 - \frac{1}{n} \end{array} \right) \left( \begin{array}{c} \sum_{k=1}^n \partial_t g_{k,1}^{[n]} \int_\Omega \varphi(x) w_k w_j \, dx \\ \sum_{k=1}^n \partial_t g_{k,2}^{[n]} \int_\Omega \varphi(x) w_k w_j \, dx \end{array} \right) _{j=1, \ldots, n}
\]

and for the second term

(3.3.57) \[
\int_\Omega \left( \begin{array}{c} P \sum_{k=1}^n g_{k,1}^{[n]} w_k \\ P \sum_{k=1}^n g_{k,2}^{[n]} w_k \end{array} \right) \cdot \partial_x \left( \begin{array}{c} w_j \\ w_j \end{array} \right) \, dx = \left( \begin{array}{c} E \partial_x \sum_{k=1}^n g_{k,1}^{[n]} w_k \\ V \partial_x \sum_{k=1}^n g_{k,2}^{[n]} w_k \end{array} \right) \cdot \partial_x \left( \begin{array}{c} w_j \\ w_j \end{array} \right) \, dx = \left( \begin{array}{c} E \partial_x \sum_{k=1}^n g_{k,1}^{[n]} w_k + V \partial_x \sum_{k=1}^n g_{k,2}^{[n]} w_k \\ V \partial_x \sum_{k=1}^n g_{k,2}^{[n]} w_k \end{array} \right) \cdot \partial_x \left( \begin{array}{c} w_j \\ w_j \end{array} \right) \, dx \right) _{j=1, \ldots, n}
\]
3. Viscoelasticity within the Framework of MM Models

\[
\begin{align*}
= & \left( \int_{\Omega} E \left( \sum_{k=1}^{n} g_{k,1}^{[n]} \partial_{x} w_{k} \partial_{x} w_{j} \right) + V \left( \sum_{k=1}^{n} \partial_{t} g_{k,2}^{[n]} \partial_{x} w_{k} \partial_{x} w_{j} \right) \right) dx \right)_{j=1,\ldots,n}.
\end{align*}
\]

Defining the notations

\[
(3.3.58) \quad \bar{g}_{i} := \begin{pmatrix} g_{1,i}^{[n]} \\ \vdots \\ g_{n,i}^{[n]} \end{pmatrix}, \quad i = 1, 2, \quad \bar{g} := \begin{pmatrix} \bar{g}_{1} \\ \bar{g}_{2} \end{pmatrix},
\]

\[
(3.3.59) \quad (G^{\rho})_{jk} := \left( \int_{\Omega} \rho(x) w_{k} w_{j} dx \right)_{jk}, \quad j, k = 1, \ldots, n,
\]

\[
(3.3.60) \quad (G^{E_{1}})_{jk} := \left( \int_{\Omega} E_{1}(x) \partial_{x} w_{k} \partial_{x} w_{j} dx \right)_{jk}, \quad j, k = 1, \ldots, n, \quad i = 1, 2,
\]

\[
(3.3.61) \quad (G^{E_{12}})_{jk} := \left( \int_{\Omega} E_{12}(x) \partial_{x} w_{k} \partial_{x} w_{j} dx \right)_{jk}, \quad j, k = 1, \ldots, n,
\]

\[
(3.3.62) \quad (G^{V_{i}})_{jk} := \left( \int_{\Omega} V_{i}(x) \partial_{x} w_{k} \partial_{x} w_{j} dx \right)_{jk}, \quad j, k = 1, \ldots, n, \quad i = 1, 2
\]

\[
(3.3.63) \quad F := \begin{pmatrix} \int_{\Omega} f_{V_{1}} w_{1} dx \\
\vdots \\
\int_{\Omega} f_{V_{n}} dx \\
\int_{\Omega} f_{V_{1}} dx \\
\vdots \\
\int_{\Omega} f_{V_{n}} dx \end{pmatrix}, \quad \bar{\alpha} := \begin{pmatrix} \alpha_{1,1}^{[n]} \\
\vdots \\
\alpha_{n,1}^{[n]} \\
\alpha_{1,2}^{[n]} \\
\vdots \\
\alpha_{n,2}^{[n]} \end{pmatrix},
\]

the problem of (3.3.54) is equivalent to:

Find functions \(g_{k,i}^{[n]}\), \(i = 1, 2, k = 1, \ldots, n\), such that the system of ordinary differential equations

\[
(3.3.64a) \quad \left( \begin{array}{cc} G^{\rho} & -\frac{1}{n} G^{\rho} \\
G^{\rho} - \frac{1}{n} G^{\rho} & G^{\rho} \end{array} \right) \partial_{t} \bar{g} + \left( \begin{array}{cc} G^{E_{1}} & G^{E_{12}} \\
G^{E_{12}} & G^{E_{2}} \end{array} \right) \bar{g} + \left( \begin{array}{cc} G^{V_{1}} & 0_{n \times n} \\
0_{n \times n} & G^{V_{2}} \end{array} \right) \partial_{t} \bar{g} = F
\]

with initial values

\[
(3.3.64b) \quad \bar{g}(0) = \bar{\alpha}, \quad \partial_{t} \bar{g}(0) = \bar{\beta}
\]

is fulfilled.
Let be
\[
G^{\text{reg}} := \begin{pmatrix}
G^0 & G^0 - \frac{1}{n} G^0 \\
G^0 - \frac{1}{n} G^0 & G^0
\end{pmatrix}.
\]

Using the substitution \( \vec{h} := \partial_t \vec{g} \), system (3.3.64) is transformed into the first order system
\[
\begin{align*}
\partial_t \vec{g} &= \vec{h} \\
\partial_t \vec{h} &= -(G^{\text{reg}})^{-1} \begin{pmatrix}
G^{E_1} & G^{E_{12}} \\
G^{E_{12}} & G^{E_2}
\end{pmatrix} \vec{g} - (G^{\text{reg}})^{-1} \begin{pmatrix}
G^{V_1} & 0_{n \times n} \\
0_{n \times n} & G^{V_2}
\end{pmatrix} \vec{h} + (G^{\text{reg}})^{-1} F
\end{align*}
\]
with initial values
\[
\begin{align*}
\vec{g}(0) &= \vec{\alpha}, \\
\vec{h}(0) &= \vec{\beta}.
\end{align*}
\]
Since the coefficients and \( F \) are not continuous, we apply a conclusion of the theorem of Carathéodory ([49, Theorem 1.45, p. 25]).

Therefore, let \( C_n : [0, T] \times \mathbb{R}^{4n} \rightarrow \mathbb{R}^{4n} \) be defined by
\[
C_n(t, y) := \begin{pmatrix}
0_{2n} \\
-(G^{\text{reg}})^{-1} \begin{pmatrix}
G^{E_1} & G^{E_{12}} \\
G^{E_{12}} & G^{E_2}
\end{pmatrix} - (G^{\text{reg}})^{-1} \begin{pmatrix}
G^{V_1} & 0_{n \times n} \\
0_{n \times n} & G^{V_2}
\end{pmatrix}
\end{pmatrix} y(t)
\]
\[
= \mathcal{A} y(t) + \begin{pmatrix}
0_{1 \times 2n} \\
(G^{\text{reg}})^{-1} F
\end{pmatrix}.
\]

We have to prove: \( C_n \) is a Carathéodory-function such that \( t \rightarrow C_n(t, y) \) is Lebesgue-measurable for all \( y, y \rightarrow C_n(t, y) \) Lipschitz-continuous for almost all \( t \in S \) and that there exists a nonnegative function \( \varphi \in L^1([0, T]) \) and \( C > 0 \) with \( |C_n(t, y)| \leq \varphi(t) + C|y| \).

**Measurability with respect to \( t \):**

\( f_V \) is Bochner-measurable as map \( S \rightarrow H \). Hence, the theorem of Pettis is applicable and the map \( t \rightarrow \int_{\Omega} f_V \varphi \, dx \) is Lebesgue-measurable for all \( \varphi \in H^* \). Moreover, \( H^* \) is dense in \( V^* \), since \( H \) is dense in \( V \) and \( V \) is reflexive (cf. [62, Sections 3 and 17]). It follows \( t \rightarrow F(t) \) is Lebesgue-measurable and, therefore, \( \Xi \) is too. Concluding, the map \( t \rightarrow C_n(t, y) \) on \([0, T]\) is Lebesgue-measurable for all \( y \in \mathbb{R}^{4n} \).

(**Lipschitz-)Continuity of \( y \rightarrow C_n(t, y) \) for almost all \( t\):**

For fixed \( t \in [0, T] \) the continuity results from the continuity of \( \mathcal{A} \) and the linearity of \( (G^{\text{reg}})^{-1} \).

\[
|C(t, y) - C(t, \overline{y})| = |\mathcal{A} y + \Xi(t) - \mathcal{A} \overline{y} - \Xi(t)| = |\mathcal{A} y - \mathcal{A} \overline{y}|
\]
\[
= |\mathcal{A} (y - \overline{y})| \leq \|\mathcal{A}\| |y - \overline{y}| \leq C|y - \overline{y}|
\]
\[
= : C
\]

**Majorant condition:**
3. Viscoelasticity within the Framework of MM Models

It holds

\[(3.3.69) \quad |C(t, y)| = |\mathcal{A}y + \Xi(t)| \leq \|\mathcal{A}\|y\| + \|\Xi(t)\| =: C[y]|s(y)\|
\]

\leq s(y) + \|\mathcal{G}\|^{-1} \max_i \|w_i\|_{H^2} 2n \|f_V(t, \cdot)\|_H =: s(y) + \varphi(t)\]

with \( s \in C(\mathbb{R}^4^n, \mathbb{R}^4^n) \) and \( \varphi \in L^1(S) \), since \( f_V \in L^2(\Omega) \) and due to the properties of the Bochner-integral \( t \to \|f_V(t, \cdot)\|_{L^2(\Omega)} \in L^2(S) \subset L^1(S) \).

A conclusion of the theorem of Carathéodory ([49, p. 25, Theorem 1.45]) ensures that the ODE system (3.3.66) has an absolutely continuous, almost everywhere differentiable solution \( y_n : [0, T] \to \mathbb{R}^4^n \) and \( u^{[n]} \in AC^1([0, T], V_n) \).

**Step 2 - A priori estimate**

We consider the regularized version of the Galerkin-equation (3.3.54),

\[(3.3.70) \quad \int_{\Omega} \varrho(x) \left( \begin{array}{c} 1 \\ 1 - \frac{1}{n} \\ 1 - \frac{1}{n} \\ 1 \end{array} \right) \partial_t u^{[n]} \cdot \left( \begin{array}{c} w_j \\ w_j \\ w_j \end{array} \right) dx + \int_{\Omega} P u^{[n]} : \partial_x u^{[n]} dx
\]

\[= \int_{\Omega} \varrho(x) \left( \begin{array}{c} f_V \\ f_V \\ f_V \\ f_V \end{array} \right) \cdot \left( \begin{array}{c} w_j \\ w_j \\ w_j \end{array} \right) dx, \quad j = 1, \ldots, n.\]

Componentwise multiplication by \((\partial_{\Omega} g_j^{[n]}, \partial_{\Omega} g_j^{[n]})^T\) and summation over \( j = 1, \ldots, n \), lead to

\[(3.3.71) \quad \int_{\Omega} \varrho(x) \left( \begin{array}{c} 1 \\ 1 - \frac{1}{n} \\ 1 - \frac{1}{n} \\ 1 \end{array} \right) \partial_t u^{[n]} \partial_x u^{[n]} dx + \int_{\Omega} P u^{[n]} : \partial_t \partial_x u^{[n]} dx
\]

\[= \int_{\Omega} \varrho(x) \left( \begin{array}{c} f_V \\ f_V \\ f_V \end{array} \right) \cdot \partial_t u^{[n]} dx.\]

Integrating over \( S_r := (0, \tau) \subseteq S \) with \( 0 < \tau \leq T \) and summing up of both equations in (3.3.71), we obtain

\[(3.3.72) \quad \int_{S_r} \int_{\Omega} \varrho(x) \left( \begin{array}{c} 1 \\ 1 - \frac{1}{n} \\ 1 - \frac{1}{n} \\ 1 \end{array} \right) \partial_t u^{[n]} : \partial_t u^{[n]} dx dt + \int_{S_r} \int_{\Omega} P u^{[n]} : \partial_t \partial_x u^{[n]} dx dt
\]

\[= \int_{S_r} \int_{\Omega} \varrho(x) \left( \begin{array}{c} f_V \\ f_V \end{array} \right) : \partial_t u^{[n]} dx dt.\]

In order to get an appropriate estimate, we consider every term separately. The first term of the left side of (3.3.72) can be reformulated as

\[(3.3.73) \quad \int_{S_r} \int_{\Omega} \varrho(x) \left( \begin{array}{c} 1 \\ 1 - \frac{1}{n} \\ 1 - \frac{1}{n} \\ 1 \end{array} \right) \partial_t u^{[n]} : \partial_t u^{[n]} dx dt
\]

\[= \int_{S_r} \int_{\Omega} \varrho(x) \left[ \partial_t u_1^{[n]} \partial_t u_1^{[n]} + \partial_t u_2^{[n]} \partial_t u_2^{[n]} \right] dx dt\]
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

\[ + \int_S \frac{\rho(x)}{2} \left( \left( \partial_t u_1^{[n]} \right)^2 + \left( \partial_t u_2^{[n]} \right)^2 \right) dxdt \]

\[ = \int_S \int_{\Omega} \left( 1 - \frac{1}{n} \right) \left[ \partial_t u_1^{[n]} \partial_t u_2^{[n]} + \partial_t u_2^{[n]} \partial_t u_1^{[n]} \right] dxdt \]

\[ + \int_S \int_{\Omega} \rho(x) \left( 1 - \frac{1}{n} \right) \partial_t u_1^{[n]} \partial_t u_2^{[n]} dxdt \]

\[ + \int_{\Omega} \rho(x) \left( 1 - \frac{1}{n} \right) \partial_r u_1^{[n]} \partial_r u_2^{[n]} \left. \right|_{t=\tau} \]

\[ - \int_{\Omega} \rho(x) \left( 1 - \frac{1}{n} \right) \partial_r u_1^{[n]} \partial_r u_2^{[n]} \left. \right|_{t=0} \]

\[ = \frac{1}{2} \left( \| \partial_r u_1^{[n]} \|^2_{L^2, L^2} + \| \partial_r u_2^{[n]} \|^2_{L^2, L^2} - \| \partial_r u_1^{[n]}(0, x) \|^2_{L^2, L^2} - \| \partial_r u_2^{[n]}(0, x) \|^2_{L^2, L^2} \right) \]

\[ + \int_{\Omega} \rho(x) \left( 1 - \frac{1}{n} \right) \partial_r u_1^{[n]}(\tau, x) \partial_r u_2^{[n]}(\tau, x) \]

\[ - \int_{\Omega} \rho(x) \left( 1 - \frac{1}{n} \right) \partial_r u_1^{[n]}(0, x) \partial_r u_2(0, x) \]

For the second term of the left side of (3.3.72), it holds

(3.3.74) \[ \int_S \left[ \mathbf{P} u^{[n]} : \partial_t x u^{[n]} \right] dxdt \]

\[ = \int_S \left[ \mathbf{E} \partial_t x u^{[n]} : \partial_t x u^{[n]} + \mathbf{V} \partial_t x u^{[n]} : \partial_t x u^{[n]} \right] dxdt \]

\[ = \int_S \left[ E_1(x) \partial_t x u_1^{[n]} \partial_t x u_1^{[n]} + E_2(x) \partial_t x u_2^{[n]} \partial_t x u_2^{[n]} \right] dxdt \]

\[ + \int_S \left[ E_{12}(x) \left( \partial_t x u_1^{[n]} \partial_t x u_2^{[n]} + \partial_t x u_2^{[n]} \partial_t x u_1^{[n]} \right) \right] dxdt \]

\[ + \int_S \left[ V_1(x) \left( \partial_t x u_1^{[n]} \right)^2 + V_2(x) \left( \partial_t x u_2^{[n]} \right)^2 \right] dxdt \]

\[ = \frac{1}{2} \left( \| \partial_t x u_1^{[n]} \|^2_{E_1, L^2} + \| \partial_t x u_2^{[n]} \|^2_{E_2, L^2} - \| \partial_t x u_1^{[n]}(0, x) \|^2_{E_1, L^2} - \| \partial_t x u_2^{[n]}(0, x) \|^2_{E_2, L^2} \right) \]

\[ + \int_S \left[ E_{12}(x) \left( \partial_t x u_1^{[n]} \partial_t x u_2^{[n]} + \partial_t x u_2^{[n]} \partial_t x u_1^{[n]} \right) \right] dxdt \]

\[ + \int_S \left[ V_1(x) \left( \partial_t x u_1^{[n]} \right)^2 + V_2(x) \left( \partial_t x u_2^{[n]} \right)^2 \right] dxdt. \]

Addition of (3.3.73) and (3.3.74) leads to

(3.3.75) \[ \frac{1}{2} \left( \| \partial_r u_1^{[n]} \|^2_{L^2, L^2} + \| \partial_r u_2^{[n]} \|^2_{L^2, L^2} \right) \]

\[ + \frac{1}{2} \left( \| \partial_r u_1^{[n]} \|^2_{E_1, L^2} + \| \partial_r u_2^{[n]} \|^2_{E_2, L^2} \right) \]

74
\[ + \int_{S_r} \int_{\Omega} V_1(x) \left( \partial_{t} u_1^{[n]} \right)^2 + V_2(x) \left( \partial_{t} u_2^{[n]} \right)^2 \, dx \, dt \]

\[ = \int_{S_r} \int_{\Omega} \theta(x) \left( \frac{f_V}{f_V} \right) : \partial_{t} u^{[n]} \, dx \, dt \]

\[ + \frac{1}{2} \left( \| \partial_{x} u_1^{[n]}(0, x) \|_{E_{1},L^2}^2 + \| \partial_{x} u_2^{[n]}(0, x) \|_{E_{2},L^2}^2 \right) \]

\[ + \int_{\Omega} \theta(x) \left( 1 - \frac{1}{n} \right) \partial_{x} u_1^{[n]}(0, x) \partial_{x} u_2^{[n]}(0, x) \, dx \]

\[ + \frac{1}{2} \left( \| \partial_{x} u_1^{[n]}(0, x) \|_{E_{1},L^2}^2 + \| \partial_{x} u_2^{[n]}(0, x) \|_{E_{2},L^2}^2 \right) \]

\[ - \int_{\Omega} \theta(x) \left( 1 - \frac{1}{n} \right) \partial_{x} u_1^{[n]}(\tau, x) \partial_{x} u_2^{[n]}(\tau, x) \, dx \]

\[ - \int_{S_r} \int_{\Omega} E_{12}(x) \left( \partial_{x} u_1^{[n]} \partial_{x} u_2^{[n]} + \partial_{x} u_2^{[n]} \partial_{x} u_1^{[n]} \right) \, dx \, dt. \]

Again, in order to estimate the terms, we look into each side of equation (3.3.72) separately. Hereby, the constants

(3.3.76) \hspace{1cm} k_{E_1,E_2}, k_{E_1,\epsilon_2}, k_{V_1,V_2}, K_f, K_{S_r}^e, K_{12,1}, K_{12,2}, K_{S_r,E_{12}}, \tilde{C}, \tilde{K}_0 > 0

are fixed, positive and independent of \( n \) and

(3.3.77) \hspace{1cm} \epsilon_f, \epsilon_{E_{12}}, \epsilon_{12}, \epsilon_1, \epsilon_2 > 0

arbitrary positive, of \( n \) independent constants. On the left side due to Poincaré’s inequality, it holds

(3.3.78) \hspace{1cm} \frac{1}{2} \left( \| \partial_{x} u_1^{[n]} \|_{E_{1},L^2}^2 + \| \partial_{x} u_2^{[n]} \|_{E_{2},L^2}^2 \right) \geq k_{E_1,E_2}^e \left( \| \partial_{x} u_1^{[n]} \|_{L^2}^2 + \| \partial_{x} u_2^{[n]} \|_{L^2}^2 \right) \]

\[ \geq k_{E_1,E_2} \| u^{[n]} \|_{W^{1,2}}^2 \]

for the elasticity terms and for the viscosity terms

(3.3.79) \hspace{1cm} \int_{S_r} \int_{\Omega} V_1(x) \left( \partial_{x} u_1^{[n]} \right)^2 + V_2(x) \left( \partial_{x} u_2^{[n]} \right)^2 \, dx \, dt \geq k_{V_1,V_2} \int_{S_r} \| \partial_{t} u^{[n]} \|_{W^{1,2}}^2 \, dt.

On the right side, the terms are estimated by Young’s and Poincaré’s inequalities,

(3.3.80) \hspace{1cm} \int_{S_r} \int_{\Omega} \theta(x) \left( \frac{f_V}{f_V} \right) : \partial_{t} u^{[n]} \, dx \, dt

\[ \leq \int_{S_r} \int_{\Omega} \left( \frac{1}{4\epsilon_f} + \frac{1}{4\epsilon_f} \right) |\theta(x) f_V(t, x)|^2 \, dx \, dt + \int_{S_r} \int_{\Omega} \epsilon_f |\partial_{t} u_1^{[n]}|^2 + \epsilon_f |\partial_{t} u_2^{[n]}|^2 \, dx \, dt \]

\[ \leq \frac{1}{2\epsilon_f} \int_{S_r} \| f_V \|_{E_{1},L^2}^2 \, dt + \epsilon_f \int_{S_r} \| \partial_{t} u \|_{L^2}^2 \, dt \]

\[ \leq \frac{1}{2\epsilon_f} \int_{S_r} \| f_V \|_{E_{1},L^2}^2 \, dt + \epsilon_f K_f \int_{S_r} \| \partial_{t} u \|_{W^{1,2}}^2 \, dt. \]
For the initial conditions holds by assumption due to (3.3.44)

\[
K_0 := \sup_n \frac{1}{2} \left( \| \partial_x u_1^n(0, x) \|_{\Omega, L^2}^2 + \| \partial_x u_2^n(0, x) \|_{\Omega, L^2}^2 \right)
+ \int_\Omega \varphi(x) \left( 1 - \frac{1}{n} \right) \partial_x u_1^n(0, x) \partial_x u_2^n(0, x) \, dx
+ \frac{1}{2} \left( \| \partial_x u_1^n(0, x) \|_{E_1, L^2}^2 + \| \partial_x u_2^n(0, x) \|_{E_2, L^2}^2 \right)
\leq \infty.
\]

The terms resulting of the regularization are estimated by Young’s inequality

\[
\left| - \int_\Omega \varphi(x) \left( 1 - \frac{1}{n} \right) \partial_x u_1^n(\tau, x) \partial_x u_2^n(\tau, x) \, dx \right| \leq \frac{1}{2} \left( \| \partial_x u_1^n \|_{\Omega, L^2}^2 + \| \partial_x u_2^n \|_{\Omega, L^2}^2 \right).
\]

Using Cauchy-Schwarz’s and Young’s inequalities, we obtain for the terms resulting from the coupling

\[
\left| - \int_{S_r} \int_\Omega E_{12}(x) \left( \partial_x u_1^n \partial_x u_2^n + \partial_x u_2^n \partial_x u_1^n \right) \, dx dt \right|
\leq K_{S_r}^* \| E_{12}(x) \|_{L^\infty} \int_{S_r} \int_\Omega \left| \partial_x u_1^n \right| \left| \partial_x u_2^n \right| \, dx dt
\leq K_{S_r,E_{12}}^* \int_{S_r} \int_{\Omega} \frac{1}{4\varepsilon_1} \left| \partial_x u_1^n \right|^2 + \frac{1}{4\varepsilon_2} \left| \partial_x u_2^n \right|^2 + \varepsilon_2 \left| \partial_x u_2^n \right|^2 + \varepsilon_1 \left| \partial_x u_1^n \right|^2 \, dx dt
\leq K_{S_r,E_{12}}^* \int_{S_r} \int_{\Omega} \| u^n \|_{W^{1,2}}^2 \, dt + K_{S_r,E_{12}} \int_{S_r} \| \partial_t u^n \|_{W^{1,2}}^2 \, dt
\leq \frac{K_{12,1}}{\varepsilon_1} \int_{S_r} \int_{\Omega} \| u^n \|_{W^{1,2}}^2 \, dt + K_{12,2} \varepsilon_1 \int_{S_r} \| \partial_t u^n \|_{W^{1,2}}^2 \, dt.
\]

From (3.3.75) to (3.3.83), it follows the inequality

\[
\frac{1}{2} \left( \| \partial_x u_1^n \|_{\Omega, L^2}^2 + \| \partial_x u_2^n \|_{\Omega, L^2}^2 \right)
+ k_{E_1,E_2} \| u^n \|_{W^{1,2}}^2 + k_{V_1,V_2} \int_{S_r} \| \partial_t u^n \|_{W^{1,2}}^2 \, dt
\leq \frac{1}{2} \left( \| \partial_x u_1^n \|_{\Omega, L^2}^2 + \| \partial_x u_2^n \|_{\Omega, L^2}^2 \right)
+ \frac{1}{2\varepsilon_1} \int_{S_r} \| f v \|_{\Omega, L^2}^2 \, dt + K_0
+ \varepsilon_1 k_{F_1} \int_{S_r} \| \partial_t u^n \|_{W^{1,2}}^2 \, dt
+ \frac{K_{12,1}}{\varepsilon_2} \int_{S_r} \| u^n \|_{W^{1,2}}^2 \, dt + K_{12,2} \varepsilon_1 \int_{S_r} \| \partial_t u^n \|_{W^{1,2}}^2 \, dt.
\]
Choosing \( \varepsilon_f \) and \( \varepsilon_{12} \) such that \( k := k_{V_1,V_2} - (\varepsilon_f K_f + K_{12,2} \varepsilon_{12}) > 0 \), we obtain

\[
(3.3.85) \quad k_{E_1,E_2} \| u^n \|_{W^{1,2}(\Omega)}^2 + k \int_{S_r} \| \partial_t u^n \|_{W^{1,2}(\Omega)}^2 dt \leq \frac{K_{12,1}}{\varepsilon_{12}} \int_{S_r} \| u^n \|_{W^{1,2}(\Omega)}^2 dt + \frac{1}{2 \varepsilon_f} \int_{S_r} \| f \|_{L^2} dt + K_0.
\]

Here, we can apply Gronwall's inequality so that with

\[
C := \frac{K_{12,1}}{\varepsilon_{19} k_{E_1,E_2}}
\]

the estimate

\[
(3.3.86) \quad \| u^n \|_{W^{1,2}(\Omega)}^2 \leq \left( \frac{1}{2 \varepsilon_f} \int_{S_r} \| f \|_{L^2}^2 dt + K_0 \right) \exp(\tau \cdot C) =: \bar{C}(\tau)
\]

holds. This leads to

\[
(3.3.87) \quad \| u^n \|_{L^\infty(S,V)}^2 \leq \bar{C} \left( \frac{1}{2 \varepsilon_f} \| f \|_{L^2}^2 + \bar{K}_0 \right) \exp(T \cdot C)
\]

and, hence, \( u^n \in [L^\infty(S,V)]^2 \) as well as

\[
(3.3.88) \quad C_1 := \sup_n \| u^n \|_{L^\infty(S,V)}^2 < \infty.
\]

Moreover, it yields by (3.3.85) and (3.3.86)

\[
(3.3.89) \quad \int_{S_r} \| \partial_t u^n \|_{W^{1,2}(\Omega)}^2 dt \leq \frac{K_{12,1}}{\varepsilon_{12}} \int_{S_r} \bar{C}(t) dt + \frac{1}{2 \varepsilon_f} \int_{S_r} \| f \|_{L^2} dt + K_0 < \bar{C}^* < \infty.
\]

Therefore, we obtain \( \partial_t u^n \in [L^2(S,V)]^2 \) and

\[
(3.3.90) \quad C_2 := \sup_n \| \partial_t u^n \|_{L^2(S,V)}^2 < \infty.
\]

**Step 3 – Limiting process \( n \to \infty \)**

The estimates (3.3.88) for \( u^n \) as well as (3.3.90) for \( \partial_t u^n \) ensure the existence of weak* (weak) convergent subsequences \( u_{nk} \) and \( \partial_t u_{nk} \) with

\[
(3.3.91) \quad u_{nk} \rightharpoonup^* u \quad \text{in} \ [L^\infty(S,V_0)]^2, \quad \partial_t u_{nk} \rightharpoonup^* \partial_t u \quad \text{in} \ [L^2(S,V_0)]^2
\]

for \( n_k \to \infty \). We multiply (3.3.55) by arbitrary functions \( \varphi \in [C^1(S)]^2 \) with \( \varphi(T) = 0 \) and obtain

\[
(3.3.93) \quad \int_{\Omega} \left( \rho(x) \left( 1 - \frac{1}{n_k} \right) \partial_t u_{nk}(t,x) \right) \cdot \varphi \left( \frac{w_j}{w_j} \right) dx + \int_{\Omega} P u_{nk}(t,x) \cdot \varphi(x) \partial_x \left( \frac{w_j}{w_j} \right) dx
\]
Therefore, we consider, for all $j$

\[ 3.3.96 \]

3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

We integrate over $S$ and perform an integration by parts in the first integral such that

\[ 3.3.97 \]

\[ 3.3.98 \]

Analogously, we obtain

\[ 3.3.99 \]

The limiting process for $n_k \to \infty$ in (3.3.94) is permitted and provides

\[ 3.3.100 \]

It remains to prove

\[ 3.3.101 \]

Therefore, we consider, for all $j \in \{1, \ldots, n\}$,

\[ 3.3.102 \]

\[ 3.3.103 \]

\[ 3.3.104 \]

Analogously, we obtain

\[ 3.3.105 \]
3. Viscoelasticity within the Framework of MM Models

\[
- \int_S \int_\Omega \varrho(x) \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} u(t,x) \partial_t \varphi(t), \begin{pmatrix} w_j \\ w_j \end{pmatrix} \, dx \, dt - \int_\Omega \varrho(x) \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} u^{(0)}(x) \varphi(0), \begin{pmatrix} w_j \\ w_j \end{pmatrix} \, dx.
\]

In (3.3.97) the weak* limiting process for \( n_k \to \infty \) is permitted and it follows

\[
\int_\Omega \left( 1 - \frac{1}{n_k} \right) u^{[n_k]}(0), \varphi(0), \begin{pmatrix} w_j \\ w_j \end{pmatrix} \, dx \to \int_\Omega \left( 1 - \frac{1}{n_k} \right) u^{(0)}, \varphi(0), \begin{pmatrix} w_j \\ w_j \end{pmatrix} \, dx.
\]

With density arguments, we can now conclude that

\[
\left( 1 - \frac{1}{n_k} \right) \int_\Omega u^{[n_k]}(0), \varphi, \begin{pmatrix} w_j \\ w_j \end{pmatrix} \, dx \to \int_\Omega u^{(0)}, \varphi, \begin{pmatrix} w_j \\ w_j \end{pmatrix} \, dx.
\]

holds. Since additionally \( u^{[n_k]}(0, x) \to u(0, x) \) and by assumption \( u^{[n_k]}(0) \to u(0) \) for \( n_k \to \infty \) hold, it yields

\[
(3.3.101)
\]

**Step 4 – Uniqueness of the solution**

Let \( v^1 \) and \( v^2 \) be solutions of the Problem 3.3.1 and \( v := v^1 - v^2 \). Then

\[
v(0, x) = 0, \quad \partial_t v(0, x) = 0
\]

holds. Due to the linearity of \( P \), it yields for all \( \varphi \in V_0 \)

\[
- \int_S \int_\Omega \varrho(x) \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \partial_t v : \partial_t \varphi \, dx \, dt + \int_S \int_\Omega (E \partial_x v + V \partial_x v) : \partial_x \varphi \, dx \, dt = 0.
\]

Let \( \chi_s \) be the characteristic function of the interval \([0, s]\) with \( 0 \leq s \leq T \). We define

\[
w_i(t, x) := \int_0^t \chi_s(\tau)v_i(\tau, x) \, d\tau + c_i, \quad i = 1, 2.
\]

The constants \( c_i \) are chosen such that \( w_i(T, x) = 0 \) and then

\[
w_i(t, x) = 0, \quad \text{for } s \leq t \leq T,
\]

\[
w_i \in AC([0, T], V),
\]

\[
\partial_t w_i(t, x) = \chi_s(t)v_i(t, x), \quad \text{almost everywhere}.
\]

We test (3.3.103) by \( w \) and obtain with positive constants \( K_E, c_V \)

\[
(3.3.106a) \quad - \int_S \int_\Omega \varrho(x) \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & \frac{1}{2} \end{pmatrix} \partial_t v : \partial_t w \, dx \, dt
\]

\[
= - \int_S \int_\Omega \varrho(x)(\partial_t v_1 \partial_t w_1 + \partial_t v_2 \partial_t w_2 + \partial_t v_1 \partial_t w_2 + \partial_t v_2 \partial_t w_2) \, dx \, dt
\]
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

\[
\begin{align*}
&= - \int_0^s \int_\Omega \varrho(x)(\partial_t v_1 v_1 + \partial_t v_1 v_2 + \partial_t v_2 v_2) \, dx \, dt \\
&= - \int_0^s \int_\Omega \varrho(x)(\partial_t v_1 + \partial_t v_2)^2 \, dx \, dt \\
&\leq 0,
\end{align*}
\]

\((3.3.106b)\)

\[
\int_S \int_\Omega E \partial_x v : \partial_x w \, dx \, dt
\]

\[
= \int_S \int_\Omega E_1 \partial_x v_1 \partial_x w_1 + E_2 \partial_x v_2 \partial_x w_2 \, dx \, dt
\]

\[
= \int_0^s \int_\Omega E_1 \partial_x v_1 \partial_x w_1 + E_2 \partial_x v_2 \partial_x w_2 \, dx \, dt
\]

\[
\leq K_E \int_0^s \|v(t, x)\|^2_{[W^{1,2}]} \, dt,
\]

\((3.3.106c)\)

\[
\int_S \int_\Omega V \partial_x v : \partial_x w \, dx \, dt
\]

\[
= \int_S \int_\Omega V_1 \partial_x v_1 \partial_x w_1 + V_2 \partial_x v_2 \partial_x w_2 \, dx \, dt
\]

\[
= - \int_0^s \int_\Omega \frac{1}{2} V_1 \partial_t (\partial_x v_1)^2 + \frac{1}{2} V_2 \partial_t (\partial_x v_2)^2 \, dx \, dt
\]

\[
\leq -c_V \|v(t, x)\|^2_{[W^{1,2}]}.
\]

Summarizing, it follows

\((3.3.107)\)

\[
\int_0^s \int_\Omega \varrho(x)(\partial_t v_1 + \partial_t v_2)^2 \, dx \, dt + c_V \|v(t, x)\|^2_{[W^{1,2}]} \leq K_E \int_0^s \|v(t, x)\|^2_{[W^{1,2}]} \, dt
\]

and by Gronwall’s inequality \(v = 0\) and, therefore, the uniqueness of the solution.

Since \(u \in [L^\infty(S, V_0)]^2\) and \(\partial_t u \in [L^2(S, V_0)]^2\) hold by [55, p.106, Proposition 1.2] it yields \(u \in [C(S, V_0)]^2\).
3. Viscoelasticity within the Framework of MM Models

Extra We obtain by (3.3.74)

\[ (3.3.108) \]
\[
\int_{S} \int_{\Omega} E \partial_{\tau} u^{[n]} : \partial_{\tau} u^{[n]} + V \partial_{\tau} u^{[n]} : \partial_{\tau} u^{[n]} \, dx \, dt
\]
\[
= \int_{S} \Omega \int_{1}^{E} \left( E_{1}(x) \partial_{\tau} u_{1}^{[n]} \partial_{\tau} u_{1}^{[n]} + E_{2}(x) \partial_{\tau} u_{2}^{[n]} \partial_{\tau} u_{2}^{[n]} + E_{12}(x) \left( \partial_{\tau} u_{1}^{[n]} \partial_{\tau} u_{2}^{[n]} + \partial_{\tau} u_{2}^{[n]} \partial_{\tau} u_{1}^{[n]} \right) \right) \, dx \, dt
\]
\[
= \frac{1}{2} \int_{\Omega} E_{1}(\partial_{\tau} u_{1}^{[n]}(\tau))^{2} + E_{2}(\partial_{\tau} u_{2}^{[n]}(\tau))^{2} + 2E_{12} \partial_{\tau} u_{1}^{[n]}(\tau) \partial_{\tau} u_{2}^{[n]}(\tau) \, dx
\]
\[
- \frac{1}{2} \int_{\Omega} E_{1}(\partial_{\tau} u_{1}^{[n]}(0))^{2} + E_{2}(\partial_{\tau} u_{2}^{[n]}(0))^{2} + 2E_{12} \partial_{\tau} u_{1}^{[n]}(0) \partial_{\tau} u_{2}^{[n]}(0) \, dx.
\]

Due to the additional constraint (3.3.49), it holds

\[ (3.3.109) \]
\[
\frac{1}{2} \int_{\Omega} E_{1}(\partial_{\tau} u_{1}^{[n]}(\tau))^{2} + E_{2}(\partial_{\tau} u_{2}^{[n]}(\tau))^{2} + 2E_{12} \partial_{\tau} u_{1}^{[n]}(\tau) \partial_{\tau} u_{2}^{[n]}(\tau) \, dx \geq c_{E} \| u^{[n]} \|^{2}_{W^{1,2}}.
\]

For \( \varepsilon_{f} > 0 \) and positive constants \( k_{V_{1},V_{2}}, K_{f} \) independently of \( n \), it follows

\[ (3.3.110) \]
\[
\frac{1}{2} \left( \| \partial_{\tau} u_{1}^{[n]} \|_{L^{2}}^{2} + \| \partial_{\tau} u_{2}^{[n]} \|_{L^{2}}^{2} \right) + c_{E} \| u^{[n]} \|^{2}_{W^{1,2}} + k_{V_{1},V_{2}} \int_{S_{r}} \| \partial_{\tau} u^{[n]} \|^{2}_{W^{1,2}} \, dt
\]
\[
\leq \frac{1}{2} \left( \| \partial_{\tau} u_{1}^{[n]} \|_{L^{2}}^{2} + \| \partial_{\tau} u_{2}^{[n]} \|_{L^{2}}^{2} \right) + \frac{1}{2\varepsilon_{f}} \int_{S_{r}} \| f_{V} \|^{2}_{L^{2}} \, dt + K_{0} + \varepsilon_{f} K_{f} \int_{S_{r}} \| \partial_{\tau} u^{[n]} \|^{2}_{W^{1,2}} \, dt.
\]

Analogously to the procedures of Step 2 and 3, we obtain the estimate

\[ (3.3.111) \]
\[
\| u \|_{L^{\infty}(S,V_{0})^{2}} + \| \partial_{\tau} u \|_{L^{2}(S,V_{0})^{2}} \leq C \left( \| u^{(0)} \|_{V_{0}^{2}} + \| u^{(1)} \|_{H^{2}} + \| f \|_{H^{2}} \right),
\]

where the constant \( C \) does not dependent on the length of the time interval. \( \square \)

Remark 3.3.5. The estimate (3.3.111) implies that there will be no ratcheting effects in regular case \( (E_{12} < E_{1}E_{2}) \).

In practice, more general boundary conditions as inhomogeneous one are of interest. In the following two sections, we consider two special problems which are applicable to describe displacement and stress driven tension and compression experiments.

3.3.3.2. Inhomogeneous Dirichlet Boundary Value Problem

Considering displacement driven experiments for a rod, cf. Fig. 3.7(b), we obtain the mathematical Problem 3.3.6.

Problem 3.3.6. Let \( S := [0,T], \, \Omega := [0,t], \, P \) and \( f \) be as in (3.3.13). Find \( u : S \times \Omega \to \mathbb{R}^{2} \) such that for given \( g, E_{1}, E_{2}, E_{12}, V_{1}, V_{2} \) and \( f_{V} \)

\[ (3.3.112) \]
\[
g(x) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \partial_{\tau} u(t,x) - \partial_{x} Pu(t,x) = g(x) f(t,x), \quad \text{in } S \times \Omega,
\]
is fulfilled for given initial condition $u^{(0)}$ and $u^{(1)}$

\begin{equation}
\tag{3.3.113a}
  u(0, x) = u^{(0)}(x), \quad \partial_t u(0, x) = u^{(1)}(x), \quad \text{in} \ \Omega,
\end{equation}

and given mixed boundary conditions, homogeneous boundary conditions at the left edge ($x = 0$)

\begin{equation}
\tag{3.3.113b}
  u(t, x) = 0, \quad \text{on} \ S \times \{0\},
\end{equation}

and for given $u^R$ inhomogeneous boundary conditions at the right edge ($x = l$)

\begin{equation}
\tag{3.3.113c}
  u(t, x) = u_1(t, l) + u_2(t, l) = u^R(t), \quad \text{on} \ S \times \{l\}.
\end{equation}

$u^R$ is a predetermined displacement at the right edge. The distribution of the predetermined displacement to both partial displacements at the edge causes problems. In order to prove the following theorem of existence and uniqueness of a weak solution of the problem, we predetermine both partial displacements $u_1$ and $u_2$ by the boundary displacements $u^R_1$ and $u^R_2$—note, then in general, the condition $\sigma = \sigma_1 = \sigma_2$ (cf. (3.3.6)) is violated. We define

\begin{equation}
\tag{3.3.113d}
  u_i(t, x) = u^R_i(t), \quad i \in \{1, 2\}, \quad \text{on} \ S \times \{l\}.
\end{equation}

For the treatment of Problem 3.3.6 concerning (3.3.113c), we refer to Section 3.3.3.4.

Setting 3.3.7.

Time & Space:

\begin{equation}
\tag{3.3.114}
  S := [0, T] \subset \mathbb{R}, \quad T > 0 \quad \text{(time)}, \quad \Omega := [0, l] \subset \mathbb{R}, \quad l > 0 \quad \text{(space)}
\end{equation}

Parameter (functions):

\begin{align}
\tag{3.3.115a}
  \rho & \in L^\infty(\Omega), \quad \exists \rho_0 \text{ f.a.a. } x \in \Omega : \quad \rho(x) > \rho_0 \quad \text{(mass density)} \\
\tag{3.3.115b}
  E_i & \in L^\infty(\Omega), \quad \exists E_{0,i} \text{ f.a.a. } x \in \Omega : \quad E_i(x) > E_{0,i}, \quad i = 1, 2 \quad \text{(elasticities)} \\
\tag{3.3.115c}
  V_i & \in L^\infty(\Omega), \quad \exists V_{0,i} \text{ f.a.a. } x \in \Omega : \quad V_i(x) > V_{0,i}, \quad i = 1, 2 \quad \text{(viscosities)} \\
\tag{3.3.115d}
  E_{12} & \in L^\infty(\Omega) \quad \text{(coupling coefficient)}
\end{align}

Function spaces:

\begin{align}
\tag{3.3.116a}
  V & := W^{1,2}(\Omega), \quad V_0 := \{ v \in W^{1,2}(\Omega) | v(0) = 0 \}, \quad H := L^2(\Omega) \\
\tag{3.3.116b}
  V & := L^2(S, V), \quad V_0 := L^2(S, V_0), \quad V_0 := L^2(S, V_0, L), \quad H := L^2(S, H)
\end{align}

Data:

\begin{equation}
\tag{3.3.117}
  f \in H^2, \quad u^{(0)} \in V_0^2, \quad u^{(1)} \in H^2, \quad u^R_1, \quad u^R_2 \in C^2(S)
\end{equation}

Theorem 3.3.8. Consider Setting 3.3.7, then there exists a unique weak solution $u$ of the Problem 3.3.6 with $u \in [C(S, V_0)]^2 \cap [L^\infty(S, V_0)]^2$.

Proof. It holds at the right edge

\begin{equation}
\tag{3.3.118}
  u_i(t, l) = u^R_i(t), \quad i = 1, 2.
\end{equation}

We apply the principle of superposition. Therefore, let be

\begin{equation}
\tag{3.3.119}
  u_i(t, x) = v_i(t, x) + w_i(t, x), \quad i = 1, 2,
\end{equation}

where $w := (w_1, w_2)^T$ takes the boundary condition of the problem into account by homogeniza-
3. Viscoelasticity within the Framework of MM Models

\[ w_i = \frac{x}{l} u_i^R(t), \quad i = 1, 2 \]

and \( v := (v_1, v_2)^T \) with

\[ v = 0 \text{ on } \partial \Omega \]

is the solution of the homogeneous Dirichlet boundary value problem

\[ \varrho(x) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \partial_{tt} v(t, x) - \partial_x P v(t, x) = \tilde{f} \]

with the initial conditions

\[ v(0, x) = v^{(0)}(x) := u^{(0)}(x) - \begin{pmatrix} \frac{x}{l} u_1^R(0) \\ \frac{x}{l} u_2^R(0) \end{pmatrix}, \]

\[ \partial_t v(0, x) = v^{(1)}(x) := u^{(1)}(x) - \partial_t \begin{pmatrix} \frac{x}{l} u_1^R(0) \\ \frac{x}{l} u_2^R(0) \end{pmatrix} \]

and

\[ \tilde{f} := \varrho(x) f - \varrho(x) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \partial_{tt} w(t, x) + \partial_x P w(t, x). \]

Since \( u^R \in C^2(S) \) holds, it yields \( \tilde{f} \in [L^2(S, H)]^2 \). By Theorem 3.3.4, there exists for the problem (3.3.122) a unique weak solution \( v \). \( v + w \) is obviously a weak solution of Problem 3.3.6 since formally

\[ \varrho(x) \partial_{tt} u - \partial_x P u = \varrho(x) \partial_{tt} v - \partial_x P v + \varrho(x) \partial_{tt} w - \partial_x P w \]

\[ = f - \varrho(x) \partial_{tt} w + \partial_x P w + \varrho(x) \partial_{tt} w - \partial_x P w = f \]

is valid.

3.3.3.3. Mixed Boundary Value Problem

Stress driven deformation experiments on a rod, cf. Fig. 3.9, are modeled by mixed boundary conditions, thus Problem 3.3.9 is applicable.

Problem 3.3.9. Let \( S := [0, T], \Omega := ]0, l[ \), \( P \) and \( f \) be as in (3.3.13). Find \( u : S \times \Omega \to \mathbb{R}^2 \) such that for given \( \varrho, E_1, E_2, E_{12}, V_1, V_2 \) and \( f \)

\[ \varrho(x) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \partial_{tt} u(t, x) - \partial_x P u(t, x) = \varrho(x) f(t, x), \quad \text{in } S \times \Omega, \]

is fulfilled for given initial condition \( u^{(0)} \) and \( u^{(1)} \)

\[ u(0, x) = u^{(0)}(x), \quad \partial_t u(0, x) = u^{(1)}(x), \quad \text{in } \Omega, \]

and given mixed boundary conditions, homogeneous boundary conditions at the left edge \( (x = 0) \)

\[ u(t, x) = 0, \quad \text{on } S \times \{0\}, \]
and for given $\sigma^R$ Robin boundary conditions at the right edge ($x = l$)

\[(3.3.126c) \quad Pu(t, l) = -E^R u(t, l) - V^R \partial_t u(t, l) + \sigma^R(t), \quad \text{on } S \times \{l\}.\]

Hereby, let be $\sigma^R = (\sigma^R, \sigma^R)^T$ and $\sigma^R$ a given stress. $E^R$ and $V^R$ describe an attached viscoelastic body at the right edge (cf. Section 3.3.2 [Robin Boundary Conditions]). If $E^R = V^R = 0$ holds, we obtain inhomogeneous Neumann boundary conditions.

Moreover, we define the Setting 3.3.10.

**Setting 3.3.10.**

**Time & Space:**

\[(3.3.127) \quad S := ]0, T[ \subset \mathbb{R}, \; T > 0 \; \text{(time)}, \quad \Omega := ]0, l[ \subset \mathbb{R}, \; l > 0 \; \text{(space)}\]

**Parameter (functions):**

\[(3.3.128a) \quad \varrho \in L^\infty(\Omega), \quad \exists \varrho_0 \text{ f.a.a. } x \in \Omega : \varrho(x) > \varrho_0 \quad \text{(mass density)}\]

\[(3.3.128b) \quad E_i \in L^\infty(\Omega), \quad \exists E_{0,i} \text{ f.a.a. } x \in \Omega : E_i(x) > E_{0,i}, \; i = 1, 2 \quad \text{(elasticities)}\]

\[(3.3.128c) \quad V_i \in L^\infty(\Omega), \quad \exists V_{0,i} \text{ f.a.a. } x \in \Omega : V_i(x) > V_{0,i}, \; i = 1, 2 \quad \text{(viscosities)}\]

\[(3.3.128d) \quad E_{12} \in L^\infty(\Omega) \quad \text{(coupling coefficient)}\]

**Function spaces:**

\[(3.3.129a) \quad V := W^{1,2}(\Omega), \quad V_0 := \{v \in W^{1,2}(\Omega) | v(0) = 0\}, \quad H := L^2(\Omega)\]

\[(3.3.129b) \quad V := L^2(\Omega), \quad V_0 := L^2(S, V_0), \quad \mathcal{H} := L^2(S, H)\]

**Data:**

\[(3.3.130) \quad f \in \mathcal{H}^2, \; u^{(0)} \in V_0^2, \; u^{(1)} \in H^2, \; \sigma^R \in [L^2(S)]^2\]

We modify the solution concept of Definition 3.3.3.

**Definition 3.3.11.** Consider Setting 3.3.10. $u \in V_0^2$ is a weak solution of Problem 3.3.9 if and only if $\partial_t u \in V_0^2$ and for all $\varphi \in V_0^2$ with $\partial_t \varphi \in \mathcal{H}^2$ and $\varphi(T) = 0$,

\[(3.3.131) \quad -\int_S \int_\Omega \left( \varrho(x) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \partial_t u(t, x) \right) \cdot \partial_t \varphi(t, x) \, dx \, dt + \int_S \int_\Omega Pu(t, x) \cdot \partial_x \varphi(t, x) \, dx \, dt - \int_S Pu(t, l) \cdot \varphi(t, l) \, dt \]

\[= \int_S \int_\Omega \varrho(x) f \cdot \varphi(t, x) \, dx \, dt + \int_\Omega \left( \varrho(x) \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u^{(1)}(x) \right) \cdot \varphi(0, x) \, dx\]

as well as $u(0, x) = u^{(0)}(x)$ hold.

**Theorem 3.3.12.** Under the assumptions of Setting 3.3.10, there exists a unique weak solution $u \in [C(S, V_0)]^2 \cap [L^\infty(S, V_0)]^2$ and $\partial_t u \in [L^2(S, V_0)]^2$ of Problem 3.3.9. Moreover, the map

\[(3.3.132) \quad S : (u^{(0)}, u^{(1)}, f, \sigma^R) \mapsto (u, \partial_t u)\]

with

\[(3.3.133) \quad S : V^2 \times H^2 \times \mathcal{H}^2 \times [L^2(S)]^2 \to [L^\infty(S, V_0)]^2 \times V^2\]
In (3.3.72), the terms in (3.3.136) leads to the Galerkin equation (3.3.135) and 

\[
\|u\|_{L^{\infty}(S,V_b)}^2 + \|\partial_t u\|_{V^2} \leq C \left( \|u(0)\|_{V^2} + \|u^{(1)}\|_{H^2} + \|f\|_{H^2} + \|\sigma^R\|_{L^2(S)^2} \right).
\]

**Proof.** The proof of Theorem 3.3.4 can be modified appropriately in some points to be applicable in this situation.

**In Step 1 - Galerkin approximation** we have to consider (3.3.54) the additional terms emerging from (3.3.126c) such that

\[
- Pu^{[n]}(t,L) = E^R u^{[n]}(t,L) + V^R \partial_t u^{[n]}(t,L) - \sigma^R(t)
\]

in (3.3.55) leads to the Galerkin equation

\[
\int_\Omega (g(x)\partial_t \left( \begin{array}{c} 1 \\ 1 - \frac{1}{n} \\ 1 \\ \end{array} \right) u^{[n]}(t,x)) \cdot \left( \begin{array}{c} w_j \\ w_j \\ \end{array} \right) dx + \int_\Omega P u^{[n]}(t,x) \partial_x \left( \begin{array}{c} w_j \\ w_j \\ \end{array} \right) dx 
+ E^R u^{[n]}(t,L) \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right) + V^R \partial_t u^{[n]}(t,L) \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right) - \sigma^R(t) \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right) 
= \int_\Omega g(x) f \cdot \left( \begin{array}{c} w_j \\ w_j \\ \end{array} \right) dx.
\]

Accordingly, the ODE system (3.3.64) is extended to

\[
\left( \begin{array}{ccc} G^0 & G^0 & G^0 \\ G^0 & -\frac{1}{n} G^0 & G^0 \\ G^0 & -\frac{1}{n} G^0 & G^0 \\ \end{array} \right) \partial_t \vec{g} + \left( \begin{array}{cc} G^{E_1} & G^{E_12} \\ G^{E_2} & G^{E_12} \\ \end{array} \right) \vec{g} + \left( \begin{array}{cc} G^{V_1} & 0_{n\times n} \\ 0_{n\times n} & G^{V_2} \\ \end{array} \right) \partial_t \vec{g} 
+ R^E \vec{g} + R^V \partial_t \vec{g} = F + \sigma^R(t) \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right).
\]

with the coefficient matrices $R^E, R^V$ which are determined by the boundary values.

**In Step 2 - A priori estimate** note the following: On the left side of (3.3.70), we add the terms

\[
- Pu^{[n]}(t,L) \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right) = E^R u^{[n]}(t,L) \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right) + V^R \partial_t u^{[n]}(t,L) \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right) - \sigma^R \cdot \left( \begin{array}{c} w_j(l) \\ w_j(l) \\ \end{array} \right)
\]

In (3.3.72), the terms

\[
\int_{S_r} E^R u^{[n]}(t,L) : \partial_t u^{[n]}(t,L) + V^R \partial_t u^{[n]}(t,L) : \partial_t u^{[n]}(t,L) dt 
= \int_{S_r} E^R \left( u^{[n]}_1(t,L) + u^{[n]}_2(t,L) \right) \partial_t u^{[n]}_1(t,L) dt 
+ \int_{S_r} E^R \left( u^{[n]}_1(t,L) + u^{[n]}_2(t,L) \right) \partial_t u^{[n]}_2(t,L) dt 
+ \int_{S_r} V^R \left( \partial_t u^{[n]}_1(t,L) + \partial_t u^{[n]}_2(t,L) \right) \partial_t u^{[n]}_1(t,L) dt
\]

85
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

\[ + \int_{S} V^{R} \left( \partial_{t} u_{1}^{[n]}(t, l) \partial_{t} + u_{2}^{[n]}(t, l) \right) \partial_{t} u_{2}^{[n]}(t, l) \, dt \]

\[ = \int_{S} \frac{E^{R}}{2} \partial_{t} \left( u_{1}^{[n]}(t, l) + v_{2}^{[n]}(t, l) \right)^{2} \, dt \]

\[ + \int_{S} V^{R} \left( \partial_{t} u_{1}^{[n]}(t, l) + \partial_{t} u_{2}^{[n]}(t, l) \right)^{2} \, dt \]

\[ = \frac{E^{R}}{2} \left[ \left( u_{1}^{[n]}(\tau, l) + u_{2}^{[n]}(\tau, l) \right)^{2} - \left( u_{1}^{[n]}(0, l) + u_{2}^{[n]}(0, l) \right)^{2} \right] \]

\[ \geq 0 \text{ to the right side} \]

\[ + \int_{S} V^{R} \left( \partial_{t} u_{1}^{[n]}(t, l) + \partial_{t} u_{2}^{[n]}(t, l) \right)^{2} \, dt \]

have to be added on the left side. Accordingly, these terms are taken into account in the estimate (3.3.84). The terms related to the inhomogeneity,

(3.3.139) \[ \int_{S} \sigma^{R}(t) : \partial_{t} u^{[n]}(t, l) \, dt, \]

are treated on the right side as follows: It holds

(3.3.140) \[ \int_{S} \sigma^{R}(t) : \partial_{t} u^{[n]}(t, l) \, dt - \int_{S} \sigma^{R}(t) : \partial_{t} u^{[n]}(t, 0) \, dt = \int_{S} \int_{\Omega} \sigma^{R}(t) : \partial_{t} u^{[n]}(t, x) \, dx \, dt. \]

Using Young’s special inequality, (3.3.139) is estimated by

(3.3.141) \[ \int_{S} \int_{\Omega} \sigma^{R}(t) : \partial_{t} u^{[n]}(t, x) \, dx \, dt \leq \frac{C}{4 \varepsilon} \| \sigma^{R} \|_{L^{2}(S)}^{2} + c \varepsilon \int_{S} \| \partial_{t} u \|_{W^{1,2}(\Omega)}^{2} \, dt. \]

\( \sigma^{R}(t) \) is assumed to be bounded and the \( \partial_{t} u \)-term can be subtracted from the left side by accordingly choice of \( \varepsilon \).

In Step 3, the limiting process, there are no additional difficulties; the terms have just to be added.

In Step 4 – Uniqueness of the solution the approach also is analogously to the proof of Theorem 3.3.4. The additional terms

(3.3.142) \[ \int_{S} E^{R} v(t, l) : \varphi(t, l) + V^{R} \partial_{t} v(t, l) : \varphi(t, l) \, dt \]

are tested as well by \( w \). We obtain

(3.3.143) \[ \int_{S} E^{R} v(t, l) : w(t, l) \, dt = \int_{0}^{S} E^{R} \partial_{t} w(t, l) : w(t, l) \, dt \]
3. Viscoelasticity within the Framework of MM Models

\[ = \int_0^s \mathbf{E}^R \partial_t (\mathbf{w}(t, l))^2 \, dt = \mathbf{E}^R (\mathbf{w}(s, l))^2 - \mathbf{E}^R (\mathbf{w}(0, l))^2 = -\mathbf{E}^R (\mathbf{w}(0, l))^2 \]

and

\[
(3.3.144) \quad \int_{S} \mathbf{V}^R \partial_t \mathbf{v}(t, l) : \mathbf{w}(t, l) \, dt = - \int_{0}^{s} \mathbf{V}^R \mathbf{v}(t, l) : \mathbf{v}(t, l) \, dt,
\]

so that in (3.3.107) the estimate holds:

\[
(3.3.145) \quad s \int_{0}^{s} \int_{\Omega} \rho(x) (\partial_t v_1 + \partial_t v_2)^2 \, dx \, dt + c \mathbf{V}\|\mathbf{v}(t, x)\|_{W^{1,2}}^2 + \mathbf{E}^R (\mathbf{w}(0, l))^2 \\
+ \int_{0}^{s} \mathbf{V}^R \mathbf{v}(t, l) : \mathbf{v}(t, l) \, dt \leq K_c \int_{0}^{s} \|\mathbf{v}(t, x)\|_{W^{1,2}}^2 \, dt.
\]

The inhomogeneity disappears since by definition \( \mathbf{v} := \mathbf{v}^1 - \mathbf{v}^2 \) holds. By Gronwall's inequality, we obtain analogously to the Dirichlet case the uniqueness of the solution.

**Remark 3.3.13.** The solutions \( \mathbf{u} \) of Problems 3.3.6 and 3.3.12 are bounded in all regular cases \( (E_{13}^2 < E_1 E_2) \) if \( u_i^R, \, i = 1, 2 \), respectively \( \sigma^R \), are bounded, cf. Theorem 3.3.4[Extra]. Especially, ratcheting will not occur in the regular cases for bounded boundary stress \( \sigma^R \) in Problem 3.3.12.

**Remark 3.3.14.** Other combinations of boundary conditions are reasonable as well and the resulting problems are treated, in principle, similarly. In case of pure Neumann boundary conditions, however, Poincaré's inequality is no longer valid so that further difficulties arise.

**Remark 3.3.15.** If the weak solutions of the problems are sufficiently smooth, they are classic solutions too.

### 3.3.3.4. Problem of Consistent Inhomogeneous Dirichlet Boundary Conditions

Concerning problems of a viscoelastic rod with inhomogeneous Dirichlet boundary conditions, it arises the problem that the predetermined boundary displacement \( u_i^R \) has to be divided appropriately into \( u_1(t, l) \) and \( u_2(t, l) \). We consider the situation of Section 3.3.3.2 so that homogeneous Dirichlet boundary conditions hold at the left edge \( (x = 0) \),

\[
(3.3.146) \quad u(t, 0) = 0.
\]

We define, here,

\[
(3.3.147) \quad u_1(t, 0) = 0 \text{ and } u_2(t, 0) = 0,
\]

since the homogeneous boundary conditions should be valid for the partial displacements as well. In principle,

\[
(3.3.148) \quad u_1(t, 0) = -u_2(t, 0)
\]

is reasonable as well. At the right edge \( (x = l) \), we pose inhomogeneous Dirichlet boundary conditions,

\[
(3.3.149) \quad u(t, l) = u^R(t),
\]

3.3.3.4. Problem of Consistent Inhomogeneous Dirichlet Boundary Conditions

Concerning problems of a viscoelastic rod with inhomogeneous Dirichlet boundary conditions, it arises the problem that the predetermined boundary displacement \( u_i^R \) has to be divided appropriately into \( u_1(t, l) \) and \( u_2(t, l) \). We consider the situation of Section 3.3.3.2 so that homogeneous Dirichlet boundary conditions hold at the left edge \( (x = 0) \),

\[
(3.3.146) \quad u(t, 0) = 0.
\]

We define, here,

\[
(3.3.147) \quad u_1(t, 0) = 0 \text{ and } u_2(t, 0) = 0,
\]

since the homogeneous boundary conditions should be valid for the partial displacements as well. In principle,

\[
(3.3.148) \quad u_1(t, 0) = -u_2(t, 0)
\]

is reasonable as well. At the right edge \( (x = l) \), we pose inhomogeneous Dirichlet boundary conditions,

\[
(3.3.149) \quad u(t, l) = u^R(t),
\]
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

for a given \( u^R \). It holds for the partial displacements due to the additive split of the displacement

\[
(3.3.150) \quad u^R(t) = u_1(t, l) + u_2(t, l).
\]

The problem is to find functions \( u_1^R \) and \( u_2^R \) such that

\[
(3.3.151) \quad u^R(t) = u_1^R(t) + u_2^R(t)
\]

with

\[
(3.3.152) \quad u_1(t, l) = u_1^R(t) \quad \text{and} \quad u_2(t, l) = u_2^R(t)
\]

holds. In order to realize the distribution, we choose the approach

\[
(3.3.153) \quad u_1(t, l) = q(t) u^R(t) \quad \text{and} \quad u_2(t, l) = (1 - q(t)) u^R(t)
\]

such that the function \( q \) has to be determined. Considering the material law

\[
(3.3.154) \quad V_1(x) \partial_{tx} u_1(t, x) + E_1(x) \partial_x u_1(t, x) + E_{12}(x) \partial_x u_2(t, x) = \sigma(t, x),
\]

\[
(3.3.155) \quad V_2(x) \partial_{tx} u_2(t, x) + E_{12}(x) \partial_x u_1(t, x) + E_2(x) \partial_x u_2(t, x) = \sigma(t, x),
\]

and subtracting (3.3.155) from (3.3.154), we obtain

\[
(3.3.156) \quad V_1(x) \partial_{tx} u_1(t, x) - V_2(x) \partial_{tx} u_2(t, x) + (E_1(x) - E_{12}(x)) \partial_x u_1(t, x) + (E_{12}(x) - E_2(x)) \partial_x u_2(t, x) = 0.
\]

If the material parameter \( E_1, E_2, E_{12}, V_1, V_2 \) are independent of the space variable \( x \),\(^{10}\) integration over \([0, l], 0 \leq y \leq l\) is permitted and it holds due to the posed homogeneous boundary condition in \( x = 0 \) with \( u_i(t, 0) = 0 \)

\[
(3.3.157) \quad V_1 \partial_t u_1(t, y) - V_2 \partial_t u_2(t, y) + (E_1 - E_{12}) u_1(t, y) + (E_{12} - E_2) u_2(t, y) = 0.
\]

We consider (3.3.157) for \( y = l \), use the approach (3.3.153) and note that in our setting \( u^R \in C^2(S) \) (see (3.3.117)). By inserting, we obtain

\[
(3.3.158) \quad (V_1 + V_2) \partial_t (q(t) u^R(t)) - V_2 \partial_t u^R(t) + (E_1 + E_2 + 2E_{12}) q(t) u^R(t) - (E_2 - E_{12}) u^R(t) = 0.
\]

We define \( g(t) := q(t) u^R(t) \) and obtain

\[
(3.3.159) \quad (V_1 + V_2) \partial_t g(t) - V_2 \partial_t u^R(t) + (E_1 + E_2 + 2E_{12}) g(t) - (E_2 - E_{12}) u^R(t) = 0,
\]

with \( g(0) = q(0) u^R(0) \). The equation (3.3.159) can now be solved (cf. [16, p. 507]). We define

\[
(3.3.160) \quad \lambda := \frac{E_1 + E_2 - 2E_{12}}{V_1 + V_2},
\]

\[
(3.3.161) \quad c := \frac{V_1(E_2 - E_{12}) + V_2(E_{12} - E_1)}{(V_1 + V_2)^2}.
\]

Then, the solution of (3.3.159) is

\[
(3.3.162) \quad g(t) = e^{-\lambda t} \left( g(0) - \frac{V_2}{V_1 + V_2} u^R(0) \right) + \frac{V_2}{V_1 + V_2} u^R(t) + c \int_0^t e^{\lambda(s-t)} u^R(s) ds.
\]

\(^{10}\)If the material parameters depend on the space variable \( x \), the problem stays open.
The original approach was \( g(t) = q(t)u^R(t) \) and we wanted to find \( q(t) \), so

\[
q(t) = \frac{g(t)}{u^R(t)}.
\]

Here, a problem arises if \( u^R \) has zeros \( t_i \in [0, T] \), because \( q(t) \) is then not defined in \( t_i \). We choose the way out by the theorem of l'Hospital. If \( t_i \) is a zero of \( u^R \), then, if \( \partial_t u^R(t_i) \neq 0 \), it holds

\[
q(t_i) = \lim_{t \to t_i} \frac{g(t)}{u^R(t)} = \lim_{t \to t_i} \frac{\partial_t g(t)}{\partial_t u^R(t)}.
\]

It holds

\[
\partial_t g(t) = \frac{V_2}{V_1 + V_2} \partial_t u^R(t) - c\lambda \int_0^t e^{\lambda(s-t)} \partial_s u^R(s) \, ds
\]

and, therefore,

\[
q(t_i) = \frac{V_2}{V_1 + V_2} - c\lambda \frac{1}{\partial_t u^R(t_i)} \int_0^{t_i} e^{\lambda(s-t)} \partial_s u^R(s) \, ds.
\]

Especially, for \( t_i = 0 \), it yields

\[
q(0) = \frac{V_2}{V_1 + V_2}.
\]

\( (3.3.167) \) is important for implementation if \( q \) is determined by \( (3.3.158) \).

### 3.3.4. Nondimensionalization and Scaling of the Mixed Boundary Value Problem

A dimensionless approach of a model offers a good insight into the possible applications of the model, since similar specific model responses will in general appear for different sets of material parameter and space and time domains.\(^{11}\)

We recall exemplarily the model approach for stress driven experiments of Section 3.3.3.3 choosing inhomogeneous Neumann boundary conditions. The problem is given by:

Find \((u_1, u_2)^T : S \times \Omega \to \mathbb{R}^2\) such that for given \( f_V \) and \( \sigma^R \)

\[
\begin{align*}
(3.3.168a) & \quad \rho \partial_t (u_1 + u_2) - \partial_x (E_1 \partial_x u_1 + E_{12} \partial_x u_2 + V_1 \partial_x u_1) = \rho f_V, \quad & \text{in } S \times \Omega, \\
(3.3.168b) & \quad \rho \partial_t (u_1 + u_2) - \partial_x (E_{12} \partial_x u_1 + E_2 \partial_x u_2 + V_2 \partial_x u_2) = \rho f_V, \quad & \text{in } S \times \Omega,
\end{align*}
\]

is fulfilled for the initial conditions

\[
(3.3.168c) \quad u_i(0, x) = 0, \quad \partial_t u_i = 0, \quad i = \{1, 2\}, \quad & \text{in } [0, l],
\]

and mixed boundary conditions – homogeneous Dirichlet boundary conditions at the left edge \((x = 0)\)

\[
(3.3.168d) \quad u_i(t, 0) = 0, \quad i = \{1, 2\}, \quad & \text{in } [0, T],
\]

\(^{11}\)A well-known application of the nondimensionalization is the Navier-Stokes equation and the Reynolds number as an indicator of the model response, cf. [33].
and inhomogeneous Neumann boundary conditions at the right edge \((x = l)\)

\[
\begin{align*}
(3.3.168e) \quad & E_1(l)\partial_x u_1(t, l) + E_{12}\partial_x u_2(t, l) + V_1(l)\partial_t u_1(t, l) = \sigma^R \quad \text{in } S, \\
(3.3.168f) \quad & E_{12}(l)\partial_x u_1(t, l) + E_2\partial_x u_2(t, l) + V_2(l)\partial_t u_1(t, l) = \sigma^R \quad \text{in } S.
\end{align*}
\]

### 3.3.4.1 Nondimensionalization

The first step is to find dimensionless products of the parameters. We assume that the parameters \(\varrho, E_1, E_2, E_{12}, V_1\) and \(V_2\) are independent on space and time. The SI units of the desired quantities and parameters are:

- \([u_i] = m\),
- \([\partial_t u_i] = m/s\),
- \([\partial_{tt} u_i] = m/s^2\),
- \([\partial_x u_i] = m/m\),
- \([\partial_{xx} u_i] = m/m^2\),
- \([\varrho] = kg/m^3\),
- \([E_{ij}] = kg/ms^2\),
- \([V_i] = kg/ms\),
- \([f_V] = m/s^2\),
- \([\sigma_R] = kg/ms^2\),
- \([l] = m\),
- \([T] = s\),
- \([x] = m\),
- \([t] = s\).

The first products can be found by collecting the parameters in groups with the same units, \(\{E_1, E_2, E_{12}\}\) and \(\{V_1, V_2\}\).

We propose to use the fractions

\[
\begin{align*}
\epsilon & := \frac{E_2}{E_1}, \quad \nu := \frac{V_2}{V_1}, \quad p := \frac{E_{12}}{\sqrt{E_1 E_2}}.
\end{align*}
\]

The choice of \(p\) is not totally arbitrary here. \(p\) is related to the determinate of the matrix

\[
\begin{pmatrix}
E_1 & E_{12} \\
E_{12} & E_2
\end{pmatrix}.
\]

Because of thermomechanical reasons the restrictions, \(E_{12}^2 \leq E_1 E_2\) has to be fulfilled. This results in \(p \in [-1, 1]\). The process quantities \(f_V\) and \(\sigma_R\) are not considered in the nondimensionalization. In order to find additional dimensionless products, we consider

\[
E_1^{\alpha_1} V_1^{\alpha_2} \varrho^{\alpha_3} t^{\alpha_4} = 1.
\]

If we substitute the parameters by their units, we obtain

\[
(kg)^{\alpha_1 + \alpha_2 + \alpha_3} (m)^{-\alpha_1 - \alpha_2 - 3\alpha_3 + \alpha_4} (s)^{-2\alpha_1 - \alpha_2 + \alpha_5} = 1.
\]

The equation holds if the equation system

\[
\begin{align*}
(3.3.172a) \quad & \alpha_1 + \alpha_2 + \alpha_3 = 0, \\
(3.3.172b) \quad & -\alpha_1 - \alpha_2 - 3\alpha_3 + \alpha_4 = 0, \\
(3.3.172c) \quad & -2\alpha_1 - \alpha_2 + \alpha_5 = 0
\end{align*}
\]

is fulfilled. We solve for \(\alpha_1, \alpha_4\) and \(\alpha_5\):

\[
\begin{align*}
(3.3.173a) \quad & \alpha_1 = -\alpha_2 - \alpha_3, \\
(3.3.173b) \quad & \alpha_4 = 2\alpha_3, \\
(3.3.173c) \quad & \alpha_5 = -\alpha_2 - 2\alpha_3.
\end{align*}
\]
We set $\alpha_2$ and $\alpha_3$ equal to 1, each of the other equal to 0. We obtain the dimensionless fractions

\[(3.3.174) \quad \eta := \frac{V_1}{E_1 T} \quad (\alpha_2 = 1, \alpha_3 = 0) \Rightarrow (\alpha_1 = -1, \alpha_4 = 0, \alpha_5 = -1),\]

\[(3.3.175) \quad E^{-1} := \frac{\rho l^2}{E_1 T^2} \quad (\alpha_2 = 0, \alpha_3 = 1) \Rightarrow (\alpha_1 = -1, \alpha_4 = 2, \alpha_5 = -2).\]

### 3.3.4.2. Scaling

Scaling is necessary to obtain comparable space and time settings. The space domain is usually transformed to a unit domain, here the interval $[0, 1]$. If higher dimensions are considered, we would transform into the unit sphere or square. The time interval is transformed to a unit interval as well. To this end, we introduce the dimensionless variables

\[(3.3.176) \quad y := \frac{x}{l} \quad \text{and} \quad s := \frac{t}{T}.\]

Moreover, let

\[(3.3.177) \quad \tilde{u}_i(t, x) := \frac{u_i(t, x)}{l} \quad \text{and} \quad v_i(s, y) := \frac{\tilde{u}_i(s \cdot T, y \cdot l)}{l}.\]

Exemplarily, we obtain for the partial derivative

\[(3.3.178) \quad \partial_t u_i(t, x) = l \partial_t \tilde{u}_i(t, x) = l \partial_s v(s, y) \frac{\partial v}{\partial t} = \frac{l}{T} \partial_s v(s, y).\]

### 3.3.4.3. Nondimensionalized and Scaled Problem

After some calculations, we obtain the transformed problem:

Find $(v_1, v_2)^T : [0, 1] \times [0, 1]$ such that for given $f_V$ and $\sigma^R$

\[(3.3.179a) \quad \partial_{ss}(v_1 + v_2) - \left( \frac{E_1 T^2}{l^2 \varrho} \partial_{yy} v_1 + \frac{E_{12} T^2}{l^2 \varrho} \partial_{yy} v_2 + \frac{V_1 T}{l^2 \varrho} \partial_{syy} v_1 \right) = \frac{T^2}{l} \varrho f_V(sT, yl),
\]

in $]0, 1]^2$,

\[(3.3.179b) \quad \partial_{ss}(v_1 + v_2) - \left( \frac{E_{12} T^2}{l^2 \varrho} \partial_{yy} v_1 + \frac{E_2 T^2}{l^2 \varrho} \partial_{yy} v_2 + \frac{V_2 T}{l^2 \varrho} \partial_{syy} v_2 \right) = \frac{T^2}{l} \varrho f_V(sT, yl),
\]

in $]0, 1]^2$

is fulfilled for the initial conditions

\[(3.3.179c) \quad v_i(0, y) = 0, \quad \partial_y v_i(0, y) = 0, \quad i = \{1, 2\}, \quad \text{in } [0, 1],\]

and mixed boundary conditions - homogeneous Dirichlet boundary conditions at the left edge $(y = 0)$

\[(3.3.179d) \quad v_i(s, 0) = 0, \quad i = \{1, 2\}, \quad \text{in } [0, 1],\]

and inhomogeneous Neumann boundary conditions at the right edge $(y = 1)$

\[(3.3.179e) \quad \frac{E_1 T^2}{l^2 \varrho} \partial_y v_1(s, 1) + \frac{E_{12} T^2}{l^2 \varrho} \partial_y v_2(s, 1) + \frac{V_1 T}{l^2 \varrho} \partial_{sy} v_1(s, 1) = \frac{T^2}{l^2 \varrho} \sigma^R, \quad \text{in } [0, 1],\]
implem ent the pure displacemen t formulation, cf. [37] and Problem 3.3.16.

We follow the nondimensional approach of Section 3.3.4, but we will not
there are several possibilities to implement the PDE system (3.3.12) and its initial and b oundary-
conditions. W e will follo w the nondimensionalized approach of Section 3.3.4, but we will not

3.3. Isothermal Viscoelasticit y (1d)  One Dimensional Rod

There are sev eral p ossibilities to implem ent the PDE system (3.3.12) and its initial and b ound-
ary conditions:

\[ \begin{align*}
\text{(3.3.181a)} & \quad \partial_s (v_1 + v_2) - (E \partial_{yy} v_1 + p \sqrt{\epsilon} E \partial_{yy} v_2 + \eta E \partial_{sy} v_1) = F(s), \quad \text{in } ]0, 1[^2, \\
\text{(3.3.181b)} & \quad \partial_s (v_1 + v_2) - (p \sqrt{\epsilon} E \partial_{yy} v_1 + \epsilon E \partial_{yy} v_2 + \nu \eta E \partial_{sy} v_2) = F(s), \quad \text{in } ]0, 1[^2 \\
\text{(3.3.181c)} & \quad v_i(0, y) = 0, \quad \partial_y v_i(0, y) = 0, \quad \text{in } [0, 1], \\
\text{(3.3.181d)} & \quad v_i(s, 0) = 0, \quad \text{in } [0, 1], \\
\text{(3.3.181e)} & \quad \partial_y v_1(s, 1) + p \sqrt{\epsilon} \partial_y v_2(s, 1) + \eta \partial_{sy} v_1(s, 1) = \zeta^R_E(s), \quad \text{in } ]0, 1[, \\
\text{(3.3.181f)} & \quad p \sqrt{\epsilon} \partial_y v_1(s, 1) + \epsilon \partial_y v_2(s, 1) + \nu \eta \partial_{sy} v_1(s, 1) = \zeta^R_E(s), \quad \text{in } ]0, 1[. \\
\end{align*} \]

The dimensionless parameters can be understood as: \( E \) – relative intensity of the elasticity in the first mechanism, \( \epsilon \) – ratio of the elasticities, \( p \) – intensity of the coupling, \( \eta \) – ratio between elasticity and viscosity, \( \nu \) – ratio between the viscosities. The permitted value ranges are

\[ \text{(3.3.182)} \quad E, \epsilon, \eta, \nu > 0, \quad -1 \leq p \leq 1. \]

3.3.5. Numerical Simulations with COMSOL 4.3a – 1d Rod

There are several possibilities to implement the PDE system (3.3.12) and its initial and boundary-
conditions. W e will follow the nondimensionalized approach of Section 3.3.4, but we will not
implement the pure displacement formulation, cf. [37] and Problem 3.3.16. Instead, we use a
mixed stress-displacement formulation of the problem to keep the affinity to the 3d implementation.

We formulate the problem in the manner we implement it with the model of 1d Mathemat-
ics Toolbox-coefficient form PDE in in COMSOL 4.3a.

Problem 3.3.17. Find \((u_1, u_2, u, \sigma)^T : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^4\) such that for given \(\sigma^R\)

\[ \begin{align*}
\text{(3.3.183a)} & \quad E \partial_x u_1 + p \sqrt{\epsilon} E \partial_x u_2 - \sigma = -\eta E \partial_{tx} u_1, \quad \text{in } ]0, 1[^2, \\
\end{align*} \]

\[ ^{12} \text{In [37], we also present an implementation with Maple}^11. \]
3. Viscoelasticity within the Framework of MM Models

\begin{align}
(3.3.183b) \quad p\sqrt{e}E\partial_x u_1 + eE\partial_x u_2 - \sigma &= -\nu\eta E\partial_x u_2, \quad \text{in }]0,1[^2, \\
(3.3.183c) \quad \partial_x u_1 + \partial_x u_2 - \partial_x u &= 0, \quad \text{in }]0,1[^2, \\
(3.3.183d) \quad \partial_t u - \partial_x \sigma &= 0, \quad \text{in }]0,1[^2, \\
\end{align}

is fulfilled for the initial conditions

\begin{align}
(3.3.183e) \quad u_i(0, x) &= 0, \quad \partial_t u_i(0, x) = 0, \quad i = \{1, 2\}, \quad \text{in } [0,1], \\
(3.3.183f) \quad u_i(t, 0) &= 0, \quad i = \{1, 2\}, \quad \text{in } [0,1], \\
(3.3.183g) \quad \sigma(t, 1) &= \sigma^R(t), \quad \text{in } [0,1].
\end{align}

We state the settings in COMSOL to conduct the dimensionless simulation. The unit system is set to None. We define in the Global Definitions node the parameters \(E, e, p, \eta, \nu\) as material parameters and \(t_{\text{step}}\) as the time step. Additional parameters will be mentioned in the following sections related to the specific numerical test. All defined material parameters can be later altered by e.g. a parameter sweep during a study. As geometry, we define an interval \(i1\) of length 1. In the Coefficient Form PDE node, we choose a PDE containing the four variables \(u_1, u_2, u\) and \(\sigma\). The equation system is implemented in the form

\begin{align}
(3.3.184a) \quad e_a \frac{\partial^2 u}{\partial t^2} + d_a \frac{\partial u}{\partial t} + \nabla \cdot (-c\nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + a u &= f, \\
(3.3.184b) \quad u &= [u_1, u_2, u, \sigma]^T, \\
(3.3.184c) \quad \nabla &= \left[ \frac{\partial}{\partial x} \right],
\end{align}

with the coefficient matrices and vectors

\begin{align}
&c = d_a = \alpha = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad f = \begin{pmatrix} -\eta E\partial_x u_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
a &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad e_a = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} E & p\sqrt{(e)}E & 0 & 0 \\ p\sqrt{(e)}E & eE & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{align}

The initial values are set to 0 for all quantities. The boundary conditions are related to the model problem. We define a User-controlled mesh such that the interval \(i1\) is divided in 1000 elements (maximum element size 0.001). We add a study, set Times to 0 : \(t_{\text{step}} : 1\) and use a relative tolerance of 0.001. This results in 8004 degrees of freedom.

3.3.5.1. Creep Test

The additional global parameter \(\sigma_C\) is defined as the value of the constant applied stress. The creep stress on the right boundary is based on a Ramp function \(\text{rm1}\) in the Global Definition.
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

The parameters are for Location 1, Slope 100, Cutoff 1 and the function is smoothed at the cutoff with a transition zone of size 0.001. As boundary conditions, we choose for each end point of the interval Dirichlet Boundary Conditions. For $x = 0$, we set the prescribed values of $u_1$, $u_2$ and $u$ to 0 and leave $\sigma$ free. For $x = 1$, we set the prescribed value of $s$ to $\sigma^R := \sigma_C \cdot \text{rm} 1(t)$ and leave the rest free.

We perform some numerical tests to get a little insight into the model’s behavior, but note that it is not our aim to treat the numerical problem fully exhaustive nor complete in the range of parameter sweeps et cetera. The most interesting parameter is $p$, since $p$ controls if the model is a regular or singular case.

**Parameter Sweep for $p$**  We fix the parameter for $E = 1$, $e = 2$, $\eta = 0.125$, $\nu = 1.1$ and $\sigma_C = 1$ and alter $p$ between the values $[-1 : 0.1 : 1]$. The results are plotted in Figs. 3.10 - 3.15. In the regular cases ($p \in [-1, 1]$) the strain tends to a constant equilibrium state, while in the singular cases ($p \in \{-1, 1\}$) the strain tends to evolving asymptotes. The equilibrium states are nearly reached within the unit interval at the boundary for this specific values, while in all other points there is a delay.

The parameter $p$ has also influence on the stress distribution inside the rod, cf. Figs. 3.11 and 3.12 as well as Figs. 3.13 - 3.15 for details on specific values of $p$. The equilibrium states on the boundary can be determined analytically if the parameters are constant in time and space, see paragraph *Analytic view*.

Figure 3.10.: Strain response ($\partial_x u$) of a 1d rod at the right edge ($x = 1$) as (2)-coupled KV model to a constant boundary stress for varying $p$
3. Viscoelasticity within the Framework of MM Models

(a) Stress for positive coupling: \( p \in \{0 : 0.1 : 1\} \)

(b) Strain for positive coupling: \( p \in \{0 : 0.1 : 1\} \)

(c) Stress for negative coupling: \( p \in \{-1 : 0.1 : 0\} \)

(d) Strain for negative coupling: \( p \in \{-1 : 0.1 : 0\} \)

Figure 3.11.: Stress (\( \sigma \)) and strain (\( \partial_x u \)) response of a 1d rod at the mid point (\( x = 0.5 \)) as (2)-coupled KV model to a constant boundary stress for varying \( p \)

Analytic view  On the right boundary (\( x = 1 \)), we can calculate the model response to the creep test explicitly, when the boundary stress is given by a continuous function, e.g. \( \sigma^R = \sigma_C \), and material parameters are constant in time and space. Solving the boundary ODE system,

\[
\partial_t v_1 = \frac{1}{\eta E} \left( \sigma^R - E v_1 - p \sqrt{e} E v_2 \right),
\]

\[
\partial_t v_2 = \frac{1}{\nu \eta E} \left( \sigma^R - p \sqrt{e} E v_1 - e E v_2 \right),
\]

we obtain for the different cases of \( p \) the following solutions: The solution are given in the uncoupled regular case \( p = 0 \) by

\[
\partial_x u(t, 1) = \int_0^t \exp \left( \frac{-1}{\eta}(t - s) \right) \frac{\sigma^R(s)}{E \eta} ds + \int_0^t \exp \left( \frac{-e}{\nu \eta}(t - s) \right) \frac{\sigma^R(s)}{E \nu \eta} ds.
\]

In the coupled regular case, \( p \in [-1, 0] \cup [0, 1] \), we obtain with \(^{13}\)

\[
R := \sqrt{1 + 2\nu (2p^2 - 1) + e^2 \nu^2}, \quad \lambda_1 := \frac{1}{2\nu} \left( -1 + \frac{e}{\nu} + R \right), \quad \lambda_2 := \frac{1}{2\nu} \left( -1 + \frac{e}{\nu} - R \right),
\]

\(^{13}\)Note that \( R = 0 \) is a special case of the uncoupled case.
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

(a) Stress for positive coupling: \( p \in \{0:0.1:1\} \)

(b) Strain for positive coupling: \( p \in \{0:0.1:1\} \)

(c) Stress for negative coupling: \( p \in \{-1:0.1:0\} \)

(d) Strain for negative coupling: \( p \in \{-1:0.1:0\} \)

Figure 3.12.: Stress (\( \sigma \)) and strain (\( \partial_x u \)) response of a 1d rod at the left edge (\( x = 0 \)) as (2)-coupled KV model to a constant boundary stress for varying \( p \)

\[
\begin{align*}
\partial_x u(t,1) &= \frac{1}{2R \nu E} \left( -1 + \frac{R}{\nu} + \frac{e}{\nu} - \frac{4p\sqrt{e}}{\nu} + \frac{e}{\nu^2} + \frac{1}{\nu} + R \right) \cdot \int_0^t \exp(\lambda_1(t-s)) \sigma^R(s) \, ds \\
&\quad + \frac{1}{2R \nu E} \left( 1 + \frac{R}{\nu} - \frac{e}{\nu} + \frac{4p\sqrt{e}}{\nu} - \frac{e}{\nu^2} - \frac{1}{\nu} + R \right) \cdot \int_0^t \exp(\lambda_2(t-s)) \sigma^R(s) \, ds,
\end{align*}
\]

in the positive singular case, \( p = 1 \), with \( \lambda_2 := (-1 - e\nu^{-1})\eta^{-1} \),

\[
\partial_x u(t,1) = \frac{(\sqrt{e} - 1)^2}{\nu E(\nu + e)} \int_0^t \sigma^R(s) \, ds + \frac{(\sqrt{e}\nu^{-1} + 1)^2}{\eta E(\nu + e)} \int_0^t \exp(\lambda_2(t-s))\sigma^R(s) \, ds
\]

and in the negative singular case, \( p = -1 \), with \( \lambda_2 := (-1 - e\nu^{-1})\eta^{-1} \),

\[
\partial_x u(t,1) = \frac{(\sqrt{e} + 1)^2}{\nu E(\nu + e)} \int_0^t \sigma^R(s) \, ds + \frac{(\sqrt{e}\nu^{-1} - 1)^2}{\eta E(\nu + e)} \int_0^t \exp(\lambda_2(t-s))\sigma^R(s) \, ds.
\]
3. Viscoelasticity within the Framework of MM Models

(a) Evolution of the stress in all material points

(b) Evolution of the strain in all material points

Figure 3.13.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod as (2)-KV model to a constant boundary stress (without coupling $p = 0$)

(a) Evolution of the stress in all material points

(b) Evolution of the strain in all material points

Figure 3.14.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod as (2)-coupled KV model to a constant boundary stress for positive singular coupling $p = 1$

In the singular cases, the parameter combinations with $e = 1$ (positive singular case) and $\sqrt{e\nu^{-1}} = 1$ (negative singular case) are worth noting. They are equivalent to a single Kelvin-Voigt model respectively a single viscous element. For a constant hold boundary stress $\sigma^R = \sigma_C$, equilibrium states of the strain will be approximated. By the equilibrium states of the strain $\partial_x u^\infty$, we understand

\[ \partial_x u^\infty(t, x) := \lim_{t \to \infty} \partial_x u(t, x). \]  

(3.3.187)

On the right boundary, the equilibrium states can be calculated by (3.3.186b), (3.3.186c) and (3.3.186d). In the regular case, the strain is bounded on the right boundary by

\[ \partial_x u^\infty(t, 1) = \frac{1 + e - 2\sqrt{ep}}{(1 - p^2)eE}\sigma_C. \]  

(3.3.188)

In the singular cases, we obtain the asymptotes

\[ \partial_x u^\infty(t, 1) = \frac{(\sqrt{e} - 1)^2}{\nu E(\nu + e)}\sigma_C \cdot t + \frac{(1 + \sqrt{e}\nu^{-1})^2}{(1 + \nu^{-1}E)^2E}\sigma_C \]  

(positive singular)
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

(a) Evolution of the stress in all material points
(b) Evolution of the strain in all material points

Figure 3.15.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod as (2)-coupled KV model to a constant boundary stress for negative singular coupling $p = -1$

\begin{align*}
\text{(3.3.190)} \quad \partial_x u^\infty(t, 1) &= \frac{\sqrt{\nu} + 1}{\nu E(\nu + e)} \sigma_C \cdot t + \frac{(1 - \sqrt{\nu} - 1)E}{(1 + \nu - 1)E} \sigma_C \quad \text{(negative singular)}.
\end{align*}

In the regular case, the strain tends for constant applied stress to a constant equilibrium state. For both singular cases, the strain evolves linearly in the equilibrium state such that constant applied stress leads to unbounded strain evolution. The equilibrium states can be compared to creep experiments with constant loading and indicated the parameter ranges for a parameter identification.

3.3.5.2. Cyclic Tension Test with Relaxation

In a cyclic tension test with relaxation, we pose $\sigma^R$ as a periodic function till the mid point of the time interval and set it afterwards to 0,

\begin{align*}
\text{(3.3.191)} \quad \sigma^R(t) := \begin{cases} 
\sigma_C (\sin(\omega t + \arcsin(c)) + c), & 0 \leq t \leq 0.5, \\
0, & 0.5 \leq t \leq 1.
\end{cases}
\end{align*}

For implementation, we choose $\sigma_C = 1$, $\omega = 20 \times pi$ (to obtain approximately 5 cycles), $c = 0.1$ and smooth the function continuously up to second derivative with a transition zone of relative size 0.01.

Parameter sweep for $p$ We fix the parameter for $E = 100$, $\nu = 1.1$ and $\sigma_C = 1$ and alter $p$ in the values $[-1 : 1]$. The results are plotted in Figs. 3.16 - 3.20. In the regular cases ($p \in [-1, 1]$), the strain tends to a periodic equilibrium oscillation, while in the singular cases ($p \in \{-1, 1\}$) the strain tends to oscillate between two evolving asymptotes. The parameter $p$ has also influence on the stress distribution inside the rod, cf. Figs. 3.17 - 3.20 as well as Figs. 3.21 - 3.23 for details on specific values of $p$. In all regular cases as well as for all material points, the strain relaxes to 0 while in the singular cases a positive non-zero equilibrium state is reached.

Parameter Sweep for $\eta$ We fix the parameter for $E = 100$, $\nu = 1.1$ and $p = 0.5$, while we alter the ratio between the elasticities and the viscosities $\eta \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^1\}$. The scope in this sweep is to show that interference effects can occur. Low values for $\eta$ enforce
3. Viscoelasticity within the Framework of MM Models

(a) Boundary Stress $\sigma_R$

(b) Strain for positive coupling: $p \in \{0:0.1:1\}$

(c) Strain for negative coupling: $p \in \{-1:0.1:0\}$

Figure 3.16.: Strain response ($\partial_x u$) of a 1d rod at the right edge ($x = 1$) as (2)-coupled KV model to a cyclic boundary stress for varying $p$

interference effects in the resulting stress and strain, while high values for $\eta$ lower the effect, cf. exemplarily Figs. 3.25 and 3.26. Larger values of $\eta$ lead to a strain evolution with lower absolute value, cf. Fig. 3.24.
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

(a) Stress for positive coupling: $p \in \{0 : 0.1 : 1\}$

(b) Strain for positive coupling: $p \in \{0 : 0.1 : 1\}$

Figure 3.17.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the mid point ($x = 0.5$) as (2)-coupled KV model to a cyclic boundary stress for varying positive $p$

(a) Stress for negative coupling: $p \in \{-1 : 0.1 : 0\}$

(b) Strain for negative coupling: $p \in \{-1 : 0.1 : 0\}$

Figure 3.18.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the mid point ($x = 0.5$) as (2)-coupled KV model to a cyclic boundary stress for varying negative $p$
3. Viscoelasticity within the Framework of MM Models

(a) Stress for positive coupling: $p \in \{0:0.1:1\}$

(b) Strain for positive coupling: $p \in \{0:0.1:1\}$

Figure 3.19.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for varying positive $p$

(a) Stress for negative coupling: $p \in \{-1:0.1:0\}$

(b) Strain for negative coupling: $p \in \{-1:0.1:0\}$

Figure 3.20.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for varying negative $p$
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

Figure 3.21.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-KV model (without coupling $p = 0$) to a cyclic boundary stress

Figure 3.22.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for positive singular coupling $p = 1$

Figure 3.23.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the left edge ($x = 0$) as (2)-coupled KV model to a cyclic boundary stress for negative singular coupling $p = -1$
3. Viscoelasticity within the Framework of MM Models

Figure 3.24.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the right edge ($x = 1$) as (2)-coupled KV model to a constant boundary stress for a regular case $p = 0.5$ and varying $\eta = \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 10^4\}$

Figure 3.25.: Stress ($\sigma$) and strain ($\partial_x u$) response of a 1d rod at the right edge ($x = 1$) as (2)-coupled KV model to a constant boundary stress for a regular case $p = 0.5$ and $\eta = 10^{-4}$
3.3. Isothermal Viscoelasticity (1d) – One Dimensional Rod

3.3.5.3. Results

The numerical simulations show the expected behavior of the mathematical analysis and the observed behavior of the 3d simulations of a narrow rod, see Section 3.2.3. In the case of

- regular coupling ($|p| < 1$; resp. $E_{12}^2 < E_1 E_2$)

for a creep test the strain evolves to a constant equilibrium state and for a cyclic tension test to a constant stress-strain orbit as long as the boundary stress is applied. The latter case is featured by a strain relaxation to zero when the applied stress is released. In the cases of

- singular coupling ($|p| = 1$; resp. $E_{12}^2 = E_1 E_2$)

for a creep test the strain evolves linearly with respect to time in the equilibrium state. For a cyclic load, we observe ratcheting behavior and strain relaxation to a constant non-zero strain level when the applied stress is released. Without stress relaxation the ratcheting would evolve.

The observed evolutions for material points inside the rod is qualitatively similar to the boundary evolutions. Although, depending on the ratio of the elasticities to the viscosities interference occurs in the strain and stress evolutions.
4. Transformation-Induced Plasticity – Experiments and Model Verifications

The phenomena of transformation-induced plasticity (TRIP) can occur for materials which undergo phase transformation while moderate stresses are applied. One example is steel in its wide variations. The definition of TRIP “can be explained as the significantly increased plasticity during a phase change. For an externally applied load for which the corresponding equivalent stress is small compared to the normal yield stress of the material, plastic deformation occurs”, see [23]. Steel possesses different crystal structures depending on its heat treatment. The temperature path respectively history determines the different mixture of crystal structures on the micro scale which results in different material behavior on the macro scale, cf. e.g. [10]. Since distortion caused by TRIP in manufacturing results in inaccurate work pieces and, therefore, in high economical costs. The TRIP phenomena is widely investigated, cf. e.g. [12] motivated by the Collaborative Research Centre (SFB) 570 “Distortion Engineering – Distortion Control in the Production Process”. One of the first contributions was made by Greenwood and Johnson in [26]. Leblond et. al. include in [40, 41] the coupling of TRIP with strain hardening phenomena. The interaction of classic plasticity and TRIP is also topic of recent works, e.g. [68, 70], with application, concerning experiments and modeling, to the steel 100Cr6 (SAE 52100). Experiments with the low alloy 51CrV4-steel are also investigated, see [31, 38, 39]. For further details, we refer to the cited literature and references within.

In our approach, we continue the works [68, 70] in the environment of the German Research Foundation (DFG) project BO 1144/4-1 “Multi-Mechanism Models: Theory and its Application to some Phenomena in the Material Behavior of Steel” conducting

- experiments characterizing the interaction of classic plasticity and TRIP of the steel 100Cr6 (SAE 52100).

In this work, we examine the TRIP behavior during the phase transformation of austenite to bainite for two temperatures; 280°C and 360°C. The influence of classic plasticity on TRIP is investigated by deformations of the specimen before the bainitic transformation takes place (pre-deformation). We provide a model approach taking the interaction of classic plasticity and TRIP into account and verify the model by parameter identifications.

In the following Sections 4.1.1 - 4.1.5, we present the experimental setup and prepare the test evaluation of Section 4.1.6. The important parts of the experiments are the predeformations at austenitizing (Section 4.1.6.2; 850°C) and at bainitizing (Section 4.1.6.4; 280°C resp. 360°C) temperature as well as the bainitic transformation (Section 4.1.6.5). The predeformations influence the material behavior during the transformation. At both bainitizing temperature, we compare groups of experiments with the same applied stress and different predeformations and groups of experiments with the same predeformation and different applied stress during the bainitic transformation, see Tab. 4.1. We highlight the most interesting observations:

- Applied stress during transformation leads to a deviation with the same sign in comparison to the experiments without applied stress.

- Predeformation at bainitizing temperature influences the evolution of the inelastic strain significantly. In comparison with experiments without predeformation, positive (resp. nega-
4.1. Experiments and Data Post Processing

4.1. Experimental Set-Up and Procedure

The experiments were provided by the Bremen Institute for Materials Testing (MPA, Materialprüfungsanstalt) – a division of the Foundation Institute of Materials Science (IWT, Stiftung Institut Werkstofftechnik) Bremen. They were performed on a Gleeble® 3500 machine. The Gleeble® 3500 machine is a combination of a tension and compression test machine with an adjacent quenching dilatometer, cf. [1] and Fig. 4.1(a).

The test specimens were hollow specimens, see Fig. 4.1(b). All specimens consisting of the steel 100Cr6 (SAE 52100) were bainitized before the experiments were conducted. The temperature measurement was taken at the middle of the test specimen. Longitudinal and transversal strain were measured by strain gauges and additionally by a laser.

For the test evaluation, the experimental procedure is split into several parts. The parts describe the austenitic transformation (Part 1), the predeformation at austenitizing temperature \( \theta_A \) (Part 2), the quenching from austenitizing temperature \( \theta_A \) to bainitizing temperature \( \theta_B \) (Part 3), the predeformation at bainitizing temperature \( \theta_B \) (Part 4), the bainitic transformation (Part 5) and the cooling to room temperature (Part 6). The predeformation parts are optional, and in our specific experiments only one predeformation as tension (positive applied stress) or compression (negative applied stress) is conducted. An overview of the experimental procedure is illustrated in Fig. 4.2. The conducted experiments are listed in Tab. 4.1. Replications of experiments were only made if a measurement was compromised. We do not present the neglected experiments.

Measurements During the experiments, time (in s), temperature (in °C), engineering stress (in MPa), engineering longitudinal and transversal strain are measured, see Section 4.1.2.

Part 1: Austenitic Transformation The test specimen is heated up from room temperature (RT) to austenitizing temperature \( \theta_A = 850°C \) within 3 minutes (Part 1a). The temperature is held for 20 minutes (Part 1b). During this part no external stress or strain are applied.
Part 2: Predeformation at $\theta_A$ (optional) The test specimen is deformed by a tension or compression in the axial direction. Hereby, the yield stress is considerably exceeded. A constant strain rate is applied for 10-15 seconds until the longitudinal strain reaches approximately $\pm 4\%$ or $\pm 6\%$. Due to experimental reasons, the measured strains are set to zero right before the predeformation. If a predeformation is performed, the applied load is released before or in the first seconds of the quenching (Part 3). This part is optional and can be skipped if no predeformation takes place at $\theta_A$. The related experiments to Part 2 are listed in Tab. 4.1 in the rows $+4\%$ SA to $-6\%$ SA.

Part 3: Quenching from $\theta_A$ to $\theta_B$ During the quenching, the specimen is cooled down to the transformation temperature $\theta_B$ (280°C resp. 360°C) within a few seconds (here: 10s). The temperature drops for some seconds below $\theta_B$, before it stabilizes at $\theta_B$.

Part 4: Predeformation at $\theta_B$ (optional) Instead of performing a predeformation at $\theta_A$ in Part 2, we can perform it at transformation temperature $\theta_B$. The predeformation starts right after the quenching is finished and before the transformation into bainite takes place. A constant strain rate is applied until the longitudinal strain reaches approximately $\pm 2\%$, $+4\%$ or $+6\%$. After the predeformation of the undercooled austenite the applied load is released. This part is
Figure 4.2.: General scheme of the experimental procedure

also optional and can be skipped if no predeformation takes place at $\theta_B$. The related experiments to Part 4 are listed in Tab. 4.1 in the rows $+2\%$ UA to $-2\%$ UA.

**Part 5: Bainitic Transformation**  The temperature $\theta_B$ is held until a complete transformation of the austenite into bainite is assumed (in general: 20 min). During the transformation an optional external stress of $\pm 25$ MPa or $\pm 50$ MPa may be applied. The stress is significantly below the yield stress. If a stress is applied, it starts not before the end of an optional predeformation at $\theta_B$.

**Part 6: Cooling to Room Temperature**  After the transformation has ended, the external heating is turned off and the specimen is cooled down to room temperature (RT).

<table>
<thead>
<tr>
<th>Predeformation</th>
<th>Applied stress in [MPa] at 280°C</th>
<th>Applied stress in [MPa] at 360°C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>+25</td>
</tr>
<tr>
<td>$+6%$ SA</td>
<td>Q98</td>
<td>Q114</td>
</tr>
<tr>
<td>$+4%$ SA</td>
<td>Q101</td>
<td>Q116</td>
</tr>
<tr>
<td>$+6%$ SA</td>
<td>Q113</td>
<td>Q118</td>
</tr>
<tr>
<td>$-4%$ SA</td>
<td>Q106</td>
<td>Q120</td>
</tr>
<tr>
<td>$-6%$ SA</td>
<td>Q110</td>
<td>Q122</td>
</tr>
<tr>
<td>$+2%$ UA</td>
<td>Q124</td>
<td>Q140</td>
</tr>
<tr>
<td>$+4%$ UA</td>
<td>Q152</td>
<td>Q213</td>
</tr>
<tr>
<td>$+6%$ UA</td>
<td>Q142</td>
<td>Q143</td>
</tr>
<tr>
<td>$-2%$ UA</td>
<td>Q135</td>
<td>Q137</td>
</tr>
</tbody>
</table>

Table 4.1.: Performed experiments: Predeformation of stable (SA) or undercooled (UA) austenite, transformation at bainitizing temperature under moderate stress ($\pm 50$ MPa to 50 MPa)
4.1.2. Measured Data

The measured data is given as \( N \) discrete points. The measured data tuple \( Data \) contains for each index \( i \in \{1, \ldots, N\} \) the current value of the time \( t^i \), temperature \( \theta^i \), engineering stress \( S^i \), engineering longitudinal \( e^i_L \) and transversal strain \( e^i_D \).

\[
Data^i = (i, t^i, S^i, e^i_L, e^i_D).
\]

(4.1.1)

In this section, we assume that every quantity is given or can be derived by the discrete measurement data and suppress the dependency of the index respectively, therefore, the time dependency.

4.1.2.1. Engineering Stress

The engineering stress \( S \) is calculated by the measured (applied) force \( F \) and the reference cross-sectional area \( A_0 \) at the beginning of the experiment, cf. Fig. 4.1(b),

\[
S = \frac{F}{A_0}.
\]

(4.1.2)

4.1.2.2. Engineering Strains

The engineering longitudinal strain is calculated by the ratio of the actual measured length of the specimen \( l \) and its reference length \( l_0 \) at the beginning of the experiment,

\[
e_L = \frac{l - l_0}{l_0}.
\]

(4.1.3)

The engineering transversal strain is calculated by the ratio of the actual measured diameter of the specimen \( d \) and its reference diameter \( d_0 \) at the beginning of the experiment,

\[
e_D = \frac{d - d_0}{d_0}.
\]

(4.1.4)

We will consider the engineering stress and strain but note that the diameter of the specimen is changing during the experiment and it would be reasonable to consider the true stress and true strain, see Appendix A.

4.1.3. Material Data

Depending on the current temperature and the manufacturing process, the steel 100Cr6 possesses several material phase fractions with different material properties. In our conducted experiments, we assume that the steel contains only two of its possible phases or a mixture of these two: austenite and bainite. This assumption is based on the investigations of [1]. During the experiments, we observe phase transformations from bainite to austenite (austenitic transformation) and from austenite to bainite (bainitic transformation). The phase fraction of austenite is denoted by \( p_A \) and the phase fraction of bainite with \( p_B \). Related parameter to austenite and bainite are denoted with a subscript \( A \) respectively \( B \).

4.1.3.1. Austenite

In [1] and [2], several material parameters for the austenitic phase of the steel 100Cr6 have been determined in dependency of the temperature \( \theta \). We cite the most important ones:

\[
\rho_A = 8041.4 - 5.75 \cdot 10^{-1} \cdot \theta + 2.6 \cdot 10^{-5} \cdot \theta^2 \left[ \frac{\text{kg}}{\text{m}^3} \right]
\]
4.1. Experiments and Data Post Processing

Elastic modulus: \( E_A = 266930 - 221.3 \cdot \theta \) [MPa]

Poisson’s ratio: \( \nu_A = 0.223 + 2.5 \cdot 10^{-4} \cdot \theta \) [-]

4.1.3.2. Bainite

For the bainitic phase, the following material parameters have been provided by the IWT and MPA and have been obtained by fits to additional experimental data:

Mass density: \( \varrho_B = 7823 - 0.2622 \cdot (\theta - 20) + 1.48510^{-4} \cdot (\theta - 20)^2 \) [kg/m\(^3\)]

Elastic modulus: \( E_B = 224217 - 4.4345 \cdot \theta - 0.1871 \cdot \theta^2 \) [MPa]

Poisson’s ratio: \( \nu_B = 0.3144 \) [-]

Alternative Material Parameters

Additionally to the elastic modulus and Poisson’s ratio, the compression and shearing modulus are used as well:

Compression modulus: \( K = \frac{E}{3(1-\nu)} \)

Shearing modulus: \( \mu = \frac{E}{2(1+\nu)} \)

<table>
<thead>
<tr>
<th>( \theta ) [°C]</th>
<th>( \varrho_A ) [kg/m(^3)]</th>
<th>( \varrho_B ) [kg/m(^3)]</th>
<th>( E_A ) [MPa]</th>
<th>( E_B ) [MPa]</th>
<th>( \nu_A ) [-]</th>
<th>( \nu_B ) [-]</th>
</tr>
</thead>
<tbody>
<tr>
<td>280</td>
<td>7876.8</td>
<td>7744.8</td>
<td>204966</td>
<td>208307</td>
<td>0.2930</td>
<td>0.3144</td>
</tr>
<tr>
<td>360</td>
<td>7831.1</td>
<td>7716.7</td>
<td>187262</td>
<td>198372</td>
<td>0.3130</td>
<td>0.3144</td>
</tr>
<tr>
<td>850</td>
<td>7562.1</td>
<td>7567.9</td>
<td>78825</td>
<td>85268</td>
<td>0.4355</td>
<td>0.3144</td>
</tr>
</tbody>
</table>

Table 4.2.: Material parameters for certain temperatures

Remark 4.1.1. Since our experimental data scatters, all material parameter from the literature have to be certified or at least to be looked at critically — especially elastic modulus and Poisson’s ratio, cf. e.g. Figs. 4.3(b) and 4.8(a).

4.1.4. Derived Data

The volume strain, the inelastic strain, the phase fractions e.g., are derived from the original measured data. The volume strain is defined by\(^1\)

\[(4.1.5) \quad e_V = e_L + 2e_D.\]

In terms of the inelastic strain \( e_I \), we have to make a model assumption. We define strain parts related to thermoelastic changes \( (e_{te}) \) and changes in the density \( (e_o) \)

\[(4.1.6) \quad e_{te} := \frac{S}{E}, \quad e_o := \sqrt{\frac{\varrho}{\varrho_0}} - 1.\]

\(^1\)For uniaxial experiments the volume strain matches the trace of the Green’s strain tensor, cf. Section 4.2.2.
Hereby, $E$ denotes the elastic modulus, $\varrho_0$ the reference mass density at the beginning of the observation, and $\varrho$ the mass density. $E$ and $\varrho$ are temperature-dependent. Now, the longitudinal and transversal strain can be expressed by

\begin{align}
\varepsilon_L &= \varepsilon_t + \varepsilon_{\varrho} + \varepsilon_I, \\
\varepsilon_D &= -\nu \varepsilon_t + \varepsilon_{\varrho} - \frac{1}{2} \varepsilon_I,
\end{align}

which leads to a formula for the inelastic strain $\varepsilon_I$

\begin{equation}
\varepsilon_I = \frac{2}{3} (\varepsilon_L - \varepsilon_D) - \frac{S}{3\mu}.
\end{equation}

$\mu$ denotes the shearing modulus. The material parameters $E$, $\nu$, $\mu$, $\varrho$ and $\varrho$ are temperature-dependent parameters and depend on the composition of the specimen (e.g., different materials or different phase fractions of the material). We recall the mixture rule (2.3.87a) defined in Section 2.3.7 for an arbitrary material parameter $C$ and a two-phase material, $J_{ph} = \{1, 2\}$, with the phase fractions $p_j$

\begin{equation}
C(\theta, p_{J_{ph}}) := \sum_{j \in J_{ph}} p_j C_j(\theta) = p_1 C_1(\theta) + p_2 C_2(\theta).
\end{equation}

### 4.1.4.1. Phase Fractions

In the considered experiments, we assume that the specimen only consists of two phases, austenite and bainite. In order to calculate the austenite and bainite phase fractions, we use the formula

\begin{equation}
\varepsilon_V = \frac{1 - 2\nu}{E} S + 3 \left( \sqrt[3]{\frac{2\varrho_0}{\varrho}} - 1 \right).
\end{equation}

We derive the formulas under the assumption of an isothermal phase transformation (cf. Section 4.1.6.5). During the observation period of a transformation $[t_S, t_E]$ (cf. Section 4.1.5.1) if one phase 1 transforms into phase 2 and only two phases exits, we assume

\begin{equation}
p_1 + p_2 = 1.
\end{equation}

The material parameters of each phase fraction are denoted by the subscripts 1 and 2. At the beginning of the transformation period, $t_S$, some reference values are needed, e.g., the reference density $\varrho_0$ at the reference time $t_S$ and the reference temperature $\theta_S = \theta(t_S)$, e.g.

\begin{equation}
\varrho_0 = \varrho_1(\theta_S).
\end{equation}

We obtain $\varrho_0 = \varrho_1$ in the isothermal case ($\theta \equiv \theta_S$). In order to determine the phase fractions by means of the measured data given as discrete data points $(i, t^i, S^i, \varepsilon^i_L, \varepsilon^i_D)$, several approaches seem reasonable. Using mixture rule (4.1.10) for the mass density, we obtain

\begin{equation}
p_2 = \left( \frac{27}{(3 + \varepsilon_V - \frac{1}{E} S)^3} - 1 \right) \frac{\varrho_1}{\varrho_2 - \varrho_1}.
\end{equation}

Obviously, the term $\frac{1}{E} S$ depends on both phase fractions. We suppose four different, discrete approaches in order to obtain the derived phase fraction $p_2$ of the forming phase as an approximation.
4.1. Experiments and Data Post Processing

1st Approach We replace $\frac{1 - \nu_i}{E_i}$ by $\frac{1 - \nu_i^{t-1}}{E_i^{t-1}}$.

\[ p_2^1 = \left( \frac{27}{(3 + e_i^{1} - \frac{1 - \nu_i^{t-1}}{E_i^{t-1}} S_i^{t})^3} - 1 \right) \frac{\varrho_1}{\varrho_2 - \varrho_1}. \]  

(4.1.15a)

2nd Approach Alternatively, we neglect the stress part,

\[ p_2 = \left( \frac{27}{(3 + e_i^1)^3} - 1 \right) \frac{\varrho_1}{\varrho_2 - \varrho_1}. \]  

(4.1.15b)

In both approaches, the terms $e_i^1 - \frac{1 - \nu_i^{t-1}}{E_i^{t-1}} S_i^{t}$ resp. $e_i^1$ are set to zero at time $t_S$, since $e_i^1$ evolves during the experiments before the transformation period begins. Therefore, we introduce auxiliary variables $D_k$, $k = 1, 2$,

(4.1.16a) 

\[ D_1^k := e_i^1 - \frac{1 - \nu_i^{t-1}}{E_i^{t-1}} S_i^{t} - e_i^1(t_S) + \frac{1 - \nu(t_S)}{E(t_S)} S(t_S), \]

and

(4.1.16b) 

\[ D_2 := e_i^1 - e_i^1(t_S). \]

During the transformation period, the forming phase fraction is then given by

\[ p_2^1 = \left( \frac{27}{(3 + D_k^1)^3} - 1 \right) \frac{\varrho_1}{\varrho_2 - \varrho_1} \left\{ \begin{array}{l} k = 1, \text{ 1st Approach}, \\ k = 2, \text{ 2nd Approach}. \end{array} \right. \]  

(4.1.17)

3rd Approach Using the reciprocal mixture rule (2.3.87b) for the density and compression modulus $K$, we obtain the polynomial equation

\[ \left( \frac{1}{3} \left( e_i^1 - \frac{S}{3} \left( \frac{1}{K_1} + \frac{1}{K_2} - \frac{1}{1 - K_1} \right) p_2 \right) \right)^3 + 1 = 1 + \left( \frac{\varrho_1}{\varrho_2 - \varrho_1} \right) p_2. \]  

(4.1.18)

Similarly as above, we have to ensure that $e_i^1 - \frac{S}{3K_1}$ is zero at the time $t_S$. This equation can be solved numerically for each index $i$.

4th Approach A more classic approach, see [66], using the reciprocal mixture rule (2.3.87b) and a linearized ansatz for the root term,

\[ \sqrt[3]{\frac{\varrho_0}{\varrho} - 1} \approx \frac{\varrho_0 - \varrho}{3\varrho_0}, \]  

(4.1.19)

leads to

\[ p_2 = \left( e_i^1 - \frac{S}{3K_1} \right) \cdot \left( \frac{\varrho_1 - \varrho_2}{\varrho_1} + \frac{S}{3} \left( \frac{1}{K_1} - \frac{1}{K_2} \right) \right)^{-1}. \]  

(4.1.20)

Neglecting the elastic part and assuming a complete transformation at $t_E$, we get the widely used form

\[ p_2 = \frac{e_i^1}{e_i^1(t_E)}. \]  

(4.1.21)
Again, we have to ensure for the third and fourth approach that \( e_V - S/(3K_1) \) respectively \( e_V \) is zero at the beginning, cf. (4.1.16).

**Remark 4.1.2.** Using the given experimental data, all presented formulas to determine the phase fractions will lead, even if a complete phase transformation is reasonable, to values of \( p_2 \) which differ from 1. Therefore, we set

\[
(4.1.22) \quad p_2 := \frac{p_2^{\text{calc}}}{p_2^{\text{calc}}(t_E)},
\]

where \( p_2^{\text{calc}} \) notes the calculated phase fraction by one of the approaches above. In order to neglect side effects and noise in the data, we use an average over the last values of \( p_2^{\text{calc}} \) instead of \( p_2^{\text{calc}}(t_E) \).

**Alternative: Volume Fractions** Transformations in steel are accompanied by characteristic volume changes, see [39],

\[
\frac{\Delta V}{V} = (1 + e_L)(1 + e_D)^2 - 1.
\]

If one can assume a complete phase transformation, the volume fraction \( w \) of the forming phase 2 can be described as

\[
(4.1.24) \quad w_2(t) = \frac{\Delta V(t)}{\Delta V(t_E)}.
\]

**4.1.5. Post Processing**

Depending on the considered experiment part, it is useful to restrict the index range and introduce reference data sets. Moreover, before utilizing the data for model fitting, it is crucial to smooth the noisy experimental data. If the data set is very large, it is useful to reduce it as well. Every post processing step manipulates the recorded data such that we document every used alteration for a correct reproducibility of the data analysis and, especially, the parameter identification.

### 4.1.5.1. Restriction to Subparts of the Experiment

As soon as a specific part of the experiment is considered, it corresponds with certain index range. If required, the measured strains have to be set to zero at the beginning of the index range in order to define a reference data set for the considered part. For every of the six parts \( \#P \in \{1, \ldots, 6\} \), cf. Section 4.1.1, we define a start index \( i_{S\#P} \) and an end index \( i_{E\#P} \). E.g., the reference data set of the longitudinal strain \( e_L \) for part \( \#P \) is defined as

\[
(4.1.25) \quad e_{L\#P,i} = e_{L,i} - e_{L,i_{S\#P}}, \quad i \in \{i_{S\#P}, \ldots, i_{E\#P}\}.
\]

**4.1.5.2. Smoothing**

The data scatters partially and contains outliers. In order to neglect these measurement effects, it is reasonable to smooth the data, e.g. by the MATLAB function `smooth`. The function provides several modes, like a *moving average* or different forms of local regression (including those which weight less or neglect outliers), see e.g. [54].

**Remark 4.1.3.** Using a smoothing method alters the data, e.g. if the smoothing span of a *moving average* is too large, the smoothed data varies to much from the original. Otherwise, abstaining a smoothing method before calculating, e.g. derivatives, results in unrealistic and noisy derived data.
4.1. Experiments and Data Post Processing

### 4.1.5.3. Reducing

If the data set is too large or less data points are sufficient to describe the experiments appropriately, reducing the data points is useful with regard to a parameter fitting to models.

**Downsample** A simple approach is to ignore most of the indices and only consider every \( j \)-th tuple, e.g. by the MATLAB function `downsample`.

**Interpolation** If the sampling rate for different experiments or even in the same part of the experiment is changing, it is useful to synchronize them while reducing. We can interpolate the given data to obtain synchronous and equidistant data sets with respect to time, e.g. by the MATLAB function `interp1`.

### 4.1.6. Data Analysis and Conclusions

This section is organized for every of the six experimental parts, see Section 4.1.1, into a statement on the observation range, the observations of each part, conclusions on the experimental part and if needed remarks on single experiments are added.

During the observations the measured data is considered (engineering strains and stress) if not stated otherwise, cf. Sections 4.1.6.5(Part 5). If we speak about the inelastic strain \( e_I \), we derive it by formula (4.1.9), but have in mind that the formula depends on the elastic modulus and the Poisson’s ratio. For now, we use the literature values, see Section 4.1.3. The usability of the literature values is addressed in Sections 4.1.6.2(Part 2) and 4.1.6.4(Part 4).

The material parameters depend on the phase fractions. The phase fractions are calculated by formula (4.1.15a). We set the phase fraction for austenite equal to 1 after Part 1 and consider only the bainitic transformation in Part 5. At the end of Part 5 the phase fraction of bainite \( p_B \) is assumed to be 1. Since the phase fractions derived by the measured data differ from 1, but are nearly stationary, we normalize \( p_B \) with ansatz (4.1.22). Instead of the last value of \( p_B \), we use the mean value of the last 5% of the considered index range of \( p_B \). Due to the later use for parameter identification, the phase fractions are smoothed again with a moving average over a span of 5% of the index range. If some \( p_B \) are greater than 1 (lower than 0), we set these to equal to 1 (resp. equal to 0). We smooth with a moving average again over a span of 5% of the index range. The considered inelastic strain \( e_I \) of Part 5 is obtained by (4.1.9) using the smoothed data (moving average over 50 data points) of \( e_L, e_D, S \) and \( p_B \) for the composition of the material parameters. For a detailed derivation, see Section 4.3.2.

For the reader’s convenience, we amend the experiment notation of Tab. 4.1, e.g. Q103 will be denoted by Q103\^{SA|+4}_180|_{-50}. The superscript denotes the predeformation and the subscript the transformation temperature and the applied stress during transformation.

### 4.1.6.1. Part 1: Austenitic Transformation

**Observation range** The start index \( i_{S1} \) denotes the index when the measurement starts and the end index \( i_{E1} \) is the minimum of the last index when \( \theta_A \) is held and the begin of a predeformation in Part 2, \( i_{E1} := \min \{i_{S2}, i_{S3}\} \).

**Observations** Beginning at 748.7°C up to 765.8°C, the volume strain decreases significantly, see Fig. 4.3(a) and Fig. 4.3(b). Then, it reaches a local minimum, before it increases to a (nearly) stationary value. The mean stationary value of the volume strain at the end of the austenitic transformation is +3.58%, while it ranges from +3.21% to +3.72%.
Conclusions for Part 1  The evolutions of the volume strains depending on the temperature show how the austenitic transformations evolve. The transformations start when the volume strains decrease due to the density changes of bainite to austenite. Hereinafter, we assume that the transformation was complete after 20 minutes. This wide scattering in the transformation start-temperature as well as in the stationary volume strains indicates that all original samples differ in their material structure. Moreover, we might only make qualitative statements about the material behavior.

Remarks on single experiments  Q103$^{SA|+4\%}$ is excluded in Fig. 4.3(a) and Fig. 4.3(b), since the measurement is not reasonable for this experimental part of the heating and austenitic transformation. The lower outlier is Q144$^{UA|+6\%}$.

![Figure 4.3.](image)

(a) Complete Part 1  (b) Zoom into the start of austenitization

4.1.6.2. Part 2: Predeformation at Austenitizing Temperature

Observation range  The start index $i_{S2}$ is chosen when the stress starts to evolve. While the end index $i_{E2}$ is set when the stress is almost equal to zero and the quenching does not start yet. The considered strains are $e_{S2}^{L}$ and $e_{S2}^{D}$.

Observations  The experiments are (longitudinal) strain controlled. Due to experimental reasons, the strain is set to zero just before the predeformation starts. The specimens are stretched or compressed with a nearly constant strain rate until the desired longitudinal strain is reached, see Tab. 4.1. The related evolutions of the inelastic strains are illustrated in Figs. 4.4 - 4.7. Afterwards, the applied stress is released and the specimen shows a little relaxation in for inelastic strain. In general, the obtained predeformation deviates from the experimental plan. The predeformations of the stable austenite can be divided in four groups based on the intensity of the predeformation: Group $+4\%$ SA, Group $+6\%$ SA, Group $-4\%$ SA and Group $-6\%$ SA.

Group $+4\%$ SA  The maximum value of the longitudinal strain $e_{L}^{S2}$ scatters from $+3.768\%$ to $+4.064\%$, while the corresponding maximum stresses $S$ lies between $+74.44$ MPa and $+84.65$ MPa, see Fig. 4.4. The end value of the inelastic strain $e_{I}^{S2}(i_{E2})$ varies from $+3.415\%$ to $+3.918\%$.
4.1. Experiments and Data Post Processing

**Figure 4.4.: Predeformation of stable austenite: Group +4% SA**

The maximum value of the longitudinal strain $e^{L2}_L$ scatters from $+5.808\%$ to $+5.932\%$, while the corresponding maximum value of the stress $S$ lies between $+78.46$ MPa and $+81.07$ MPa, see Fig. 4.5. The end value of the inelastic strain $e^{S2}_{I}(i_{E2})$ varies from $+5.494\%$ to $+5.666\%$.

**Group +6% SA** The maximum value of the longitudinal strain $e^{L2}_L$ scatters from $+5.808\%$ to $+5.932\%$, while the corresponding maximum value of the stress $S$ lies between $+78.46$ MPa and $+81.07$ MPa, see Fig. 4.5. The end value of the inelastic strain $e^{S2}_{I}(i_{E2})$ varies from $+5.494\%$ to $+5.666\%$.

**Group -4% SA** The minimum value of the longitudinal strain $e^{L2}_L$ scatters from $-4.112\%$ to $-3.822\%$, while the corresponding maximum value of the stress $S$ lies between $-88.97$ MPa and $-78.18$ MPa, see Fig. 4.6. The end value of the inelastic strain $e^{S2}_{I}(i_{E2})$ varies from $-4.4458\%$ to $-3.731\%$.

**Group -6% SA** The minimum value of the longitudinal strain $e^{L2}_L$ scatters from $-5.924\%$ to $-5.874\%$, while the corresponding maximum value of the stress $S$ lies between $-87.84$ MPa and $-85.1$ MPa, see Fig. 4.7. The end value of the inelastic strain $e^{S2}_{I}(i_{E2})$ varies from $-6.497\%$ to $-5.843\%$.

Most of the measured stress curves show noise and occasionally the strain measurements contain outliers, e.g. Fig. 4.6. Shortly after the first loading above the yield stress, the stress
Figure 4.6.: Predeformation of stable austenite: Group -4% SA

Figure 4.7.: Predeformation of stable austenite: Group -6% SA

Figure 4.8.: Predeformation of stable austenite: Exemplarily comparison of experiments vs. literature values (cf.[1] and Section 4.1.3)
4.1. Experiments and Data Post Processing

slightly decreases in absolute value for all experiments. Thereafter, the stress nearly linearly increases in absolute value under ongoing applied stress.

Conclusions for Part 2 Although, the strain rates of the applied loading are alike, they scatter and we cannot exclude strain rate dependency. This fact might be one of the reasons of the differences in the stress-strain-curves. Moreover, from the conducted experiments we cannot conclude, whether the strain hardening saturates and whether the strain hardening is of kinematic or isotropic type.

The $e_{S1}^2$-S-graph, see Fig. 4.8(a) for a comparison with the literature value as given in [1], indicates that the material does not experience ideal linear elastic material behavior for small strains. It implies that a very small elastic range exists for this steel (if at all) and that the yield stress is low. Furthermore, the $e_{V}^S2$-S-graph evolves similar to the stress-strain-curves and the $e_{L}^S2$-S-curves scatter widely (related to this window range). In theory, an $e_{V}^S2$-S-graph should show a linear dependency (cf. (4.1.5), (4.1.7), (4.1.8)) but we cannot observe this in the experiments, see Fig. 4.8(b). So far, we neglect this observation further on. Due to the noise, the data should be smoothed for further data analysis.

Remarks on single experiments The measured stress is very noisy for Q149 $^{\text{SA}|-4}$ $^{\text{360}|+50}$. In Q118 $^{\text{SA}|-6}$ $^{\text{280}|+25}$ and Q121 $^{\text{SA}|-4}$ $^{\text{280}|+25}$ there are outliers in the $e_L$-measurement probably due to swarms of the specimen which fly into the laser beam of the measuring equipment. The strain rates of Q210 $^{\text{SA}|-4}$ $^{\text{360}|\pm 0}$ and Q211 $^{\text{SA}|-4}$ $^{\text{360}|\pm 0}$ are higher than the other strain rates.

4.1.6.3. Part 3: Quenching from Austenitizing to Bainitizing Temperature

Observation range The index range $i_{S3}$ upto $i_{E3}$ starts when the temperature $\theta$ starts decreasing from $\theta_A$ and ends when $\theta$ is constantly equal to $\theta_B$ - the drop below $\theta_B$ is included. The considered strains are $e_{L}^S3$ and $e_{D}^S3$.

Observations The volume strain decreases nearly linearly with decreasing temperature. The values of $e_{V}^S3(i_{E3})$ vary at $\theta_B =280^\circ\text{C}$ from -4.368% to -3.575% and at $\theta_B =360^\circ\text{C}$ from -3.587% to -3.29%, see also Fig. 4.9(a) and Fig. 4.9(b). In some experiments, the predeformation of Part 4 starts before $\theta_B$ is reached. In these cases, we take the $e_{V}^S3$-value at the first time when $\theta_B$ is reached. In general, the obtained predeformation deviates

Conclusions for Part 3 This experimental part verifies the density of austenite in the literature (red line in Fig. 4.9(a) and Fig. 4.9(b)). The scattering of the volume strain seems randomly and neither connected to the austenitic transformation nor to the predeformation in Part 2.

Remarks on single experiments Q138 $^{\text{UA}|-2}$ $^{\text{280}|\pm 10}$ is an outlier for nearly the whole observation range.

4.1.6.4. Part 4: Predeformation at Bainitizing Temperature

Observation range The start index $i_{S4}$ is chosen when the stress starts to evolve. While the end index $i_{E4}$ is set when the stress is almost equal zero. The considered strains are $e_{L}^S4$ and $e_{D}^S4$.

Observations The experiments are (longitudinal) strain controlled. Due to experimental reasons, the strain is set equal to zero just before the predeformation starts. The specimens are stretched or compressed with a nearly constant strain rate until the desired predeformation is approximately reached, see Tab. 4.1. Afterwards, the applied stress is released and the specimen shows a little relaxation for inelastic strain. In general, the obtained predeformation deviates
4. Transformation-Induced Plasticity – Experiments and Model Verifications

Figure 4.9.: Quenching from $\theta_A = 850^\circ C$: Comparison of experiments vs. literature values (Q138 excluded)

from the experimental plan. The predeformations of the undercooled austenite are divided in six groups (different intensity and different temperature): $+2\%$ UA$_{280^\circ C}$, $+4\%$ UA$_{280^\circ C}$, $+6\%$ UA$_{280^\circ C}$, $-2\%$ UA$_{280^\circ C}$ and $+4\%$ UA$_{360^\circ C}$. In several experiments, the inelastic strain relaxes slightly after the stress is released.

**Group $+2\%$ UA$_{280^\circ C}$** In Q124$_{UA|+2|280\pm0}$ and Q125$_{UA|+2|280\pm50}$, the predeformation takes place before the quenching is completed. Therefore, the longitudinal strain decreases although the stress starts to increase. The maximum value of the longitudinal strain lies between $+1.741\%$ and $+1.932\%$, the maximum value of the stress between $+402.4$ MPa and $+448.9$ MPa, see Fig. 4.10. The end value of the inelastic strain $e^S_{4i}(t_{E4})$ varies from $+1.381\%$ to $+1.627\%$.

Figure 4.10.: Predeformation of undercooled austenite: Group $+2\%$ UA$_{280^\circ C}$

**Group $+4\%$ UA$_{280^\circ C}$** In Q152$_{UA|+4|280\pm0}$, the predeformation starts while the quenching is in progress, and furthermore the stress drops for some indices. During the relaxation in Q212$_{UA|+4|280\pm50}$
4.1. Experiments and Data Post Processing

a small drop in the $e_{L}^{S4}$-measurement is observable. The maximum value of the longitudinal strain scatters from $+3.372\%$ to $+3.798\%$, while the maximum value of the stress lies between $+455.3$ MPa and $+483.0$ MPa, see Fig. 4.11. The end value of the inelastic strain $e_{I}^{S4}(t_{E4})$ varies from $+3.017\%$ to $+3.559\%$.

Figure 4.11.: Predeformation of undercooled austenite: Group $+4\%$ UA$_{280^\circ C}$

**Group $+6\%$ UA$_{280^\circ C}$** The maximum value of the longitudinal strain scatters from $+5.693\%$ to $+5.741\%$, while the maximum value of the stress lies between $+514.9$ MPa and $+562.0$ MPa, see Fig. 4.12. The end value of the inelastic strain $e_{I}^{S4}(t_{E4})$ varies from $+4.803\%$ to $+5.358\%$.

Figure 4.12.: Predeformation of undercooled austenite: Group $+6\%$ UA$_{280^\circ C}$

**Group $-2\%$ UA$_{280^\circ C}$** Q138$^{UA|-2}_{280\text{-}25}$ contains several outliers in the $e_{L}^{S4}$-measurement. The minimum value of the longitudinal strain scatters from $-1.836\%$ to $-1.727\%$, while the maximum value of the stress lies between $-472.5$ MPa and $-446.0$ MPa, see Fig. 4.13. The end value of the inelastic strain $e_{I}^{S4}(t_{E4})$ varies from $-2.075\%$ to $-1.695\%$.

**Group $+4\%$ UA$_{360^\circ C}$** In Q154$^{UA|+4}_{360\text{-}50}$ the predeformation starts while the quenching is in progress. The maximum value of the longitudinal strain scatters from $+3.713\%$ to $+3.785\%$, ...
4. Transformation-Induced Plasticity – Experiments and Model Verifications

Figure 4.13.: Predeformation of undercooled austenite: Group $-2\%$ UA$_{280^\circ C}$

Figure 4.14.: Predeformation of undercooled austenite: Group $+4\%$ UA$_{360^\circ C}$

Figure 4.15.: Predeformation of undercooled austenite: Comparison of experiments vs. literature values (cf. [1] and Section 4.1.3)
4.1. Experiments and Data Post Processing

(a) Elastic range

while the maximum value of the stress lies between +417.2 MPa and +422.0 MPa, see Fig. 4.14.
The end value of the inelastic strain $e_{S4}^{l_s}(i_{E4})$ varies from +3.418% to +3.537%.

Conclusions for Part 4 Only a few experiments can be used to verify the elastic modulus, see
Fig. 4.15(a) and Fig. 4.16(a) – at least for isothermal fits. In every group (except +4% UA$_{300}$C)
the stress scatters over a wide range but in comparison to the experiments at $\theta_A$ with less noise
(relatively). Again, like for the predeformation at $\theta_A$, the $e_{S4}^{S4}$-S-graphs are far from a linear
dependency, see Fig. 4.15(b) and Fig. 4.16(b).

Remarks on single experiments In Q124$_{280}^{UA|+2}$, Q125$_{280}^{UA|+2}$, Q152$_{280}^{UA|+4}$, and Q154$_{300}^{UA|+4}$
the predeformation starts while the quenching is still in progress. In Q124$_{280}^{UA|+2}$ and Q125$_{280}^{UA|+2}$
the inelastic strain is influenced by the quenching. Q138$_{280}^{UA|+2}$ contains several outliers. In Q152$_{280}^{UA|+4}$
and Q212$_{280}^{UA|+4}$ there are big steps in longitudinal strain, while in Q130$_{280}^{UA|+2}$ we observe a little
drop in the stress. The measurement of the transversal strain of Q144$_{280}^{UA|+6}$ does not match
the other observation since it increases with increasing $e_L$, this results in an unordinary inelastic
strain.

4.1.6.5. Part 5: Bainitic Transformation

Observation range & Observations There are several possibilities when to start the observation.
On the one hand, after $\theta_B$ remains constant (or after predeformation), and on the other
hand when the transformation is starts. One way to obtain the start index of the transformation
period is to consider the evolution of the volume strain $e_V$ after quenching, since the transforma-
tion of austenite into bainite is accompanied by a characteristic volume change of the specimen,
cf. Fig. 4.17.

Still, defining reference strains which only consider transformation related effects is a nontrivial
task, since effects of the quenching are still observable after $\theta_B$ is reached, as well as the ap-
plied stress during transformation and the predeformation effects the volume strain. If $e_{S3}^{S3}$ stays
(nearly) constant for some time, it is easy to set the starting point, but in most experiments
the evolution of the volume strain (slowly) increases and we cannot separate the density effects
of the transformation effects, cf. Fig. 4.18(a). Therefore, we will define the starting point of the transformations under applied stress by the inelastic strain $e_I$, elsewhere we use the volume strain $e_V$. The inelastic strain depends on elastic moduli, Poisson’s ratios and on the phase fractions –

![Figure 4.17: Transformation at $\theta_B = 280^\circ$C: $e_V$-evolution after quenching for all experiments with predeformation of stable austenite](image1)

![Figure 4.18: Transformation at $\theta_B = 280^\circ$C: After quenching for all experiments with predeformation of stable austenite and without applied stress during transformation](image2)

all of those values respectively post-processing data have to be reviewed and verified carefully.

The experiments can be classified into ten groups for this part of the experimental procedure: Group $\pm 0$ MPa$_{280}$°C, Group +25 MPa$_{280}$°C, Group +50 MPa$_{280}$°C, Group -25 MPa$_{280}$°C, Group -50 MPa$_{280}$°C, Group $\pm 0$ MPa$_{360}$°C, Group +25 MPa$_{360}$°C, Group +50 MPa$_{360}$°C, Group -25 MPa$_{360}$°C and Group -50 MPa$_{360}$°C.

As in all other parts of the experiment, the data scatters a lot, see e.g. $e_V$ Fig. 4.17. The differences might also be influenced by pretreatment, but exemplarily in the Group $\pm 0$MPa$_{380}$°C there is no observable trend, see Fig. 4.18(b). A comparison of both considered transformation
temperatures shows that at 360°C the transformation is clearly faster and the volume strain increases to a lower stationary value than at 280°C; see exemplarily Figs. 4.19(a) and 4.19(b). For all experiments with negative applied stress, we see at the beginning a small elastic effect while for experiments with positive loading a noticeable jump is observable.

In the experiments Q145^{SA}_{360}+30, Q153^{UA}_{360}+40, Q155^{UA}_{360}+40, Q230^{SA}_{360}+25 and Q145^{SA}_{360}+40, little humps in the inelastic strain are visible before the evolution tends to a stable value.

In this part of the experiments, we have no repetition and therefore we cannot validate the experimental results. We will just describe the qualitative behavior of the material and its trends. The experimental results for the inelastic strain are illustrated in Figs. 4.20 - 4.26 and for the influence of predeformations on the resulting inelastic strains in Figs. 4.29 - 4.31.

**Influence of applied stress** In the groups with the same predeformation, positive applied stress leads to positive deviation of \( \varepsilon _{V}^{S5} \) related to the experiment without applied stress. Negative applied stress leads to negative deviation. Larger absolute stress leads to larger absolute deviation. The dependency of the stationary inelastic strain at the end of the transformation on the average applied stress is approximately linear considering a group of the same predeformation, cf. Figs. 4.27 and 4.28.

![Graphs showing the evolution of phase fractions and volume strains](image)

Figure 4.19.: Transformation at \( \theta _{B} \): \( p_{B} \) and \( \varepsilon _{V}^{S5} \) evolutions for all considered experiments

**Influence of predeformation at bainitizing temperature** At 280°C as well as at 360°C, the predeformation of the undercooled austenite influences the evolution of the inelastic strain significantly. For all applied stresses, positive predeformation of the undercooled austenite results in positive deviation, while negative predeformation results in negative deviation related to the blind test with no predeformation. Larger absolute value in predeformation results in larger absolute deviation. Positive as well as negative predeformation accelerates the transformation. Positive predeformation has greater influence, except in the experiments with +50 MPa applied stress, see Fig. 4.24.

**Influence of predeformation at austenitizing temperature** The trends of the experiments with predeformation at austenitizing temperature are not that conclusive. First of all, the influence
is one dimension smaller ($10^{-1}$) than for experiments with predeformation at $\theta_B$.

Applying no stress, negative predeformation leads to negative deviation, while small positive (Q101$_{280}^{SA|-4}$) results in small positive and larger positive predeformation (Q112$_{280}^{SA|+6}$) in negative deviation. For -25 MPa all predeformations lead to negative deviation, where the positive ones have larger influence. The same holds for +25 MPa whereas the negative predeformation have larger influence. In case of -50 MPa applied stress, the deviation for negative predeformation is in positive. Small positive predeformation results in positive deviation as well (Q105$_{280}^{SA|+4}$) and larger positive predeformation leads to negative deviation (Q105$_{280}^{SA|+6}$). For +50 MPa the situation is clearer - negative predeformation results in positive deviation and positive deformation in negative deviation. At 360°C, the influences are similar but the data situation less informative.

At 280°C, the transformation is accelerated for the Groups ±25 MPa$_{280}$ and +50 MPa$_{280}$, but slowed for the Groups ±0 MPa$_{280}$ and -50 MPa$_{280}$. Clear trends are not observable at 360°C. In the Group +50 MPa$_{360}$, the predeformation has no influence, where for Group -50 MPa$_{360}$ the transformation is slowed down and in Group ±0 MPa$_{360}$ negative predeformation leads to acceleration and positive one to a slow down.

**Conclusions for Part 5**  The predeformation of the undercooled austenite strongly influences the evolution of the inelastic strain. Considering Fig. 4.30, we can suggest a linear dependency between predeformation of the undercooled austenite and the resulting inelastic strain at 280°C. Since the influence of the predeformation of the stable austenite at 850°C is much smaller and the trends are partially inconsistent (cf. Fig. 4.29), the results are susceptible and might be influenced otherwise, e.g. little deviations in the heat treatment, deviations in material structure.

Summarized, in order to obtain conclusive quantitative statements, the number of conducted experiments with predeformations at austenitizing temperature is too small. This also holds for experiments with transformation at 360°C, see Fig. 4.31.

For now, we do not have an explanation for the little humps in some of the experiments at 360°C.

**Remarks on single experiments** Several experiments include outliers in this experimental part, e.g. Q106$_{280}^{SA|-4}$, Q119$_{280}^{SA|+6}$, Q122$_{280}^{SA|+6}$. In Q116$_{280}^{SA|+4}$, the applied stress for the transformation part begins before $\theta_B$ is constantly reached. Q111$_{280}^{SA|-50}$, Q114$_{280}^{SA|+25}$ and Q122$_{280}^{SA|-25}$ contain steps in their evolutions of the inelastic strain due to measurement errors. The inelastic strain of Q146$_{360}^{SA|-50}$ evolves positively. Due to the applied negative stress, we would expect a negative evolution instead.
4.1. Experiments and Data Post Processing

Figure 4.20.: Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_I^{SS}$ and the phase fractions $p_B$ for the Group $\pm 0$ MPa$_{280^\circ}$C

Figure 4.21.: Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_I^{SS}$ and the phase fractions $p_B$ for the Group $+25$ MPa$_{280^\circ}$C
4. Transformation-Induced Plasticity – Experiments and Model Verifications

Figure 4.22.: Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_1^{SS}$ and the phase fractions $p_B$ for the Group $+50$ MPa$_{280^\circ}$C

Figure 4.23.: Transformation at $\theta_B = 280^\circ$C: Evolutions of the (crawled) inelastic strains $e_1^{SS}$ and the phase fractions $p_B$ for the Group $-25$ MPa$_{280^\circ}$C
4.1. Experiments and Data Post Processing

Figure 4.24.: Transformation at $\theta_B = 280^\circ C$: Evolutions of the (crawled) inelastic strains $e_I^{SS}$ and the phase fractions $p_B$ for the Group $-50$ MPa$_{280^\circ C}$

Figure 4.25.: Transformation at $\theta_B = 360^\circ C$: Evolutions of the (crawled) inelastic (TRIP) strains $e_I^{SS}$ and the phase fractions $p_B$ for the Groups $\pm 0$ MPa$_{360^\circ C}$ and $\pm 25$ MPa$_{360^\circ C}$

128
4. Transformation-Induced Plasticity – Experiments and Model Verifications

Figure 4.26.: Transformation at $\theta_B = 360^\circ$C: Evolutions of the (crawled) inelastic (TRIP) strains $e_{I}^{S5}$ and the phase fractions $p_B$ for the Group $\pm 50$ MPa$_{360^\circ}$C

Figure 4.27.: Transformation at $\theta_B = 280^\circ$C: Average applied stresses vs. inelastic (TRIP) strains $e_{I}^{S5}$ at the end of transformation
4.1. Experiments and Data Post Processing

Figure 4.28.: Transformation at $\theta_B = 360^\circ C$: Average applied stresses vs. inelastic (TRIP) strains $e_{I5}$ at the end of transformation

Figure 4.29.: Transformation at $\theta_B = 280^\circ C$: Inelastic (plastic) strains $e_{I2}(i_{E2})$ of predeformation at $850^\circ C$ vs. inelastic (TRIP) strains $e_{I5}$ at the end of transformation

Figure 4.30.: Transformation at $\theta_B = 280^\circ C$: Inelastic (plastic) strains $e_{I4}(i_{E4})$ of predeformation at $280^\circ C$ vs. inelastic (TRIP) strains $e_{I5}$ at the end of transformation
4. Transformation-Induced Plasticity – Experiments and Model Verifications

Figure 4.31.: Transformation at $\theta_B = 360^\circ$C: Inelastic (plastic) strains $e^{S2}_{p(E2)}$ resp. $e^{S4}_{p(E4)}$ of predeformation at 360°C vs. inelastic (TRIP) strains $e^{S5}_I$ at the end of transformation

4.1.6.6. Part 6: Cooling to Room Temperature

Observation range The observation for this part starts when the external heating is turned off and temperature slowly decreases towards room temperature. The considered strains are $e^{S6}_L$ and $e^{S6}_D$.

Observations The volume strain nearly linearly decreases for all experiments, see Figs. 4.32(a) and 4.32(b).

Conclusions for Part 6 All experiments (excluding Q146$_{SA}\pm0_{360\pm50}$) verify the density of bainite, cf. red line in Figs. 4.32(a) and 4.32(b). The dependency of the volume strain $e_V$ to the temperature $\theta$ is nearly linear, may be slightly quadratic. There are no clear trends of deviations related to the pretreatment.

Remarks on single experiments In Q146$_{SA}\pm0_{360\pm50}$ the external heating is turned off, while the applied stress of the transformation part is still active. Due to this fact, we observe a positive deviation in the volume strain.

4.2. Thermoelastic-Plastic-TRIP Model

We recall the model ansatz of [68] to take the interaction between classic plasticity and transformation-induced plasticity into account. The model is a 3M model with one criterion as a combination of an uncoupled thermoelastic element (cf. Section 2.3.1) with a coupling of a plastic Chaboche model with one back stress and isotropic hardening (cf. Section 2.3.4.2) and a TRIP mechanism (cf. Section 2.3.7).

The conducted experiments concentrate on the mechanical behavior of the material, therefore we only consider the mechanical part of the model and will neglect the thermal coupling with the heat equation. Furthermore, we derive the phase fractions from the experimental data and do not consider evolution equations for them nor do we assume the phase fractions as inner variables. Moreover, only two phase fractions are considered – austenite and bainite. Extended considerations are for future work.

First, we state a simplified 3d model under the assumptions from above, then we reduce the model to the uniaxial experiment situation. The reduced model is used to conduct a parameter
identification for selected experiments.

4.2.1. MM Model for Interaction of Classic Plasticity and TRIP

We embed the outline of the model into the concept presented in Chapter 2. We recollect the basic equations - valid in a time-space domain $]0, T[\times \Omega$ - of the balance of momentum (2.2.1a) and the balance of energy (2.2.1b)

\begin{align}
(4.2.1) & \quad \rho \partial_t u - \text{Div} \sigma = f, \\
(4.2.2) & \quad \rho \partial_t \varepsilon + \text{div} q = \sigma : \partial_t \varepsilon + r,
\end{align}

the definition of the Green strain tensor (2.2.2), the assumption of the additive split of the strain tensor in a sum of 3 partial strains $\varepsilon_\mathcal{N}$ (2.2.4), $\mathcal{N} := \{0, 1, 2\}$, and the definition of partial stresses $\sigma_\mathcal{N}$ (2.2.11)

\begin{equation}
(4.2.3) \quad \varepsilon = \frac{1}{2} (\nabla u + \nabla u^T), \quad \varepsilon = \sum_{j \in \mathcal{N}} A_j \varepsilon_j, \quad \sigma_j = A_j \sigma, j \in \mathcal{N}
\end{equation}

and the relations between the internal energy, free energy and entropy

\begin{equation}
(4.2.4) \quad e = \psi + \theta \eta, \quad \eta = -\frac{\partial \psi}{\partial \theta}.
\end{equation}

The partial strains are identified by the thermoelastic strain $\varepsilon_{te} := \varepsilon_0$, the plastic strain $\varepsilon_p := \varepsilon_1$ and the TRIP strain $\varepsilon_{tp} := \varepsilon_2$. Other related quantities are denoted by the related subscript. Both inelastic strains, $\varepsilon_p$ and $\varepsilon_{tp}$, possess one tensorial inner variable, $\alpha_p$ respectively $\alpha_{tp}$, to cover kinematic hardening effects, thus $\mathcal{J}_a := \{1, 2\}$. The plastic strain is related to one scalar inner variable, $r_p := a_1$, as well to include isotropic hardening. We consider a two-phase situation and define $p_A := a_2$, $p_B := a_3$, so $\mathcal{J}_a := \{1, 2, 3\}$. For simplicity, we fixed the weights of the mechanisms,

\begin{equation}
(4.2.5) \quad A_{te} = A_p = A_{tp} = 1.
\end{equation}
such that
\begin{equation}
\sigma = \sigma_{te} = \sigma_p = \sigma_{tp}.
\end{equation}
We consider the thermoelastic mechanism uncoupled from the plastic and TRIP mechanisms. The free energy is a split of a thermoelastic part \( \psi_{te} \), an inelastic part \( \psi_{in} \) and a chemical part \( \psi_{ch} \),
\begin{equation}
\psi = \psi_{te}(\theta, \varepsilon_{te}, p_A, p_B) + \psi_{in}(\theta, \alpha_p, \alpha_{tp}, r_p, p_A, p_B) + \psi_{ch}(\theta, \varepsilon_{te}, \alpha_p, \alpha_{tp}, r_p, p_A, p_B).
\end{equation}

The chemical part \( \psi_{ch} \) represents the chemically stored energy with respect to phase transformations and the explicit interaction with the mechanisms, cf. e.g. [44] for specifications. We define the parts of the free energy as follows,
\begin{align}
\psi_{te} &:= \frac{1}{2\theta} (\varepsilon_{te} : \mathbf{E}_{te} : \varepsilon_{te} + \gamma_{te} : \varepsilon_{te}) - C_\theta(\theta, p_{\text{ph}}), \\
\psi_{in} &:= \frac{1}{2\theta} (\alpha_p : C_p : \alpha_p + 2\alpha_p : C_{\text{int}} : \alpha_{tp} + \alpha_{tp} : C_{tp} : \alpha_{tp}) + \gamma_p r_p^2, \\
\psi_{ch} &:= \frac{1}{2\theta} \left( a_{AP}^2 + a_{AB}^2 P_{AB}^2 + \gamma_A^p p_A + \gamma_B^p p_B + \sum_{k=1}^3 (d_{k,te,A} p_A + d_{k,te,B} p_B) J^k(\varepsilon_{te}) \\
&+ \sum_{k=1}^3 (d_{k,te,A} p_A + d_{k,te,B} p_B) J^k(\alpha_p) + \sum_{k=1}^3 (d_{k,te,A} p_A + d_{k,te,B} p_B) J^k(\alpha_{tp}) \right).
\end{align}

Note the parameters notation of Sections 2.3.1, 2.3.4.2 and 2.3.7. In our concept, we have to ensure the convexity of the free energy with respect to the inner variables for fixed temperature. Hence, the coefficients underlie certain restrictions. We will specify them later. All coefficients may depend on the phase fractions \( p_A, p_B \) and the temperature \( \theta \). We recall once again the mixture rule (2.3.87a) defined in Section 2.3.7 for two phase fractions
\begin{equation}
C(\theta, p_A, p_B) := C_A(\theta) p_A + C_B(\theta) p_B = C_B(\theta) + (C_A(\theta) - C_B(\theta)) p_A = C_A(\theta) + (C_B(\theta) - C_A(\theta)) p_B.
\end{equation}

As usual the thermomechanical forces are given by
\begin{equation}
X_{te} := \varrho \frac{\partial \psi}{\partial \varepsilon_{te}}, \quad X_p := \varrho \frac{\partial \psi}{\partial \alpha_p}, \quad X_{tp} := \varrho \frac{\partial \psi}{\partial \alpha_{tp}}, \quad R_p := \varrho \frac{\partial \psi}{\partial r_p}, \quad Z_A := \varrho \frac{\partial \psi}{\partial p_A}, \quad Z_B := \varrho \frac{\partial \psi}{\partial p_B}.
\end{equation}

The thermomechanical forces \( Z_A \) and \( Z_B \) are chemical forces, cf. [44].

With regard to the later application to the experiments described in Section 4.1, we simplify our approach. We neglect the chemical part of the free energy \( \psi_{ch} \). Moreover, the phase fractions of austenite \( p_A \) and bainite \( p_B \) are determined by the experiments and are considered therefore as given parameters. This is obviously a restriction to the model, but since we do not cover thermal effects in the experiment these simplifications seem reasonable. The bainitic transformation can be described by a Johnson-Mehl-Avrami approach, see e.g. [32]. Furthermore, we assume the isotropic case such that the thermoelastic mechanism is modified by
\begin{align}
\varepsilon_{te} : \mathbf{E}_{te} : \varepsilon_{te} &= 2\mu \varepsilon_{te} + \lambda_L(\varepsilon_{te} : I)^2, \\
\gamma_{te} &= -3K \left( \sqrt[3]{\frac{\varrho_0}{\varrho(\theta, p_A, p_B)}} - 1 \right) I.
\end{align}

Hereby, \( \mu \) is the shearing modulus, \( \lambda_L \) the second Lamé coefficient and \( K \) the compression
4.2. Thermoelastic-Plastic-TRIP Model

These parameters can be reformulated by the elastic modulus $E$ and Poisson’s ratio $\nu$

$$\mu = \frac{E}{2(1+\nu)}, \quad \lambda_L = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad K = \lambda_L + \frac{2}{3}\mu = \frac{E}{3(1-2\nu)}.$$  

The coefficients of the inelastic mechanisms are simplified to

$$C_p := \frac{2}{3}c_p I^{[4]}, \quad C_{int} := \frac{2}{3}c_{int} I^{[4]}, \quad C_{tp} := \frac{2}{3}c_{tp} I^{[4]}.$$  

$I^{[4]}$ denotes the unit tensor of 4-th order such that $\alpha : I^{[4]} : \beta = \alpha : \beta$ for tensors $\alpha, \beta$ of second order. The coefficients $c_p, c_{int}$ and $c_{tp}$ are scalar. The factor $\frac{2}{3}$ is introduced to ensure the compatibility of identified parameters by uniaxial experiments to the 3d model, cf. e.g. [76].

The parameters $c_p, c_{tp}$ and $c_{int}$ have to fulfill the restriction

$$c_{int}^2 \leq c_pc_{tp}$$

due to the convexity of the free energy for fixed temperature. We assume that inelastic material effects are volume preserving such that

$$\text{tr}(\varepsilon_p) = 0, \quad \text{tr}(\varepsilon_{tp}) = 0, \quad \text{tr}(X_p) = 0, \quad \text{tr}(X_{tp}) = 0.$$  

The evolution functional related to the thermoelastic strain is defined by

$$F_{\varepsilon}^{\varepsilon}(\theta, \sigma, \varepsilon_{te}) := \sigma - X_{te} = \sigma - \left(2\mu \varepsilon_{te} + \lambda_L \varepsilon_{te} : I - 3K \left(\sqrt{\frac{\eta_0}{\varrho(\theta, p_A, p_B)}} - 1\right) I\right).$$

The plastic evolution is determined by

$$F_{\varepsilon}^{\varepsilon}(\theta, \sigma, \varepsilon_p, X_p, R) := \partial_t \varepsilon_p - \left(\lambda_p \frac{\sigma - X_p^*}{\|\sigma - X_p^*\|_M}\right) \cdot (f_p(\sigma^* - X_p^*, R) = 0)$$

with

$$f_p(\sigma - X_p, R) := \|\sigma^* - X_p^*\|_M - (R + R_0),$$

$R_0$ being the initial yield stress. $R$ describes the isotropic hardening and is determined by the evolution functional related to the inner variable $r_p$

$$F^{\alpha}(\theta, \varepsilon_p, r_p, R) := \partial_t r_p - \left(1 - \frac{\beta_p}{\gamma_p} R\right) \lambda_p.$$  

The evolution of the isotropic hardening variable $r_p$ is a special case of the formula presented in Section 2.3.4 ($g_1 = 1, \quad b_1 = \beta_p, \quad c_{11} = \gamma_p$). We define an Armstrong-Frederick approach for the kinematic hardening,

$$F_{\varepsilon}^{\alpha}(\theta, \varepsilon_p, X_p) := \partial_t \varepsilon_p - \partial_t \alpha_p - \partial_t \varepsilon_p + \frac{3}{2} \frac{a_p}{c_p} X_p \partial_t s_p.$$  

We recall parts of Section 2.3.7 for the TRIP mechanism. The evolution functional related the the TRIP strain is given by

$$F_{\varepsilon}^{\alpha}(\theta, \sigma, \varepsilon_{tp}, X_{tp}) := \partial_t \varepsilon_{tp} - \frac{3}{2} \kappa(\theta, p_A, p_B)(\sigma^* - X_{tp}^*),$$
where $\kappa$ denotes the function
\begin{equation}
(4.2.25) \quad \kappa(\theta, p_A, p_B) := \kappa_0 A(\theta) \frac{\partial \Phi_A}{\partial p_A} (\partial_t p_A) + \kappa_0 B(\theta) \frac{\partial \Phi_B}{\partial p_B} (\partial_t p_B)
\end{equation}

and $\kappa_0 A$ and $\kappa_0 B$ are the Greenwood-Johnson parameters of austenite respectively bainite. Note that just the part of the forming phase is active. For the saturation functions $\Phi_i$, $i = A, B$, we consider three approaches:
\begin{align}
(4.2.26a) \quad & \Phi_i(p_i) = p_i, \quad \text{Tanaka,} \\
(4.2.26b) \quad & \Phi_i(p_i) = p_i(1 - \ln(p_i)), \quad \text{Leblond,} \\
(4.2.26c) \quad & \Phi_i(p_i) = p_i(2 - p_i), \quad \text{Desalos, Denis.}
\end{align}

The hardening variable related to TRIP evolves also by an Armstrong-Frederick approach
\begin{equation}
(4.2.27) \quad F^\alpha tp(\theta, \varepsilon tp, X tp) := \partial_t \alpha tp - \partial_t \varepsilon tp + \frac{3}{2} a tp c tp X tp \partial_t s tp.
\end{equation}

We state the equations for the thermomechanical forces given by their definition (4.2.12)
\begin{align}
(4.2.28) \quad & X_p = \frac{2}{3} c_p \alpha p + \frac{2}{3} c_{int} \alpha tp, \\
(4.2.29) \quad & X tp = \frac{2}{3} c_{int} \alpha p + \frac{2}{3} c tp \alpha tp, \\
(4.2.30) \quad & R = \gamma_p r p.
\end{align}

The coupling of the back stresses by the inner variables determines the coupling of the model. In order to complete the model, we have to define initial conditions for all differential equations,
\begin{equation}
(4.2.31) \quad \varepsilon p(0) := \varepsilon^0 p, \quad \varepsilon tp(0) := \varepsilon^0 tp, \quad \alpha p(0) := \alpha^0 p, \quad \alpha tp(0) := \alpha^0 tp, \quad r p(0) := r^0 p.
\end{equation}

For undeformed materials these values are set to 0.

### 4.2.2. 1d Model for Uniaxial Experiments

In case of uniaxial experiments, the material laws simplify. In our conducted experiments, the direction of interest is the $x$-direction. Under the assumption of traceless inelastic quantities, this results in
\begin{align}
(4.2.32) \quad & \sigma = \begin{pmatrix} S & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \sigma^* = \begin{pmatrix} \frac{2}{3} S & 0 & 0 \\ 0 & -\frac{1}{3} S & 0 \\ 0 & 0 & -\frac{1}{3} S \end{pmatrix}, \\
(4.2.33) \quad & \varepsilon_{in} = \begin{pmatrix} \varepsilon_{in}^{11} \\ 0 \\ 0 \end{pmatrix}, \quad X_{in} = \begin{pmatrix} X_{in}^{11} \\ 0 \\ 0 \end{pmatrix}, \quad \varepsilon_{in} = \begin{pmatrix} \varepsilon_{in}^{11} \\ 0 \\ 0 \end{pmatrix}, \quad X_{in} = \begin{pmatrix} X_{in}^{11} \\ 0 \\ 0 \end{pmatrix},
\end{align}

where $\varepsilon_{in}$ and $X_{in}$ stand for plastic as well as the TRIP quantities. The longitudinal strain $\varepsilon_L$ and the transversal strain $\varepsilon_D$ are given by
\begin{align}
(4.2.34) \quad & \varepsilon_L = \frac{S}{E} + \left( \sqrt{\frac{20}{q}} - 1 \right) + \frac{1}{2} \varepsilon_{in}^{11} = \varepsilon^{11}, \\
(4.2.35) \quad & \varepsilon_D = -\nu S \frac{E}{E} + \left( \frac{3}{2} \sqrt{\frac{20}{q}} - 1 \right) - \frac{3}{2} \varepsilon_{in}^{11} = \varepsilon^{22} = \varepsilon^{33}.
\end{align}
4.2. Thermoelastic-Plastic-TRIP Model

The equation for the inelastic strain in $x$-direction is then derived as

\[(4.2.36)\]
\[
e_{in} := \varepsilon_{in}^{11} = \frac{2}{3}(e_L - e_D) - \frac{S}{3\mu}.
\]

We recall the definition of the volume strain $e_V$,

\[(4.2.37)\]
\[
e_V = \text{tr}(\varepsilon) = e_L + 2e_D = \left(1 - \frac{\nu}{E}\right)S + 3\left(\sqrt{\frac{\rho_0}{\rho}} - 1\right).
\]

Moreover, we define the new 1d quantities for the back stresses, inelastic strains and inner variables

\[(4.2.38)\]
\[
x_p := \frac{3}{2}X_p^{11}, \quad x_{tp} := \frac{3}{2}X_{tp}^{11}, \quad e_p := \varepsilon_{p}^{11}, \quad e_{tp} := \varepsilon_{tp}^{11}, \quad \alpha_p := \alpha_p^{11}, \quad \alpha_{tp} := \alpha_{tp}^{11}.
\]

In the uniaxial case, it holds for the accumulated strains

\[(4.2.39)\]
\[
\partial_t s_p = |\partial_t e_p|, \quad \partial_t s_{tp} = |\partial_t e_{tp}|.
\]

The evolution equations of the 3d model are modified according to the above derivation. The plastic strain is given by

\[(4.2.40)\]
\[
\partial_t e_p = \lambda_p \frac{S - x_p}{|S - x_p|} \cdot (f_p(S - x_p, R) = 0), \quad e_p(0) = e_p^0,
\]

with the yield function

\[(4.2.41)\]
\[
f_p(S - x_p, R) = |S - x_p| - (R + R_0).
\]

Based on the yield condition, the plastic multiplier $\lambda_p$ can be determined as

\[(4.2.42)\]
\[
\lambda_p = |\partial_t e_p| = \partial_t s_p.
\]

The isotropic hardening law does not change since it is already a scalar equation,

\[(4.2.43)\]
\[
\partial_t r_p = \left(1 - \frac{\beta_p}{\gamma_p} R\right) \cdot R = \gamma_p r_p, \quad r_p(0) = r_p^0.
\]

The isotropic hardening saturates to $\frac{\gamma_p}{\beta_p}$ and for isothermal without any hardening history situations ($r_p(0) = 0$), we can solve the hardening law to

\[(4.2.44)\]
\[
R = \frac{\gamma_p}{\beta_p} \left(1 - e^{-\beta_p s_p} \right).
\]

The kinematic hardening law is determined by

\[(4.2.45)\]
\[
\partial_t \alpha_p = \partial_t e_p - \frac{a_p}{c_p}x_p\partial_t s_p, \quad \alpha_p(0) = \alpha_p^0.
\]

The TRIP mechanism is given in the uniaxial case by

\[(4.2.46)\]
\[
\partial_t e_{tp} = \kappa(\theta, p_A, p_B)(S - x_{tp}), \quad e_{tp}(0) = e_{tp}^0,
\]

with the related hardening law

\[(4.2.47)\]
\[
\partial_t \alpha_{tp} = \partial_t e_{tp} - \frac{a_{tp}}{c_{tp}}x_{tp}\partial_t s_{tp}, \quad \alpha_{tp}(0) = \alpha_{tp}^0.
\]
The scalar function $\kappa$ matches the 3d cases, see (4.2.25). The interaction of the mechanism is given by

\begin{align}
  x_p &= c_p \alpha_p + c_{\text{int}} \alpha_{tp}, \\
  x_{tp} &= c_{\text{int}} \alpha_p + c_{tp} \alpha_{tp}.
\end{align}

### 4.2.2.1. Numerical Scheme of the 1d Model for Uniaxial Experiments

The set of equations of Section 4.2.2 is the basis for a parameter identification of the in Section 4.1 presented experiments. In order to conduct a parameter identification, we have to state a numerical scheme of the 1d model.

We adapt the used algorithm to our situation and separate the plastic and TRIP part, since plasticity and TRIP are not occurring at the same time in the experiment. The algorithm for the plastic parts of the experiments is similar to the return mapping algorithm for plasticity, cf. [56]. We implement the plastic material law with a strain driven semi implicit algorithm, see Alg. 1.

In case of calculating TRIP, we use an explicit stress driven algorithm, see Alg. 2. The coupling is considered by passing the end value of $\alpha_p$ of Alg. 1. The plastic inner variable does not evolve during the transformation, since the yield stress is not exceeded and is therefore fixed. The saturation function $\kappa$ can be partially precalculated and is just arbitrary in $\kappa_{OB}$. In case of the Leblond ansatz, we have to ensure $p_B \neq 0$. For a detailed discussion including the singularity, we refer to [75]. The part of the saturation function $\kappa$ depending on $p_B$, $\kappa^i_w$, see Alg. 2, can and should be precalculated and smoothed. We calculate it by

\begin{align}
  \kappa^i_w &:= \kappa^i \Phi \cdot (p^i_B - p^{i-1}_B) / (t^i - t^{i-1}), \quad k^1_w := 0.
\end{align}

Both choices of for the algorithms seem reasonable, since in the experiments the predeformation is strain controlled, whereas during the transformation a constant stress is applied.

If the index is suppressed for a variable or parameter, we understand it as data vector and the operation is component-wise.

---

\footnote{We also tried a semi implicit stress driven, an explicit strain driven and an explicit stress driven algorithm.}

The presented algorithm is the most stable one with the best performance related to our data.
4.2. Thermoelastic-Plastic-TRIP Model

Algorithm 1 Semi implicit strain driven algorithm for classic plasticity with back stress

1: load $N$ (number of data points), $t$, $\theta$, $e_L$ \hspace{1cm} \triangleright \text{prepared measurement data}$
2: load $\varrho$, $E$, $\nu$, $R_0$ \hspace{1cm} \triangleright \text{material parameter}$
3: set $a_p$, $c_p$, $\beta_p$, $\gamma_p$ \hspace{1cm} \triangleright \text{parameter to be optimized}$
4: initialize $e^i_{te} = 0$, $e^i_p = 0$, $x^i_p = 0$, $s^i_p = 0$, $R^i = 0$, $\alpha^i_p = 0$, $S^i = 0$
5: $e_\varrho = \sqrt{\frac{\rho^i_p}{\varrho}} - 1$
6: for $i = 2 : N$ do
7: \hspace{1cm} $S_i = E^i (e^i_L - e^{i-1}_p - e^i_\varrho)$ \hspace{1cm} \triangleright \text{trial stress}$
8: \hspace{1cm} if $|S_i - x^{i-1}_p| - (R^i_0 + R^{i-1}) < 0$ then \hspace{1cm} \triangleright \text{elastic branch}$
9: \hspace{1.5cm} $e^i_p = c^i_p$, $x^i_p = x^{i-1}_p$, $R^i = R^{i-1}$, $s^i_p = s^{i-1}_p$, $\alpha^i_p = \alpha^{i-1}_p$
10: \hspace{1cm} else \hspace{1cm} \triangleright \text{plastic branch}$
11: \hspace{1.5cm} if $i=2$ then
12: \hspace{2cm} $\alpha^i_p = \alpha^{i-1}_p + (t^i - t^{i-1}) \cdot AF\left(x^{i-1}_p, \frac{e^{i-1}_p}{\rho_p}, \frac{a_p}{\gamma_p}\right)$
13: \hspace{1.5cm} else
14: \hspace{2cm} $\alpha^i_p = \alpha^{i-1}_p + (t^i - t^{i-1}) \cdot AF\left(x^{i-1}_p, \frac{e^{i-1}_p - e^{i-2}_p}{\rho_p - E^{i-1} + \rho_p}, \frac{a_p}{\gamma_p}\right)$
15: \hspace{1cm} end if
16: \hspace{1cm} $x^i_p = c_p \alpha^i_p$
17: \hspace{1cm} $s^i_p$: solve (numerically)
18: \hspace{1cm} $|S_i - E^i \text{sign}(S_i - x^{i-1}_p)(s^i_p - s^{i-1}_p) - x^i_p| - R^i_0 - \left(\frac{c_p}{\beta_p} (1 - \exp(-\beta_p s^i_p)) \right)$ \hspace{1cm} for $s^i_p$
19: \hspace{1cm} $R^i = \frac{c_p}{\beta_p} (1 - \exp(-\beta_p s^i_p))$
20: \hspace{1cm} $e^i_p = e^{i-1}_p + \text{sign}(S^i_p - x^i_p)(s^i_p - s^{i-1}_p)$
21: \hspace{1cm} end if
22: \hspace{1cm} $e^i_{te} = e^i_L - e^i_\varrho - e^i_p$
23: $S^i = E^i e^i_{te}$
24: end for
25: $e_D = \frac{\rho^i}{E^i} S - \frac{1}{2} e_p$
26: $e_I = e_p$
27: return $S$, $e_D$, $e_{te}$, $e_\varrho$, $e_p$, $e_I$, $x_p$, $s_p$, $R$, $\alpha_p$

28: function $AF(x,y,c)$ \hspace{1cm} \triangleright \text{Armstrong-Frederick type evolution}$
29: \hspace{1cm} $AF(x, y, b) = y - b \cdot x \cdot |y|$  
30: end function
Algorithm 2: Explicit stress driven algorithm for TRIP with back stress and interaction to classic plasticity

1: load \( N, t, \theta, S, p_B \)  \( \triangleright \) prepared measurement data
2: load \( \rho, E, \nu, \mu, \kappa_w \)  \( \triangleright \) precalculated material parameter
3: load \( \alpha_p^* \)  \( \triangleright \) precalculated inner variable of plasticity
4: set \( \kappa_0 B, \alpha_{tp}, c_{tp}, c_{int} \)  \( \triangleright \) parameter to optimize
5: initialize \( e_{te} = 0, e_{tp} = 0, x_{tp} = 0, \alpha_{tp} = 0, S^i = 0, x_{tp} = c_{int}\alpha_p^* \)
6: \( e_{\varphi} = \sqrt{\frac{\rho}{\rho}} - 1 \)
7: \( e_{te} = \frac{S}{E} \)
8: for \( i = 2 : N \) do
9:  if \( i = 2 \) then
10:      \( \alpha_{tp}^{i} = \alpha_{tp}^{i-1} + (t^{i} - t^{i-1}) \cdot AF\left(x_{tp}^{i-1}, e_{tp}^{i-1}, \kappa_{p}^{i}, \alpha_{tp}^{i}\right) \)
11:  else
12:      \( \alpha_{tp}^{i} = \alpha_{tp}^{i-1} + (t^{i} - t^{i-1}) \cdot AF\left(x_{tp}^{i-1}, e_{tp}^{i-1}, e_{tp}^{i-2}, \kappa_{p}^{i}, \alpha_{tp}^{i}\right) \)
13:  end if
14:  \( x_{tp}^{i} = c_{int}\alpha_p^* + c_{tp}\alpha_{tp}^{i} \)
15:  \( s_{tp}^{i} = s_{tp}^{i-1} + |e_{tp}^{i} - e_{tp}^{i-1}| \)
16:  \( e_{tp}^{i} = e_{tp}^{i-1} + (t^{i} - t^{i-1}) \cdot \kappa_0 B \kappa_w (S^i - x_{tp}^{i}) \)  \( \triangleright \) see (4.2.50) for \( \kappa_w \)
17: end for
18: \( e_L = e_{te} + e_{\varphi} + e_{tp} \)
19: \( e_D = -\nu S - \frac{1}{2} e_{tp} \)
20: \( e_I = e_{tp} \)
21: return \( e_L, e_D, e_{te}, e_{\varphi}, e_{tp}, e_I, x_{tp}, \alpha_{tp}, s_{tp} \)
22: function \( AF(x, y, c) \)  \( \triangleright \) Armstrong-Frederick type evolution
23:     \( AF(x, y, b) = y - b \cdot x \cdot |y| \)
24: end function
4.3. Parameter Identification

The parameter identification is conducted for selected experiments of Section 4.1 with the Alg. 1 (predeformation, Part 2 resp. Part 4) and Alg. 2 (TRIP, Part 5) of Section 4.2.2.1. The units of the sought parameters are MPa for \(c_p\), \(c_{tp}\), \(c_{int}\) and \(\gamma_p\), MPa for \(\kappa_0B\), while \(a_p\), \(a_{tp}\) and \(\beta_p\) are unitless.

4.3.1. Identification for Predeformation

The aim is to obtain three suitable parameter sets for \(a_p\), \(c_p\), \(\beta_p\) and \(\gamma_p\) at 850°C, 280°C resp. 360°C such that the experiments are well described by the strain driven semi implicit algorithm. The optimization is obtained by minimizing the cost functional

\[
F_p := \sum_{j \in \mathcal{N}} \frac{N_j}{N^*_N} \left( \frac{\|e_p^{exp} - e_p^{calc}\|^2_2}{\|e_p^{exp}\|^2_2} + \frac{\|S^{exp} - S^{calc}\|^2_2}{\|S^{exp}\|^2_2} \right)
\]

for every \(i\)-th experiment, \(N^*_N := \sum_{j \in \mathcal{N}} N_j\), \(e_p^{exp}\) the data vector of the prepared inelastic (plastic) strain, \(e_p^{calc}\) the data vector of the calculated inelastic strain, \(S^{exp}\) the data vector of the prepared stress and \(S^{calc}\) the data vector of the calculated stress.

Preparing Data Due to the noisy data, especially for the stress, we prepare the data by smoothing using the MATLAB routine `smooth` with the option `moving` for a moving average over 40 data points for the longitudinal and transversal strains (≈ 2-5% (SA) and 1-2% (UA) of the considered index range) and 100 data points for the stress (≈ 6-12% (SA) and 2-6% (UA) of the considered index range). We assume that the test specimens contain only austenite after austenitization \(p_A \equiv 1\) and precalculate the mass density \(\rho\), elastic modulus \(E\), Poisson’s ratio \(\nu\), shearing modulus \(\mu\) and the yield stress \(R_0\) by the given material data of Section 4.1.3. Since no material data is available for the yield stress \(R_0\), we approximate it for our experiments with \(R_0 := E e^{ref}_{L} \) and \(e^{ref}_{L} := 5 \cdot 10^{-4}\) being a reference strain to which we can assume elastic material behavior. At 850°C, this leads to a yield stress of 39.41 MPa, (280°C, \(R_0 = 102.48 \text{ MPa}; 360°C, R_0 = 93.63 \text{ MPa}\). The inelastic strain is calculated by (4.1.9) with the prepared remaining data. Every alteration in this preparing steps will lead to deviations in the parameter optimization.

Optimization Procedure It is not convenient to start right away optimizing for all experiments of one temperature. First, we have to reject experiments which are not suitable for optimization due to scattering, measurement errors or unusual experiment process, see Tab. 4.3. Our optimization procedure is based on four steps for every of the three temperatures:

1. Select one group of experiments with the same predeformation and one representative experiment within with the most characteristic measurements of the group. Conduct the optimization for 80 start parameters for this representative experiment.

2. Selected the parameter combinations with the 25 smallest errors of Step 1 and optimize over the entire group with the same deformation.

3. Selected the parameter combinations with the 10 smallest errors of Step 2 and optimize over the group of experiments with the same absolute deformation.

4. Selected the parameter combinations with the 5 smallest errors of Step 3 and optimize all experiments of one temperature.

\(3\) We have chosen the following nonstandard options `MaxFunEvals',10e3,'MaxIter',10e3,'TolX',1e-8.
Table 4.3.: Performed experiments. QXYZ representativ experiment for a group of predeformation, QXYZ unsuitable for optimization for predeformation, QXYZ unsuitable for optimization for transformation part

<table>
<thead>
<tr>
<th>Predeformation</th>
<th>Applied stress in [MPa] at 280°C</th>
<th>Applied stress in [MPa] at 360°C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
<td>+25</td>
</tr>
<tr>
<td>+0% SA</td>
<td>Q98</td>
<td>Q114</td>
</tr>
<tr>
<td>+4% SA</td>
<td>Q101</td>
<td>Q116</td>
</tr>
<tr>
<td>+6% SA</td>
<td>Q113</td>
<td>Q118</td>
</tr>
<tr>
<td>-4% SA</td>
<td>Q106</td>
<td>Q120</td>
</tr>
<tr>
<td>-6% SA</td>
<td>Q110</td>
<td>Q122</td>
</tr>
<tr>
<td>+2% UA</td>
<td>Q124</td>
<td>Q140</td>
</tr>
<tr>
<td>+4% UA</td>
<td>Q152</td>
<td></td>
</tr>
<tr>
<td>+6% UA</td>
<td>Q142</td>
<td>Q143</td>
</tr>
<tr>
<td>-2% UA</td>
<td>Q135</td>
<td>Q137</td>
</tr>
</tbody>
</table>

\[ A_{X_p} := \frac{c_p}{\sigma_p}, \quad A_R := \frac{\gamma_p}{\beta_p}. \] (4.3.2)

Therefore, we have to ensure that the optimization starts with parameters such that the stationary values are not exceed by the calculation.\(^4\) The minimum limiting value for the sum of \(A_{X_p}\) and \(A_R\) is

\[ G := \max\{S_{Q_{rep}}^{exp}\} - R_0, \] (4.3.3)

with \(S_{Q_{rep}}^{exp}\) being data set of the prepared stress of the representative experiment, so that we choose our starting parameter by

\[ G_{safe}^{i} = A_{X_p} + A_R, \quad G_{i}^{safe} := \delta_i G \] (4.3.4)

with \(\delta_1 = 1.5, \delta_2 = 2.5\). We set \(A_{X_p}\) and calculate \(A_R\). The starting values for \(A_X\) are \(\{1, G_{1}^{safe}/4, G_{1}^{safe}/2, G_{2}^{safe}/4, G_{2}^{safe}/2\}\). Moreover, we chose in case of predeformation at 850°C \(\sigma_p \in \{40, 70, 90, 200\}\), \(\beta_p \in \{50, 100, 250, 500, 731\}\) MPa and in case of predeformation at 280°C resp. 360°C \(\sigma_p \in \{10, 50, 100, 1000\}\), \(\beta_p \in \{500, 1000, 1300, 2000\}\) MPa and calculate the starting values of \(c_p\) and \(\gamma_p\) by the set parameters. The choice of parameters is influenced by manual fitting tries.

In the following sections, we present the results of the parameter identification using the stated optimization procedure. The results are discussed in Section 4.3.1.4.

4.3.1.1. Results for 850°C

We state the best resulting parameter sets for every step while choosing different starting groups concerning the predeformation at 850°C. The considered experiments are selected according to

\(^4\)Since we only use the explicit formulation for isotropic hardening, these stationary values can be overshooted leading to unreasonable numeric evaluations.
4.3. Parameter Identification

Starting with Group +4% SA  \(Q_{103}^{SA|+4}_{280|-50}\) is chosen as the representative experiment of this group. The results are stated in Tab. 4.4. The value of the cost functional (4.3.1) for the parameter set of Step 4 is \(5.4065 \cdot 10^{-5}\).

<table>
<thead>
<tr>
<th>Step</th>
<th>(a_p) [-]</th>
<th>(c_p) [MPa]</th>
<th>(\beta_p) [-]</th>
<th>(\gamma_p) [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>206.86</td>
<td>3471.01</td>
<td>393.41</td>
<td>6963.80</td>
<td>Q103</td>
</tr>
<tr>
<td>2</td>
<td>32.64</td>
<td>1.81</td>
<td>265.49</td>
<td>8897.57</td>
<td>Group +4% SA</td>
</tr>
<tr>
<td>3</td>
<td>100.79</td>
<td>440.61</td>
<td>334.36</td>
<td>10088.16</td>
<td>Groups ±4% SA</td>
</tr>
<tr>
<td>4</td>
<td>239.10</td>
<td>8218.27</td>
<td>61.07</td>
<td>14.61</td>
<td>Groups ±4%, ±6% SA</td>
</tr>
</tbody>
</table>

Table 4.4.: Results of parameter optimization starting with Group +4% SA

Starting with Group -4% SA  \(Q_{106}^{SA|-4}_{280|\pm0}\) is chosen as the representative experiment of this group. The results are stated in Tab. 4.4. The value of the cost functional (4.3.1) for the parameter set of Step 4 is \(5.4610 \cdot 10^{-5}\).

<table>
<thead>
<tr>
<th>Step</th>
<th>(a_p) [-]</th>
<th>(c_p) [MPa]</th>
<th>(\beta_p) [-]</th>
<th>(\gamma_p) [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>87.38</td>
<td>1330.70</td>
<td>618.68</td>
<td>13792.58</td>
<td>Q106</td>
</tr>
<tr>
<td>2</td>
<td>22.42</td>
<td>87.58</td>
<td>399.24</td>
<td>13711.76</td>
<td>Group -4% SA</td>
</tr>
<tr>
<td>3</td>
<td>64.55</td>
<td>413.99</td>
<td>399.70</td>
<td>11434.06</td>
<td>Groups ±4% SA</td>
</tr>
<tr>
<td>4</td>
<td>199.78</td>
<td>5887.76</td>
<td>741.00</td>
<td>5014.49</td>
<td>Groups ±4%, ±6% SA</td>
</tr>
</tbody>
</table>

Table 4.5.: Results of parameter optimization starting with Group -4% SA

Starting with Group +6% SA  \(Q_{104}^{SA|+6}_{280|+50}\) is chosen as the representative experiment of this group. The results are stated in Tab. 4.4. The value of the cost functional (4.3.1) for the parameter set of Step 4 is \(5.4272 \cdot 10^{-5}\).

<table>
<thead>
<tr>
<th>Step</th>
<th>(a_p) [-]</th>
<th>(c_p) [MPa]</th>
<th>(\beta_p) [-]</th>
<th>(\gamma_p) [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>178.03</td>
<td>6411.16</td>
<td>57.68</td>
<td>29.23</td>
<td>Q104</td>
</tr>
<tr>
<td>2</td>
<td>217.62</td>
<td>7771.79</td>
<td>73.33</td>
<td>124.64</td>
<td>Group +6% SA</td>
</tr>
<tr>
<td>3</td>
<td>193.50</td>
<td>7431.29</td>
<td>69.62</td>
<td>32.22</td>
<td>Groups ±6% SA</td>
</tr>
<tr>
<td>4</td>
<td>217.62</td>
<td>7771.79</td>
<td>73.33</td>
<td>124.64</td>
<td>Groups ±4%, ±6% SA</td>
</tr>
</tbody>
</table>

Table 4.6.: Results of parameter optimization starting with Group +6% SA

Starting with Group -6% SA  \(Q_{112}^{SA|-6}_{280|-50}\) is chosen as the representative experiment of this group. The results are stated in Tab. 4.4. The value of the cost functional (4.3.1) for the parameter set of Step 4 is \(5.5643 \cdot 10^{-5}\).

<table>
<thead>
<tr>
<th>Step</th>
<th>(a_p) [-]</th>
<th>(c_p) [MPa]</th>
<th>(\beta_p) [-]</th>
<th>(\gamma_p) [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>206.86</td>
<td>3471.01</td>
<td>393.41</td>
<td>6963.80</td>
<td>Q103</td>
</tr>
<tr>
<td>2</td>
<td>32.64</td>
<td>1.81</td>
<td>265.49</td>
<td>8897.57</td>
<td>Group +4% SA</td>
</tr>
<tr>
<td>3</td>
<td>100.79</td>
<td>440.61</td>
<td>334.36</td>
<td>10088.16</td>
<td>Groups ±4% SA</td>
</tr>
<tr>
<td>4</td>
<td>239.10</td>
<td>8218.27</td>
<td>61.07</td>
<td>14.61</td>
<td>Groups ±4%, ±6% SA</td>
</tr>
</tbody>
</table>

Table 4.3.
Figure 4.33.: Starting with Group +4% SA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg. 1 using the resulting parameter set of Step 4 ($a_p = 239.10$, $c_p = 8218.27$ MPa, $\beta_p = 61.07$, $\gamma_p = 14.61$ MPa)
4.3. Parameter Identification

(a) Stress-strain graph

(b) Evolution of inelastic strain

(c) Calculated evolution of back stress

Figure 4.34.: Starting with Group-6% SA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg. 1 using the resulting parameter set of Step 4 ($a_p = 43.78$, $c_p = 855.73$ MPa, $\beta_p = 500.67$, $\gamma_p = 12360.50$ MPa)
4. Transformation-Induced Plasticity – Experiments and Model Verifications

<table>
<thead>
<tr>
<th>Step</th>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>29.62</td>
<td>609.51</td>
<td>460.26</td>
<td>12807.58</td>
<td>Q112</td>
</tr>
<tr>
<td>2</td>
<td>42.37</td>
<td>829.28</td>
<td>795.30</td>
<td>20561.68</td>
<td>Group -6% SA</td>
</tr>
<tr>
<td>3</td>
<td>42.37</td>
<td>829.28</td>
<td>795.30</td>
<td>20561.68</td>
<td>Groups ±6% SA</td>
</tr>
<tr>
<td>4</td>
<td>43.78</td>
<td>855.73</td>
<td>500.67</td>
<td>12360.50</td>
<td>Groups ±4%, ±6% SA</td>
</tr>
</tbody>
</table>

Table 4.7.: Results of parameter optimization starting with Group -6% SA

4.3.1.2. Results for 280°C

We state the best resulting parameter sets for every step while choosing different starting groups concerning the predeformation at 280°C. The considered experiments are selected by Tab. 4.3.

**Starting with Group +2% UA$_{280°C}$**  
Q141$^{UA|+2}_{260|25}$ is chosen as the representative experiment of this group. The results are stated in Tab. 4.8. The value of the cost functional (4.3.1) for the parameter set of Step 4 is $4.8645 \cdot 10^{-5}$.

<table>
<thead>
<tr>
<th>Step</th>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.27</td>
<td>3134.73</td>
<td>938.19</td>
<td>259498.85</td>
<td>Q141</td>
</tr>
<tr>
<td>2</td>
<td>24.39</td>
<td>3117.14</td>
<td>1007.44</td>
<td>275087.76</td>
<td>Group +2% UA$_{280°C}$</td>
</tr>
<tr>
<td>3</td>
<td>31.86</td>
<td>4305.12</td>
<td>1013.92</td>
<td>274159.15</td>
<td>Groups ±2% UA$_{280°C}$</td>
</tr>
<tr>
<td>4</td>
<td>31.86</td>
<td>4367.14</td>
<td>1023.71</td>
<td>276219.31</td>
<td>Groups ±2%, ±4%, ±6% UA$_{280°C}$</td>
</tr>
</tbody>
</table>

Table 4.8.: Results of parameter optimization starting with Group +2% UA$_{280°C}$

**Starting with Group +4% UA$_{280°C}$**  
Q213$^{UA|+4}_{260|50}$ is chosen as the representative experiment of this group. The results are stated in Tab. 4.9. The value of the cost functional (4.3.1) for the parameter set of Step 4 is $5.0427 \cdot 10^{-5}$.

<table>
<thead>
<tr>
<th>Step</th>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16.45</td>
<td>3468.29</td>
<td>881.35</td>
<td>243943.71</td>
<td>Q213</td>
</tr>
<tr>
<td>2</td>
<td>236.98</td>
<td>50622.73</td>
<td>2600.43</td>
<td>294826.15</td>
<td>Q213</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>skipped</td>
</tr>
<tr>
<td>4</td>
<td>14.88</td>
<td>3098.15</td>
<td>901.30</td>
<td>238101.16</td>
<td>Groups ±2%, ±4%, ±6% UA$_{280°C}$</td>
</tr>
</tbody>
</table>

Table 4.9.: Results of parameter optimization starting with Group +4% UA$_{280°C}$

**Starting with Group +6% UA$_{280°C}$**  
Q143$^{UA|+6}_{260|25}$ is chosen as the representative experiment of this group. The results are stated in Tab. 4.10. The value of the cost functional (4.3.1) for the parameter set of Step 4 is $4.8903 \cdot 10^{-5}$.

**Starting with Group -2% UA$_{280°C}$**  
Q135$^{UA|-2}_{260|0}$ is chosen as the representative experiment of this group. The results are stated in Tab. 4.11. The value of the cost functional (4.3.1) for the parameter set of Step 4 is $5.4610 \cdot 10^{-5}$.
Figure 4.35.: Starting with Group-2% UA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg 1 using the resulting parameter set of Step 4 ($a_p = 31.86$, $c_p = 4367.14$ MPa, $\beta_p = 1023.71$, $\gamma_p = 276219.31$ MPa)
4. Transformation-Induced Plasticity – Experiments and Model Verifications

<table>
<thead>
<tr>
<th>Step</th>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>23.60</td>
<td>4281.40</td>
<td>786.51</td>
<td>223348.07</td>
<td>Q143</td>
</tr>
<tr>
<td>2</td>
<td>23.29</td>
<td>6008.86</td>
<td>1057.08</td>
<td>250637.52</td>
<td>Group +6% UA$_{280^\circ C}$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>skipped</td>
</tr>
<tr>
<td>4</td>
<td>23.15</td>
<td>5890.09</td>
<td>1066.03</td>
<td>2255636.87</td>
<td>Groups ±2%,+4%,+6% UA$_{280^\circ C}$</td>
</tr>
</tbody>
</table>

Table 4.10.: Results of parameter optimization starting with Group +6% UA$_{280^\circ C}$

<table>
<thead>
<tr>
<th>Step</th>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>15.37</td>
<td>3344.22</td>
<td>980.06</td>
<td>288837.43</td>
<td>Q135</td>
</tr>
<tr>
<td>2</td>
<td>15.24</td>
<td>3313.94</td>
<td>980.06</td>
<td>283658.28</td>
<td>Group -2% UA$_{280^\circ C}$</td>
</tr>
<tr>
<td>3</td>
<td>36.88</td>
<td>5155.48</td>
<td>1083.32</td>
<td>286021.31</td>
<td>Groups ±2% UA$_{280^\circ C}$</td>
</tr>
<tr>
<td>4</td>
<td>16.54</td>
<td>4250.16</td>
<td>982.52</td>
<td>257598.34</td>
<td>Groups ±2%,+4%,+6% UA$_{280^\circ C}$</td>
</tr>
</tbody>
</table>

Table 4.11.: Results of parameter optimization starting with Group -2% UA$_{280^\circ C}$

4.3.1.3. Results for 360°C

We state the best resulting parameter sets for every step while choosing different starting groups concerning the predeformation at 360°C. The considered experiments are selected by Tab. 4.3. Since we only conducted experiments with the same predeformation, Step 3 and 4 are skipped in the optimization procedure.

Starting with Group +4% UA  Q153$_{360^\circ C}$ is chosen as the representative experiment of this group. The results are stated in Tab. 4.11. The value of the cost functional (4.3.1) for the parameter set of step 2 is $2.0086 \cdot 10^{-5}$.

<table>
<thead>
<tr>
<th>Step</th>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
<th>Experiments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>40.35</td>
<td>3834.25</td>
<td>796.26</td>
<td>192267.23</td>
<td>Q153</td>
</tr>
<tr>
<td>2</td>
<td>40.35</td>
<td>3834.25</td>
<td>796.26</td>
<td>192267.23</td>
<td>Group +4% UA$_{360^\circ C}$</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>skipped</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>skipped</td>
</tr>
</tbody>
</table>

Table 4.12.: Results of parameter optimization starting with Group +4% UA$_{360^\circ C}$
4.3. Parameter Identification

Figure 4.36.: Starting with Group +4% UA: Comparison of all considered (smoothed) experiments with their strain driven reconstruction by Alg. 1 using the resulting parameter set of Step 2 ($a_p = 40.35$, $c_p = 3834.25$ MPa, $\beta_p = 796.26$, $\gamma_p = 192267.23$ MPa)
4. Transformation-Induced Plasticity – Experiments and Model Verifications

4.3.1.4. Discussion of the Conducted Parameter Identifications for Predeformations

The used Chaboche plasticity model with Armstrong-Frederick-like kinematic and isotropic hardening is capable of reproducing the qualitative material behavior at all of the three predeformation temperatures.

Predeformation at 850°C The optimization procedure starting with Group +4% SA minimizes the cost functional \( F_p \) the most considering all utilizable experiments with predeformation at 850°C. Since the Group +4% SA contains the most utilizable experiments, the result is within expectation, see Tab. 4.13. For the found locally optimal parameter set \((a_p = 239.10, c_p = 8218.27 \text{ MPa}, \beta_p = 61.07, \gamma_p = 14.61 \text{ MPa})\), the used algorithm Alg. 1 underestimates the elastic relaxation for almost all experiments after the stress is released, see Fig. 4.33(a). The positive predeformations are approximated sufficiently whereas the negative predeformation are underestimated. The same applies for inelastic strain, see Fig. 4.33(b). Unfortunately, the parameter set with the smallest found value of the cost functional \( F_p \) results in back stress evolutions which tend to the same stationary value for different predeformations with the same sign, see Fig. 4.33(c). We observe that different predeformation experimentally results in different interaction with TRIP such that the resulting back stress at the end of the predeformation should differ for different intensities of the predeformations. The parameter set \((a_p = 43.78, c_p = 855.73 \text{ MPa}, \beta_p = 500.67, \gamma_p = 12360.50 \text{ MPa})\) obtained by starting with Group -6% SA provides the desired evolutions of the back stresses. Although this parameter set is not optimal in sense of minimizing the cost functional \( F_p \), it results in optical sufficient results, see Figs. 4.34(a), 4.34(b) and 4.34(c).

<table>
<thead>
<tr>
<th>Starting Group</th>
<th>( a_p ) [\text{-}]</th>
<th>( c_p ) [\text{MPa}]</th>
<th>( \beta_p ) [\text{-}]</th>
<th>( \gamma_p ) [\text{MPa}]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>+4% SA</td>
<td>239.10</td>
<td>8218.27</td>
<td>61.07</td>
<td>14.61</td>
<td>5.4065 \cdot 10^{-5}</td>
</tr>
<tr>
<td>-4% SA</td>
<td>199.78</td>
<td>5887.76</td>
<td>741.00</td>
<td>5014.49</td>
<td>5.4610 \cdot 10^{-5}</td>
</tr>
<tr>
<td>+6% SA</td>
<td>217.62</td>
<td>7771.79</td>
<td>73.33</td>
<td>124.64</td>
<td>5.4272 \cdot 10^{-5}</td>
</tr>
<tr>
<td>-6% SA</td>
<td>43.78</td>
<td>855.73</td>
<td>500.67</td>
<td>12360.50</td>
<td>5.5643 \cdot 10^{-5}</td>
</tr>
</tbody>
</table>

Table 4.13.: Results of the parameter optimization for predeformations at 850°C

Predeformation at 280°C Starting with Group +2% UA280°C minimizes the cost functional the most \( F_p \) considering all utilizable experiments with predeformation at 280°C, see Tab. 4.14. The used algorithm Alg. 1 underestimates the elastic relaxation for experiments with negative predeformation and overestimates it for experiments with positive predeformation after the stress is released for the found locally optimal parameter set \((a_p = 31.86, c_p = 4367.14 \text{ MPa}, \beta_p = 1023.71, \gamma_p = 276219.31 \text{ MPa})\), see Fig. 4.35(a). The predeformations are approximated sufficiently, but slightly underestimated in absolute value. The inelastic strains of the Groups +2% UA280°C and +4% UA280°C are well reproduced, whereas there are some deviation for the Groups -2% UA280°C and +6% UA280°C. But since both of these groups scatter in their experimental evolution of the inelastic strain, see Fig. 4.35(b), a better overall approximation cannot be expected. The evolutions of the back stresses are reasonable, within expectation and usable for the later cause, see Fig. 4.35(c).

Predeformation at 360°C The inelastic strains of the Group +4% UA360°C is well reproduced, see Fig. 4.36(b). The found locally optimal parameter set is \((a_p = 40.35, c_p = 3834.25 \text{ MPa}, \beta_p = 796.26, \gamma_p = 192267.23 \text{ MPa})\). The stress-strain-graphs are overestimated in the beginning and underestimated it in the end of the predeformation, see Fig. 4.36(a). The evolutions of the
### 4.3. Parameter Identification

#### Starting Group

<table>
<thead>
<tr>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+2%$ UA $280^\circ$C</td>
<td>31.86</td>
<td>4367.14</td>
<td>1023.71</td>
<td>276219.31</td>
</tr>
<tr>
<td>$+4%$ UA $280^\circ$C</td>
<td>14.88</td>
<td>3098.15</td>
<td>901.30</td>
<td>238101.16</td>
</tr>
<tr>
<td>$-6%$ UA $280^\circ$C</td>
<td>23.15</td>
<td>5890.09</td>
<td>1066.03</td>
<td>2255636.87</td>
</tr>
<tr>
<td>$-2%$ UA $280^\circ$C</td>
<td>16.54</td>
<td>4250.16</td>
<td>982.52</td>
<td>257598.34</td>
</tr>
</tbody>
</table>

Table 4.14.: Results of the parameter optimization for predeformations at $280^\circ$C

back stresses are reasonable, within expectation and usable for the later cause, see Fig. 4.36(c).

We recommend to use the parameter sets of Tab. 4.15 in order to approximate the predeformation with Alg. 1.

#### Table 4.15.: Recommended parameter sets to approximate the predeformation with Alg. 1

<table>
<thead>
<tr>
<th>Predeformation at</th>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$280^\circ$C</td>
<td>31.86</td>
<td>4367.14</td>
<td>1023.71</td>
<td>276219.31</td>
</tr>
<tr>
<td>$360^\circ$C</td>
<td>40.35</td>
<td>3834.25</td>
<td>238101.16</td>
<td>192267.23</td>
</tr>
<tr>
<td>$850^\circ$C</td>
<td>43.78</td>
<td>855.73</td>
<td>500.67</td>
<td>12360.50</td>
</tr>
</tbody>
</table>

#### 4.3.2. Identification for Bainitic Transformation

The aim is to obtain four suitable parameter sets for $\kappa_{0B}$, $a_{tp}$, $c_{tp}$ and $c_{int}$ at $280^\circ$C (resp. $360^\circ$C) such that the experiments with predeformation at $850^\circ$C (resp. at bainitizing temperature ($280^\circ$C and $360^\circ$C)) are well described by the stress driven explicit Alg. 2. The interaction between classic plasticity and TRIP is implemented by passing the calculated end value of the plastic inner variable $\alpha_p$ related to the predeformation. In order to calculate the value of plastic inner variable $\alpha_p$, we use Alg. 1 with the parameter sets stated in Tab. 4.15 based on Section 4.3.1.4. The optimization is obtained by minimizing the cost functional

$$F_{tp} := \sum_{j \in \mathcal{N}} \frac{N_j}{N} \cdot \left( \frac{\|e_{tp}^{exp} - e_{tp}^{calc}\|_2^2}{\|e_{tp}^{exp}\|_2^2} \right)$$

(4.3.5)

with the MATLAB routine *lsqnonlin* related to $\kappa_{0B}$, $a_{tp}$, $c_{tp}$ and $c_{int}$. $\mathcal{N}$ denotes the set of considered experiments, see Tab. 4.1, $N_j$ the number of data points for the $j$-th experiment, $N := \sum_{j \in \mathcal{N}} N_j$, $e_{tp}^{exp}$ the data vector of the prepared inelastic (TRIP) strain and $e_{tp}^{calc}$ the data vector of the calculated inelastic (TRIP) strain.

#### Preparing Data

The original noisy data is not suitable for an use concerning the parameter identification. Especially, the derivation of the bainite phase fraction is important, since we consider no own phase fraction evolution in our model and we need a suitable approximation as well for the part of the saturation function $\kappa_w$, cf. Alg. 2. We use a smoothing moving average of 50 data points for $\theta$, $S$, $e_L$ and $e_D$, downsample each set by taking just every second data point into account and synchronize all experiments by the MATLAB function *interp1* with a time step of 1s.

---

$^5$We have chosen the following nonstandard options 'MaxFunEvals',10e3,'MaxIter',10e3,'ToI',1e-8.
Phase Fraction A derivation of the phase fraction by Section 4.1.4 with the approaches (4.1.15a) and (4.1.22) is not sufficient. We use the calculated phase fraction $p^1_B$ by (4.1.15a) and normalize by $p^N_B$, $J_N := \{(i_{E5} - [0.05]E5) : i_{E5}\}$

\[
p^N_B := \frac{1}{|J_N|} \sum_{i \in J_N} (p^1_B)^i, \quad \text{so}\]

\[
p^2_B := \frac{p^1_B}{p^N_B}.
\]

Unfortunately, the derived data is very noisy at the beginning of the phase evolution, but reasonable later on. Instead of smoothing the data set $p^2_B$ directly with a moving average, we set the first 20 indices to zero ($p^2_B(1 : 20) := 0$) and smooth the reverse set $p^R_B := p^2_B(i_{E5} : -1 : i_{S5})$ with a span of 20 data points - this implies $p^R_B(i_{S5}) = 0$. Again, we smooth the set with a span of 5% of the index range and cut off all data points below 0 and above 1 setting them equal to 0 respectively equal to 1. After a last smoothing with a span of 15% of the index range, we obtain a usable approximation of the phase fraction $p_B$, cf. Figs. 4.20 - 4.26. The inelastic strain is then derived by (4.1.9) with the crawled data.

Saturation Function The material parameter $\kappa$ is given in our situation by

\[
\kappa = \kappa_{0B} \frac{\partial \Phi_B}{\partial p_B} \langle \partial_t p_B \rangle.
\]

Knowing the phase fractions $p_B$, only the Greenwood Johnson parameter $\kappa_{0B}$ is free, where

\[
\kappa_w := \frac{\partial \Phi_B}{\partial p_B} \langle \partial_t p_B \rangle
\]

can be precalculated by an approximation,

\[
\left(\frac{\partial \Phi_B}{\partial p_B}\right)^i := \begin{cases} 1, & \text{Tanaka}, \\ -\ln(\max\{0.01, p^i_B\}), & \text{Leblond}, \\ 2 - 2p^i_B, & \text{Desalos / Denis}, \end{cases}
\]

\[
\langle(\partial_t p_B)\rangle^i := \left\{\frac{p^i_B - p^{i-1}_B}{t^i - t^{i-1}}\right\}, i \in \{i_{S5} + 1 : i_{E5}\}. (\langle(\partial_t p_B)\rangle)^{i_{S5}} := \frac{p^{i_{S5}}_B}{p^{i_{S5}}_B}.
\]

In case of the Leblond model, we approximate the singularity in 0 by assuming a minimum phase fraction of bainite of 0.01 for all indices. The numeric time derivative of $p_B$ is smoothed with a moving average of 20 data points for the reverse set and setting the first 20 values to zero, as above. A comparison of the precalculated $\kappa_w$ depending on the three presented approaches are illustrated in Fig. 4.37 exemplarily for the experiments $Q99^{SA | 50}_{280 | 50}$ and $Q100^{SA | 50}_{280 | 50}$.

Optimization Procedure It is not convenient to start the parameter identification with all experiments at once. Our optimization procedure is based on the following steps:

1. Select suitable experiments without predeformation and transformation under applied stress at the same temperature (280°C or 360°C). Conduct the optimization for the parameters $\kappa_{0B}$, $a_{tp}$ and $c_{tp}$ with 81 start parameter combinations.

2. Select suitable groups of experiments with predeformation (at same temperature; SA or UA) and each transformation under the same applied stress at the same transformation temperature chosen at step 1. Select the 5 best approximating parameter sets of Step 1 and
4.3. Parameter Identification

Figure 4.37.: Predeformation at 280°C: Comparison of $\kappa_w$ for Q99 (±0%, +50 MPa) and Q100 (±0%, -50 MPa)

add for each set the starting parameter set for $c_{int}$ – choosing $c_{int} \in \{\pm \frac{1}{3} \sqrt{c_{tp} c_p}, \pm \frac{2}{3} \sqrt{c_{tp} c_p}\}$ MPa. $c_{tp}$ are the obtain values of Step 1 of each parameter set and $c_p$ by the parameter optimization for the predeformation. This results in 20 new starting parameters. We conduct a parameter optimization for $\kappa_0B$, $a_{tp}$, $c_{tp}$ and $c_{int}$ and the experiments with predeformation (SA, UA280°C or UA360°C) and transformation under the same applied stress.

3. Combine groups of experiments with same absolute applied stress and predeformation at one temperature and perform a parameter optimization for $c_{int}$ by using the best 5 resulting parameter set of Step 2 of each group with the same applied stress as starting parameters.

4. Select all parameter sets of Step 3 and conduct an optimization for all four parameters $\kappa_0B$, $a_{tp}$, $c_{tp}$ and $c_{int}$ for all used experiments of Steps 1-3.

Starting Parameters In Step 1 of our optimization procedure, we choose $\kappa_0B \in \{10^{-4}, 10^{-3}, 10^{-2}\}$ 1/MPa and $a_{tp} \in \{10^{-4}, 10^{-3}, 10^{-2}, 10^{-1}, 1, 10^1, 10^2, 10^3, 10^4\}$. Similar to the parameter optimization of the predeformation, we calculate $c_{tp}$ by selecting stationary values of the TRIP back stress $X_{tp}$.

\[
A_{X_{tp}} := \frac{c_{tp}}{a_{tp}} \in \{5, 10, 20\} \text{ MPa.}
\]

For the Steps 2 to 4, the choice of $c_{int}$ is not arbitrary due to the restriction of thermomechanical consistency,

\[
c_{int}^2 \leq c_p c_{tp}.
\]

We select $c_{int} \in \{\pm \frac{1}{3} \sqrt{c_{tp} c_p}, \pm \frac{2}{3} \sqrt{c_{tp} c_p}\}$ MPa.

Two optimizations are conducted for illustration – both for the transformation temperature 280°C. We consider only experiments with applied stress of ±50 MPa during transformation and select two groups of experiments.

\[\text{As in plasticity, we can determine the limiting value of } X_{tp} \text{ in isothermal case for experimental situation with no predeformation}.\]
4. Transformation-Induced Plasticity – Experiments and Model Verifications

- Group (SA ±50MPa): Q99_{SA|±0}^{280|±50}, Q100_{SA|±0}^{280|±50}, Q102_{SA|+4}^{280|+50}, Q104_{SA|+6}^{280|+50}, Q107_{SA|+6}^{280|+50}, Q105_{SA|+6}^{280|+50}, Q108_{SA|+4}^{280|+50}, Q112_{SA|+6}^{280|+50}

- Group (UA ±50MPa): Q99_{UA|±4}^{280|±50}, Q100_{UA|±4}^{280|±50}, Q213_{UA|+4}^{280|+50}, Q132_{UA|+6}^{280|+50}, Q136_{UA|+2}^{280|+50}, Q139_{UA|+2}^{280|+50}, Q212_{UA|+4}^{280|+50}, Q134_{UA|+6}^{280|+50}

The first optimization is conducted for Group (SA ±50MPa), experiments with predeformation at 850°C, and the second one for Group (UA ±50MPa), experiments with predeformation at 280°C. The first step is a optimization for Q99_{SA|±0}^{280|±50} and Q100_{SA|±0}^{280|±50}, thus it is used for both groups.

4.3.2.1. Results for transformation temperature 280°C with predeformation at 850°C:
Group (SA ±50MPa)

In Step 1, we consider the experiments Q99_{SA|±0}^{280|±50} and Q100_{SA|±0}^{280|±50}. Step 2 is divided in two steps. In Step 2a (+50 MPa applied stress), we consider additionally Q105_{SA|+4}^{280|+50}, Q104_{SA|+6}^{280|+50}, Q107_{SA|+6}^{280|+50}, besides Q99_{SA|±0}^{280|±50}. In Step 2b (-50 MPa applied stress), we consider additionally Q103_{SA|+6}^{280|±50}, Q106_{SA|+4}^{280|±50}, Q111_{SA|+6}^{280|±50} besides Q100_{SA|±0}^{280|±50}. The complete Group (SA ±50MPa) is used for the Steps 3 and 4.

The results are illustrated for the different approaches for the saturation functions in Tabs. 4.16 - 4.18. The approach of Desalos and Denis leads to the lowest value of the cost functional $\mathcal{F}_{tp}$ for this selection of experiments. Tanaka’s approach predicts a significantly back stress evolutions, see Fig. 4.38, whereas the approaches of Leblond and Desalos/Denis predict back stress evolutions tending to zero, see exemplarily 4.39(Desalos/Denis).

<table>
<thead>
<tr>
<th>Tanaka</th>
<th>$\kappa_B$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>1.4708 · 10^{-4}</td>
<td>5.2953 · 10^{-5}</td>
<td>9.6255 · 10^{3}</td>
<td>0</td>
<td>7.7500 · 10^{-7}</td>
</tr>
<tr>
<td>Step 2a</td>
<td>1.2526 · 10^{-4}</td>
<td>1.7305 · 10^{-6}</td>
<td>8.5983 · 10^{3}</td>
<td>291.5933</td>
<td>1.6148 · 10^{-6}</td>
</tr>
<tr>
<td>Step 2b</td>
<td>1.2930 · 10^{-4}</td>
<td>5.7403 · 10^{-7}</td>
<td>8.8016 · 10^{3}</td>
<td>197.9865</td>
<td>1.5117 · 10^{-6}</td>
</tr>
<tr>
<td>Step 3</td>
<td>1.2975 · 10^{-4}</td>
<td>2.0915 · 10^{-7}</td>
<td>8.9435 · 10^{3}</td>
<td>242.2621</td>
<td>1.7317 · 10^{-6}</td>
</tr>
<tr>
<td>Step 4</td>
<td>1.3193 · 10^{-4}</td>
<td>1.3974 · 10^{-4}</td>
<td>8.9793 · 10^{3}</td>
<td>243.5331</td>
<td>1.5425 · 10^{-6}</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Leblond</th>
<th>$\kappa_B$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>7.7167 · 10^{-5}</td>
<td>0.00457</td>
<td>1.0222 · 10^{-12}</td>
<td>0</td>
<td>3.3496 · 10^{-6}</td>
</tr>
<tr>
<td>Step 2a</td>
<td>7.7458 · 10^{-5}</td>
<td>4.8482 · 10^{3}</td>
<td>1.0399 · 10^{-12}</td>
<td>2.3692 · 10^{3}</td>
<td>3.7896 · 10^{-6}</td>
</tr>
<tr>
<td>Step 2b</td>
<td>8.2974 · 10^{-5}</td>
<td>4.8866 · 10^{3}</td>
<td>1.0254 · 10^{-12}</td>
<td>2.3574 · 10^{3}</td>
<td>3.7962 · 10^{-6}</td>
</tr>
<tr>
<td>Step 3</td>
<td>7.9975 · 10^{-5}</td>
<td>4.8598 · 10^{3}</td>
<td>4.3892 · 10^{-8}</td>
<td>2.3549 · 10^{3}</td>
<td>4.3756 · 10^{-5}</td>
</tr>
<tr>
<td>Step 4</td>
<td>7.8923 · 10^{-5}</td>
<td>5.1932 · 10^{3}</td>
<td>4.5629</td>
<td>2.3557 · 10^{3}</td>
<td>4.1611 · 10^{-6}</td>
</tr>
</tbody>
</table>

Table 4.16: Transformation at 280°C: Results of the optimization procedure using Tanaka’s approach and the experiments of Group (SA ±50MPa)

Table 4.17: Transformation at 280°C: Results of the optimization procedure using Leblond’s approach and the experiments of Group (SA ±50MPa)
4.3. Parameter Identification

Figure 4.38.: Transformation at 280°C: Results of step 4 of the optimization procedure using Tanaka’s approach and the experiments of Group (SA ±50MPa)

Table 4.18.: Transformation at 280°C: Results of the optimization procedure using the approach by Desalos and Denis and the experiments of Group (SA ±50MPa)

<table>
<thead>
<tr>
<th>Step</th>
<th>$\kappa_0B$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>$7.7519 \cdot 10^{-5}$</td>
<td>0.2126</td>
<td>$1.0222 \cdot 10^{-12}$</td>
<td>0</td>
<td>$8.3719 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>Step 2a</td>
<td>$7.8762 \cdot 10^{-5}$</td>
<td>4.9415 $\cdot 10^3$</td>
<td>$1.0222 \cdot 10^{-12}$</td>
<td>$2.3545 \cdot 10^3$</td>
<td>$6.3154 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>Step 2b</td>
<td>$8.0138 \cdot 10^{-5}$</td>
<td>6.5340 $\cdot 10^3$</td>
<td>$1.1127 \cdot 10^{-9}$</td>
<td>$2.3731 \cdot 10^3$</td>
<td>$7.0573 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>Step 3</td>
<td>$7.9417 \cdot 10^{-5}$</td>
<td>4.8219 $\cdot 10^3$</td>
<td>$1.0224 \cdot 10^{-12}$</td>
<td>$2.3686 \cdot 10^3$</td>
<td>$8.4498 \cdot 10^{-7}$</td>
</tr>
<tr>
<td>Step 4</td>
<td>$7.8710 \cdot 10^{-5}$</td>
<td>4.1913 $\cdot 10^3$</td>
<td>0.0018</td>
<td>$2.3708 \cdot 10^3$</td>
<td>$8.7383 \cdot 10^{-7}$</td>
</tr>
</tbody>
</table>

Figure 4.39.: Transformation at 280°C: Results of step 4 of the optimization procedure using the approach by Desalos and Denis and the experiments of Group (SA ±50MPa)
4.3.2.2. Results for transformation temperature 280°C with predeformation at 280°C: Group (UA ±50MPa)

In Step 1, we consider the experiments Q99\textsuperscript{SA}|\pm 0\textsuperscript{280|+50} and Q100\textsuperscript{SA}|\pm 0\textsuperscript{280|+50}. Step 2 is divided in two steps. In Step 2a (+50 MPa applied stress), we consider additionally Q213\textsuperscript{UA}|+4\textsuperscript{280|+50}, Q133\textsuperscript{UA}|+6\textsuperscript{280|+50}, and Q136\textsuperscript{UA|2}\textsuperscript{280|+50} besides Q99\textsuperscript{SA|0}\textsuperscript{280|+50}. In Step 2b (-50 MPa applied stress), we consider additionally Q139\textsuperscript{UA|2}\textsuperscript{280|+50}, Q212\textsuperscript{UA|4}\textsuperscript{280|+50}, and Q134\textsuperscript{UA|6}\textsuperscript{280|+50} besides Q100\textsuperscript{SA|0}\textsuperscript{280|+50}. The complete Group (UA ±50MPa) is used for the Steps 3 and 4.

The results are illustrated for the different approaches for the saturation functions in Tabs. 4.19 - 4.21. Tanaka’s approach leads to the lowest value of the cost functional $F_{tp}$ for this selection of experiments, cf. also Fig. 4.40.

<table>
<thead>
<tr>
<th>Tanaka</th>
<th>$\kappa_{0B}$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>1.4708 · 10^{-4}</td>
<td>5.2953 · 10^{-5}</td>
<td>9.6255 · 10^{-3}</td>
<td>0</td>
<td>7.7500 · 10^{-7}</td>
</tr>
<tr>
<td>Step 2a</td>
<td>2.5491 · 10^{-4}</td>
<td>5.4390 · 10^{-9}</td>
<td>5.9338 · 10^{-3}</td>
<td>-2.7345 · 10^{-3}</td>
<td>1.2718 · 10^{-6}</td>
</tr>
<tr>
<td>Step 2b</td>
<td>1.5040 · 10^{-4}</td>
<td>219.6080</td>
<td>2.6200 · 10^{-3}</td>
<td>-3.6488 · 10^{2}</td>
<td>8.5742 · 10^{-7}</td>
</tr>
<tr>
<td>Step 3</td>
<td>1.4894 · 10^{-4}</td>
<td>1.2430 · 10^{-12}</td>
<td>3.7347 · 10^{3}</td>
<td>-2.8158 · 10^{-3}</td>
<td>1.3003 · 10^{-5}</td>
</tr>
<tr>
<td>Step 4</td>
<td>1.5607 · 10^{-4}</td>
<td>1.0222 · 10^{-12}</td>
<td>6.6628 · 10^{-3}</td>
<td>-2.8197 · 10^{-3}</td>
<td>1.8464 · 10^{-5}</td>
</tr>
</tbody>
</table>

Table 4.19.: Transformation at 280°C: Results of the optimization procedure using Tanaka’s approach and the experiments of Group (UA ±50MPa)

<table>
<thead>
<tr>
<th>Leblond</th>
<th>$\kappa_{0B}$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>7.7167 · 10^{-5}</td>
<td>0.0457</td>
<td>1.0222 · 10^{-12}</td>
<td>0</td>
<td>3.3496 · 10^{-6}</td>
</tr>
<tr>
<td>Step 2a</td>
<td>1.2996 · 10^{-4}</td>
<td>7.6292 · 10^{-9}</td>
<td>1.0621 · 10^{-12}</td>
<td>-2.7780 · 10^{-3}</td>
<td>3.6914 · 10^{-6}</td>
</tr>
<tr>
<td>Step 2b</td>
<td>1.4209 · 10^{-4}</td>
<td>296.1441</td>
<td>1.7589 · 10^{-5}</td>
<td>-3.8372 · 10^{-3}</td>
<td>1.0156 · 10^{-7}</td>
</tr>
<tr>
<td>Step 3</td>
<td>1.1246 · 10^{-4}</td>
<td>3.2082 · 10^{-10}</td>
<td>1.0347 · 10^{-10}</td>
<td>-2.8447 · 10^{-3}</td>
<td>1.4763 · 10^{-5}</td>
</tr>
<tr>
<td>Step 4</td>
<td>9.5527 · 10^{-5}</td>
<td>1.0222 · 10^{-12}</td>
<td>1.0334 · 10^{-12}</td>
<td>-2.8436 · 10^{-3}</td>
<td>2.2069 · 10^{-5}</td>
</tr>
</tbody>
</table>

Table 4.20.: Transformation at 280°C: Results of the optimization procedure using Leblond’s approach and the experiments of Group (UA ±50MPa)

<table>
<thead>
<tr>
<th>Desalos / Denis</th>
<th>$\kappa_{0B}$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>7.7519 · 10^{-5}</td>
<td>0.2126</td>
<td>1.0222 · 10^{-12}</td>
<td>0</td>
<td>8.3719 · 10^{-7}</td>
</tr>
<tr>
<td>Step 2a</td>
<td>1.3097 · 10^{-4}</td>
<td>1.5822 · 10^{-9}</td>
<td>3.6737 · 10^{-12}</td>
<td>-2.7618 · 10^{-3}</td>
<td>1.4217 · 10^{-6}</td>
</tr>
<tr>
<td>Step 2b</td>
<td>1.3289 · 10^{-4}</td>
<td>277.4774</td>
<td>2.4975 · 10^{-9}</td>
<td>-3.7842 · 10^{-3}</td>
<td>4.8413 · 10^{-6}</td>
</tr>
<tr>
<td>Step 3</td>
<td>1.1062 · 10^{-4}</td>
<td>1.0222 · 10^{-10}</td>
<td>1.2589 · 10^{-10}</td>
<td>-2.8332 · 10^{-3}</td>
<td>1.3893 · 10^{-5}</td>
</tr>
<tr>
<td>Step 4</td>
<td>9.5043 · 10^{-5}</td>
<td>1.0224 · 10^{-12}</td>
<td>1.0352 · 10^{-12}</td>
<td>-2.8336 · 10^{-3}</td>
<td>1.9452 · 10^{-5}</td>
</tr>
</tbody>
</table>

Table 4.21.: Transformation at 280°C: Results of the optimization procedure using approach by Desalos and Denis and the experiments of Group (UA ±50MPa)
4.3. Parameter Identification

Figure 4.40.: Transformation at 280°C: Results of step 4 of the optimization procedure using Tanaka’s approach and the experiments of Group (UA ±50MPa)

4.3.3. Conclusions of the Parameter Identifications

The presented algorithms of the 1d model can reproduce the qualitative material behavior of the steel 100Cr6 for predeformations of the austenite and its influence on TRIP during the bainitic phase transformation.

The proposed optimization procedure in Section 4.3.1 yields sufficient results for an approximation of the predeformation of austenite at 280°C, 360°C and 850°C, cf. Section 4.3.1.4.

Using our optimization procedure for the parameter identifications for the TRIP behavior, the approach by Desalos and Denis leads to the best results by means of the smallest value of the cost functional $\mathcal{F}_{tp}$ for the experiments with predeformation at 850°C and transformation under ±50 MPa applied stress, see Tabs. 4.16 - 4.18. All three approaches for the saturation functions can reproduce the qualitative material behavior, see Figs. 4.39(a) and 4.38(a), but lack in accurateness. The models considering a Leblond or Desalos/Denis approach predict back stress evolutions near zero, where the approach by Tanaka predicts a significant evolution.

For the experiments with predeformation at 280°C, the approach by Tanaka results in the best approximation, see Tabs. 4.19 - 4.21. The resulting approximation is not very accurate, see Fig. 4.38(a). Although, the change in interaction by switching form predeformation at 850°C to 280°C is reproduced – note the sign of $c_{int}$.

Since the restriction $c_{int}^2 < c_p c_{tp}$ is violated in case of the approaches of Leblond and Desalos/Denis, we recommend to use the obtained parameters by Tanaka’s approach. Unfortunately, the restriction could not be considered within the optimization but only by considering correct start parameters.

The combined parameter sets to reproduce the material behavior of the interaction of classic plasticity and TRIP by the Algs. 1 and 2 with Tanaka’s approach are for the experiments with

- predeformation at austenitizing temperature 850°C and bainitic transformation at 280°C:

<table>
<thead>
<tr>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>43.78</td>
<td>855.73</td>
<td>500.67</td>
<td>12360.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\kappa_B$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.3193 \cdot 10^{-4}$</td>
<td>1.3974 $\cdot 10^{-4}$</td>
<td>8979.3</td>
<td>243.5331</td>
</tr>
</tbody>
</table>
predeformation at bainitizing temperature 280°C and bainitic transformation at 280°C:

<table>
<thead>
<tr>
<th>$a_p$ [-]</th>
<th>$c_p$ [MPa]</th>
<th>$\beta_p$ [-]</th>
<th>$\gamma_p$ [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>16.54</td>
<td>4250.16</td>
<td>982.56</td>
<td>257598.34</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\kappa_0B$ [1/MPa]</th>
<th>$a_{tp}$ [-]</th>
<th>$c_{tp}$ [MPa]</th>
<th>$c_{int}$ [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.3193 \cdot 10^{-4}$</td>
<td>$1.0222 \cdot 10^{-12}$</td>
<td>6662.8</td>
<td>$-2.8197 \cdot 10^3$</td>
</tr>
</tbody>
</table>

We cannot expect a better approximation of the experiments with our procedure at this point due to several reasons. Obviously, we only have conducted few experiments and this may compromise the reproducibility of the experimental results such that the scattering results in unreliable parameters. The scattering is based on the measurement and on material deviations in the test specimens. Still, we use some fixed literature material parameters to abstain the conflict to past works. We observe slightly different temperature paths in the experiments as well. Moreover, the noisy data require post processing like smoothing for at least the derived data, e.g. $p_B$, $\partial_t p_B$, $\kappa_w$, and this leads to alterations of the derived data itself. If we conduct the parameter identification for fewer smaller groups, e.g. 2 to 3 experiments the approximations are better, see e.g. Step 2b T abs. 4.19 - 4.21.

The model ansatz does not capture all kind of dependencies – we assume plastic material behavior for the predeformation parts, where a viscoplastic approach is also reasonable. We neglect effects of the quenching too. Furthermore, the results are strongly depending on the used numerical algorithms and the optimization routines. In both cases, there might be more suitable choices.

Summarized, we presented an effective procedure to capture a qualitative description of the interaction between classic plasticity and TRIP. The presented procedure can be used to obtain a more accurate material behavior approximation by considering more experimental results, including alterations of the model ansatz, e.g. alternative back stress evolutions for the predeformation or transformation part (e.g. linear kinematic hardening), alterations in optimization routines and more.
5. Summary and Outlook

Multi-mechanism models (MM models) had raised more and more interest in the recent decades. The three main topics of this work were

- the modeling with MM models of serial type,
- the mathematical investigation of the related boundary value problems of a viscoelastic MM model, and
- the verification of a MM model considering the interaction of classic plasticity and transformation-induced plasticity (TRIP) by experiments.

**Modeling with MM models** The concept of modeling with MM models of serial type was generalized in Chapter 2. In past approaches, the thermoelastic element was considered separately from the inelastic parts. By not separating the thermoelastic part, we extended the basic concept of MM models and were able to introduce a model of *coupled thermoelastic mechanisms with hardening*. Past approaches of viscous, creep, TRIP, and plastic (ideal and Chaboche) models were embedded in the new concept. Allowing a coupling between the elastic elements, we presented two generalized viscoelastic mechanisms based on rheological elements. We recalled a model approach for a plastic MM model with \( m \) (plastic) mechanisms and \( n \) yield criteria (here: \( m\text{MnC-B} \)) and proposed an alternative model (here: \( m\text{MnC-A} \)) altering the yield criteria by using for all mechanisms the same effective stress. Moreover, we collected the presented example mechanisms and models under one *MM model with Chaboche ansatz*; stating the dependency of the model’s partial strains to algebraic and differential kind of evolution equations.

Possible extensions of this work are to consider more general evolution equations like of integral, integro-differential or fractional kind. Thermomechanical consistency was not addressed in total generality and more statements could be possible. Our model approach was based on the idea of series connections of rheological models. New approaches could extend the concept on basis of parallel connections. Moreover, small deformation were considered and an embedding of the presented model approach within the framework of large deformations is also of great interest.

**Mathematical Investigation of Linear Viscoelasticity** A detailed modeling of linear viscoelasticity within the framework of MM models was presented in Chapter 3 for 3d and 1d situations. In the 3d case, the related boundary value problem of an *isothermal-(N)-coupled KV model* was transformed into a system of integro-differential equations. The transformed problem was investigated according to its weak solvability. We proved the existence and uniqueness of a weak solution via Galerkin’s method. To the author’s knowledge, this is a new approach within this framework. A detailed mathematical treatment was provided for related mathematical problems of the *isothermal-(2)-coupled-KV model* of a 1d rod. Existence and uniqueness of weak solutions were proven via Galerkin’s method dealing with a singular mass matrix for three different kind of boundary conditions. In both cases, 3d and 1d, numerical simulations underlined the new quality of the *isothermal-(N)-coupled KV model* covering material effects like ratcheting without predicting instantaneous elasticity.

Open for future work are the mathematical investigation of the provided mathematical problem to linear thermoelasto-viscoelasticity within the framework of MM models – covering the
coupling to the heat equation (including the full dissipation terms). Moreover, a mathematical investigation of the related problem of the isothermal (N)-coupled KV model without transformation into an integro-differential equation system by e.g. fix point arguments would be of interest. Extensions to nonlinear viscosities are as well. In order to develop experimental test situations for parameter identifications, an investigations of the asymptotic analysis of the problems and, therefore, their attractor sets could also be helpful. Furthermore, experiments should be conducted to verify the model approach on real material.

**MM Model Verification by Experiments (TRIP)** In Chapter 4, we verified a MM Model taking the interaction of classic plasticity and TRIP into account by experiments.

A detailed test evaluation of experiments characterizing the interaction of classic plasticity and TRIP of the steel 100Cr6 (SAE 52100) was presented. The main observations were that predeformation at austenitizing and bainitizing temperature influences the evolution of the inelastic strain – especially under applied stress during transformation. Predeformations at bainitizing temperature were leading to deviations in the inelastic strain evolution with the same sign. For predeformation at austenitizing temperature, no conclusive statements on the influences could be made. The prevailing trends were that positive (resp. negative) predeformations were leading to negative (resp. positive) deviations in the evolutions of the inelastic strain. Every predeformation led to acceleration of the bainitic phase transformation.

A MM model considering the interaction of classic plasticity and TRIP was recalled and embedded in our presented approach for MM models of Chapter 2. The 3d model was simplified to a 1d model which was suitable to describe the conducted uniaxial experiments. We provided numerical algorithms of the 1d model and conducted a systematical parameter identification. Using the numerical scheme of the model, we were capable to qualitatively reproduce the observed material behavior. The parameter identification resulted in usable parameter sets to approximate the material behavior of the steel 100Cr6 (SAE 52100) under the conducted experimental situation for selected experiments.

In order to achieve a more accurate description of the TRIP behavior of the steel 100Cr6 (SAE 52100), more experiments should be conducted to ensure the reproducibility of the experimental results. The preparation and post processing of the experimental data altered the original data. Its influence should be analyzed more carefully. We only considered the engineering strains and stresses while considering true strain and, especially, true stresses could be more reasonable. The presented MM model considering the interaction of classic plasticity and TRIP included Armstrong-Frederick-like approaches for the kinematic and isotropic hardening laws. According to the experimental results, non-saturating hardening laws would be a reasonable choice as well.

Conducting a sensitivity analysis for the parameters, altering the numerical schemes of the model (e.g. alternative discretization) and the optimization routines could improve the parameter identification. The elastic modulus as well as the Poisson’s ratio and initial yield stress were considered as predetermined parameters. In order to enhance the results, they also could be considered for optimization. Although, the presented results of the proposed procedure for parameter identification lacked in accurateness, the effective optimization procedure could be a suitable basis for further investigations. An implementation of the 3d model with the obtained parameter set could be used to conduct numerical simulation for 3d work pieces.
A. Concept of True Stress and True Strain

A.1. True Stress

Since the diameter of the specimen constantly changes during the experiment, the cross section changes too. Based on this, the actual acting stress differs from the measured one. Therefore, the concept of true stresses $^tS$ is used such that the actual measured force is related to the actual cross-section $A$, cf. [47],

(A.1.1) \[ ^tS = \frac{F}{A}. \]

The cross-section $A$ can be calculated in the following manner. We assume a constant wall thickness $w$ and a measurable inner and outer diameter $d_I$ resp. $d_O$. Since the wall thickness is constant, we can assume an average diameter $d_M$, $d_{M,0}$ the reference average diameter, so that

(A.1.2) \[ d_I = d_M - w, \quad d_O = d_M + w \]

and

(A.1.3) \[ A = \frac{\pi}{4} \left( (d_M + w)^2 - (d_M - w)^2 \right) = \frac{\pi}{4} 4wd_M \]

If we assume moreover a constant dilatation over the cross-section, it yields

(A.1.4) \[ d_M = d_{M,0}(1 + e_D). \]

Therefore, the ratio of the reference cross-section and the actual cross-section is

(A.1.5) \[ \frac{A_0}{A} = \frac{1}{1 + e_D}. \]

The true stress can now be calculated by the provided data of engineering stress and transversal strain,

(A.1.6) \[ ^tS = \frac{F}{A} = \frac{F}{A_0} \cdot \frac{A_0}{A} = S \cdot \frac{1}{1 + e_D}. \]

A.2. True Strain

We derive the true strain expression exemplary for the longitudinal strain. Let $dl$ be the incremental change in the specimen length and $l$ be the reference length at the beginning of the considered increment. The strain increment is then defined by

(A.2.1) \[ d'e = \frac{dl}{l}. \]
The total longitudinal strain (true strain) is given by

\[(A.2.2) \quad \varepsilon_L = \int_0^{\varepsilon_L} d\varepsilon_L = \int_{l_0}^{l} \frac{dl}{l} = \ln \frac{l}{l_0} = \ln(1 + \varepsilon_L). \]

**Remark A.2.1.** As far as we know, the general published material parameters are usually obtained by using engineering stresses and strains. Obviously, if true stresses and strains are considered, we have to take this fact into account.
Bibliography


Multi-mechanism models (MM models) are used studying various materials and mechanical effects. 

In this work, a general concept of modeling with MM models of serial type is introduced within the framework of continuum mechanics. Contrary to many authors, the thermoelastic strain is not regarded as a special separated strain. The modular principle of MM models is illustrated by several basic, advanced and generalized mechanisms and MM models. We specify the modeling of linear viscoelasticity with MM models and apply the concept of MM models to the phenomena of transformation-induced plasticity. 

In the case of viscoelasticity, we present the 3d mathematical problems of a thermoelastic-(N)-coupled-Kelvin-Voigt-element model and an isothermal-(N)-coupled-Kelvin-Voigt-element model. The latter model is mathematically analyzed according to its weak solvability as a mixed boundary value problem by transforming the original problem into an equivalent system of integro-differential equations. Therefore, the presented approach of proving an existence and uniqueness result is nonstandard. In case of a 1d rod, the modeling and the mathematical treatment are provided related to the isothermal-(2)-coupled-Kelvin-Voigt-element model. We conduct numerical simulations of the isothermal-(2)-coupled-Kelvin-Voigt-element model for 3d and 1d situations. The results underline the quality of the introduced viscoelastic model covering material effects like ratcheting without predicting instantaneous elasticity. 

The phenomena of transformation-induced plasticity can occur for materials which undergo phase transformations when moderate stresses are applied. A detailed test evaluation for experiments characterizing the interaction of classic plasticity and transformation-induced plasticity of the steel 100Cr6 (SAE 52100) is presented. A MM model considering the interaction of classic plasticity and transformation-induced plasticity is verified by these experiments. Therefore, we develop numerical schemes of the model suited for uniaxial situations and perform systematical parameter identifications for some selected experiments.