Dynamics and transport of instabilities in magnetized quasi-Keplerian Taylor-Couette flows

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M. Sc. Anna Guseva

Gutachter: Prof. Dr. Marc Avila
Prof. Dr. rer. nat. Bruno Eckhardt

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Anna Guseva

1. **Reviewer**
   Prof. Dr. Marc Avila
   Center of Applied Space Technology and Microgravity
   University of Bremen

2. **Reviewer**
   Prof. Dr. Bruno Eckhardt
   Department of Physics
   Philipps University of Marburg

**Supervisor**
Prof. Dr. Marc Avila
Anna Guseva

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Reviewers: Prof. Dr. Marc Avila and Prof. Dr. Bruno Eckhardt
Supervisor: Prof. Dr. Marc Avila

University of Bremen

Fluid Mechanics and Simulation Group
Center for Applied Space Technology and Microgravity
Faculty of Production Engineering
Am Fallturm 2
28195 Bremen
Abstract

The stability and transition to turbulence in canonical shear flows have since long been an outstanding scientific problem. One of the most exciting examples of shear flow is Keplerian motion of gas and dust in accretion disks. Although the Keplerian velocity profile is linearly stable, the presence of magnetic fields gives rise to the magnetorotational instability (MRI). MRI is considered one of the most powerful sources of turbulence in hydrodynamically stable quasi-Keplerian flows, however obtaining observational evidence of its operation is challenging. Although the linear stability of Keplerian flows with applied external magnetic fields has been studied for decades, the influence of the instability on the outward angular momentum transport, an inherent prerequisite for accretion to occur, is still far from understood. The aim of this thesis was to provide a better understanding of angular momentum transport and nonlinear properties of the MRI.

Motivated by recent laboratory experiments, the MRI driven by an azimuthal magnetic field in an electrically conducting fluid sheared between two concentric rotating cylinders (Taylor–Couette flow) was explored. The instability was studied numerically with both linear stability analysis and fully resolved direct numerical simulations of the Navier–Stokes and induction equations. It was found that at low magnetic Prandtl numbers, as those in liquid metals, the laminar Couette flow becomes unstable to a wave rotating in the azimuthal direction and standing in the axial direction via a supercritical Hopf bifurcation. Subsequently, the flow features a catastrophic transition to spatio-temporal chaos which is mediated by a subcritical Hopf bifurcation. The results are in quantitative agreement with the PROMISE experiment and dramatically extend its realizable parameter range. Subsequently, the enhancement of angular momentum transport by turbulent stresses in the highly turbulent flow regimes was determined. One regime is dominated by magnetically triggered inertial waves, with transport mostly due to velocity fluctuations, and another by magnetocoriolis waves, where magnetic field fluctuations prevail. The magnetic Reynolds number defines the type of turbulence, with a crossover around the critical value of 100. The results give a comprehensive picture of transport enhancement by MRI spanning from low (as in liquid metals) to high (as in plasma) magnetic Prandtl numbers. In the latter case the existence of a finite-amplitude dynamo was demonstrated. This suggests that accretion disks can operate self-sustaining MHD-turbulence and thereby transport angular momentum efficiently without the need of considering imposed magnetic fields.
Zusammenfassung

des Merkmal aller AS darstellt, bisher nur unzureichend gut verstanden. Das Ziel der vorliegenden Arbeit war es, zu einem besseren Verständnis des TDI und der nichtlinearen Eigenschaften der MRI beizutragen.

Motiviert durch jüngste Laborexperimente, ist es Gegenstand der vorliegenden Arbeit die Mechanismen der MRI zu untersuchen, welche durch ein azimuthales Magnetfeld in einem elektrisch leitenden Fluid erzeugt werden. Das Fluid wird dabei zwischen zwei konzentrisch rotierenden Zylindern geschert, was als klassische Taylor-Couette-

Ein weiterer Teil dieser Arbeit zielte darauf ab, die Erhöhung des TDI aufgrund von turbulenten Spannungen im hoch-turbulenten Parameterbereich zu untersuchen. Dabei wurden zwei grundsätzlich unterschiedliche Regime entdeckt. Ein Regime ist dominiert durch magnetisch angeregte Trägheitswellen, wobei der TDI hauptsächlich durch Geschwindigkeitsfluktuationen hervorgerufen wird. Das andere Regime ist gekennzeichnet durch Magnetokorioliswellen, wobei hier hauptsächlich die Fluktua-
tionen im magnetischen Feld für den TDI verantwortlich sind. Es wurde weiterhin herausgefunden, dass der Übergang zwischen diesen beiden Regimen durch die magnetische Reynolds-Zahl gekennzeichnet, der kritischer Wert auf etwa 100 bestimmt wurde. Die Gesamtheit der Ergebnisse liefert ein umfassendes Bild über die Verstärkung des TDI sowohl bei großen als auch bei kleinen \( Pm \), wobei ersteres für Plasmen und letzteres für Flüssigmetalle charakteristisch ist. Dadurch wird weiterhin der Schluss nahegelegt, dass AS durchaus einen autonomen turbulenten Zustand einnehmen können, der sich aufgrund von MHD-Mechanismen selbst erhält und außerdem eine wesentliche Ursache des TDI darstellt.
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Shear flows are ubiquitous in nature and technology, from flows in microchannels and around airplane wings, to geophysical flows in the atmosphere of the Earth. Under certain conditions laminar shear flows may become unstable, and this changes dramatically their properties, for example, angular momentum, mass and heat transfer. While some shear flows are linearly unstable, like the flow of two liquid layers with different velocities and densities (Kelvin–Helmholtz instability), other need strong perturbations to become unstable nonlinearly (pipe flow). From the numerous examples of shear flows, one stands out especially due to its enormous lengthscales and observational difficulty: flow of gas and dust in accretion disks. Accretion disks are astrophysical systems that consist of ionised gas and dust orbiting a massive body. An image of such disk observed with Hubble space telescope is given in Figure 1.1a. Planets and stars are formed from the initially dispersed matter in accretion disks. The physical mechanism of accretion is straightforward: a parcel of viscous fluid in the differentially rotating disk loses its angular momentum over time and falls onto the central object (Fig. 1.1b). However, the motion of gas in accretion disks cannot be laminar because viscous (molecular) outward transport is too slow for accretion to occur at the observed rates. Shakura & Sunyaev suggested the presence of turbulent motion and parameterized momentum transport by an effective turbulent eddy-viscosity in their early $\alpha$-model (Shakura and Sunyaev, 1973). In so-called Keplerian disks the angular velocity profile of gas follows the law

$$\Omega \sim r^{-3/2}. \quad (1.1)$$

This velocity profile is hydrodynamically stable according to the Rayleigh criterion for rotating fluids (Rayleigh, 1917), which states that angular momentum must increase with radius for the flow to be stable. Ionized accretion disks, however, are necessarily magnetized. Assuming the disk matter is conducting, the disk flow may become unstable due presence of magnetic fields. The magnetorotational instability (MRI) may act in rotating flows, provided the angular velocity decreases with radius, which is true of Keplerian flows (1.1).

The MRI is of great importance in astrophysics. First discovered by Velikhov (1959) in 1959, it remained largely unnoticed until 1991 when Balbus and Hawley (1991) realised its application to accretion disk theory. The growth rates of the MRI and the parameter ranges in which it acts were determined in several linear analyses (Balbus and Hawley, 1991; Ogilvie and Pringle, 1996; Balbus and Hawley, 1998; Hollerbach and Rüdiger, 2005; Hollerbach et al., 2010), but these do not provide information about the flow structure and scaling of angular momentum transport after nonlinear saturation.
1.1 Stability of rotating fluids

1.1.1 Rayleigh criterion for stability

Let us first revisit under which conditions velocity profile (1.1) becomes unstable, following the derivation of (Rayleigh, 1917; Landau and Lifshitz, 2013). Consider the fluid element rotating at a radius \( r_0 \) from the axis of rotation, like in Figure 1.1b. Its angular momentum is \( L_0 = mr_0^2\Omega \), and the corresponding centrifugal force is \( L_0^2/mr_0^3 \). Imagine this element was displaced to another radius \( r > r_0 \). The element still possesses the same amount of angular momentum \( L_0 \), but the acting centrifugal force at the new radius changes: \( L_0^2/mr^3 \). In order for the element to return to its initial orbit, this force has to be smaller than the equilibrium centrifugal force \( L^2/mr^3 \). In other words, \( L_0^2 \) should be smaller than \( L^2 \):

\[
L^2 - L_0^2 > 0. \tag{1.2}
\]

Expanding \( L \) in Taylor series around the point \( r_0 \), we get the stability condition in more general form:

\[
L \frac{dL}{dr} > 0. \tag{1.3}
\]
If all fluid elements in the system rotate in one direction, clockwise or counterclockwise, the axis of rotation can be chosen so that angular momentum is positive $L > 0$. Then
\[ \frac{dL}{dr} \sim \frac{d(r^2\Omega)}{dr} > 0. \] (1.4)
That is, the flow is stable to the small perturbations, if the angular momentum increases with radius.

1.1.2 Keplerian flows in the absence of magnetic fields

Now let us have a closer look at Keplerian rotation. Imagine that some fluid is rotated at angular velocity $\Omega(r)$ that depends on radial position of fluid element. In astrophysical context, gravitational force makes the matter follow curved paths and keeps the objects on their orbits:
\[ mv^2/r = m\Omega^2 = \frac{G_{gr}Mm}{r^2}, \] (1.5)
\[ \Omega = \sqrt{\frac{G_{gr}M}{r^3}} \propto r^{-3/2}. \] (1.6)

We revisit the stability of rotating flows with a more formal approach following the derivation of (Balbus and Hawley, 1998). Consider the small departures from the circular motion $(\xi_r, \xi_\phi)$ in the plane $(r, \phi)$ perpendicular the axis of rotation. For convenience, let us replace the inertial reference frame to a frame rotating with the fluid element at the radius $r$ with the angular velocity $\Omega(r)$. In the rotating reference frame, two additional (fictitious) forces act on the fluid element: Coriolis and centrifugal force. The equations of motion in this case are:
\[ \ddot{\xi}_r - 2\Omega \dot{\xi}_\phi = -\frac{d\Omega^2}{d\ln r} \xi_r, \] (1.7)
\[ \dot{\xi}_\phi + 2\Omega \xi_r = 0. \] (1.8)

Considering infinitesimal disturbances of the form $\xi_{r,\phi} \sim \exp(i\omega t)$ in equations (1.7), (1.8) the following expression for angular frequency $\omega$ is derived:
\[ \omega^2 = 4\Omega^2 + r \frac{d\Omega^2}{d\ln r} = \frac{1}{r^3} \frac{d(r^4\Omega^2)}{dr}. \] (1.9)

The principal condition for stability of the disturbance $\exp(i\omega t)$ is the positive value in the right hand side of (1.9): $d(r^4\Omega^2)/dr > 0$. Indeed, if $\omega^2$ is positive, $\omega$ from equation (1.9) is real and the fluid particle will continue small stable circular motions around its radius of rotation $r$. On the contrary, if $\omega^2$ is negative, $\omega$ is imaginary and even small disturbance will grow exponentially and make the fluid element depart farther and farther from the origin. Note that $r^4\Omega^2 = L^2$ and we have recovered the Rayleigh stability criterion for rotating fluids (1.3).
In the case of Keplerian rotation $\Omega \sim r^{-3/2}$, the angular momentum increases with radius ($L \sim r^{1/2}$) and thus Keplerian rotation is stable to infinitesimally small axisymmetric disturbances. This would imply for Keplerian flows to be laminar, and angular momentum transport there to be very slow - unless there were another mechanism of destabilisation, either nonlinear or involving additional physical concepts. Quasi-Keplerian flows are considered to be nonlinearly stable up to Reynolds numbers of $10^6$ (Ji et al., 2006; Lopez and Avila, 2017).

1.1.3 Keplerian flows in the presence of an axial magnetic field

Balbus and Hawley (1991) were the first to note the importance of magnetic fields in the astrophysical context. Galaxies, accretion disks and stars consist of ionized gases (plasmas). Hence magnetic fields are frequent companions of these astrophysical objects and arises there through dynamo action (Brandenburg and Subramanian, 2005). We follow here the discussion on magnetorotational instability from (Balbus and Hawley, 1998). The evolution of a magnetic field in magnetohydrodynamics is described by the so-called induction equation. If a perfectly conducting fluid with zero resistivity is considered (i.e. no dissipation effects), this equation takes the form of

$$\frac{\partial B}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad (1.10)$$

In terms of small but finite motions $\xi$ equation (1.10) can be rewritten as

$$\delta B = \nabla \times (\xi \times \mathbf{B}), \quad \xi = \nu \delta t. \quad (1.11)$$

For simplicity, we consider a constant magnetic field and divergence-free displacement $\nabla \cdot \xi = 0$. This transforms (1.11) into

$$\delta B = (\mathbf{B} \cdot \nabla)\xi. \quad (1.12)$$

In other words, the magnetic field will change only if there is motion in the fluid. Furthermore, if the magnetic field is uniform and directed along the axis of rotation $\mathbf{B} = B_0 \mathbf{e}_z$ and the perturbation $\xi$ is axisymmetric and depends on time and $z$ only $\xi \sim \exp(i\omega t + ikz)$, equation (1.12) can be linearized as

$$\delta B = ikB_0\xi. \quad (1.13)$$

The magnetic field, in its turn, exerts Lorentz force acting on the fluid:

$$\frac{1}{\mu_0 \rho} (\nabla \times \mathbf{B}) \times \mathbf{B} = -\nabla \left( \frac{B^2}{2\mu_0 \rho} \right) + \frac{1}{\mu_0 \rho} (\mathbf{B} \cdot \nabla)\mathbf{B}. \quad (1.14)$$

The first term on the right hand side of (1.14) is a potential, which for incompressible problems can be neglected, and the second term is the magnetic tension force. The
latter can be described as a fictitious stress, acting on the surface of the fluid element (the so-called Maxwell stresses). If finite, axisymmetric motions are considered, with the use of (1.13) the magnetic tension becomes:

\[
\frac{1}{\mu_0 \rho} (\mathbf{B} \cdot \nabla) \delta \mathbf{B} = \frac{ikB_0 \delta \mathbf{B}}{\mu_0 \rho} = -\frac{k^2 B_0^2}{\mu_0 \rho} \xi. \tag{1.15}
\]

Note that the magnetic tension force described by (1.15) is restoring force, since it is proportional to the fluid displacement \(\xi\) but directed oppositely. The restoring constant \(k^2 B_0^2 / (\mu_0 \rho)\) is the strength of this force. This force acts on fluid elements in the same way that restoring force of a spring acts on a mass attached to it.

Adding the force (1.15) to equations (1.7)-(1.8), we get:

\[
\ddot{\xi}_r - 2\Omega \dot{\xi}_\phi = - \left( \frac{d\Omega^2}{d \ln r} + \frac{k^2 B_0^2}{\mu_0 \rho} \right) \xi_r, \tag{1.16}
\]

\[
\ddot{\xi}_\phi + 2\Omega \dot{\xi}_r = - \frac{k^2 B_0^2}{\mu_0 \rho} \xi_\phi. \tag{1.17}
\]

Defining

\[
\kappa \equiv 4\Omega^2 + r \frac{d\Omega^2}{d \ln r}, \quad K \equiv \frac{k^2 B_0^2}{\mu_0 \rho}, \tag{1.18}
\]

substituting them into (1.16) and (1.17) and considering again perturbations with time dependence \(\xi \sim \exp(i\omega t)\), the following equation (“dispersion relation”) for the oscillation frequency \(\omega\) results:

\[
\omega^4 - \left( 2K + \kappa^2 \right) \omega^2 + K \left( K + \frac{d\Omega^2}{d \ln r} \right) = 0. \tag{1.19}
\]

This is a 4th-order equation in respect to \(\omega\), valid for small displacements in the plane of rotation of an accretion disk. The epicyclic frequency \(\kappa\) is the frequency at which a radially displaced fluid parcel will oscillate. \(K\) can be re-written as \(K = k^2 v_0^2\), where \(v_0 = B / \sqrt{\mu_0 \rho}\) is the speed of magnetohydrodynamic oscillation of fluid parcel due to the tension of magnetic field lines (Alfven velocity). Now we shall show that the last term in (1.19) \(K + \frac{d\Omega^2}{d \ln r}\) now defines the stability of the flow. As discussed in the previous section, for the flow to be unstable, \(\omega\) has to be imaginary so that \(\omega^2\) is negative. The solution of (1.19) gives two roots for \(\omega^2:\)

\[
\omega_{1,2}^2 = \frac{2K + \kappa^2 \pm \sqrt{(2K + \kappa^2)^2 - 4K \left( K + \frac{d\Omega^2}{d \ln r} \right)}}{2}. \tag{1.20}
\]
As it is clear from (1.20), $\omega^2 < 0$ if

$$2K + \kappa^2 < \sqrt{(2K + \kappa^2)^2 - 4K \left( K + \frac{d\Omega^2}{d\ln r} \right)}, \quad \text{or} \quad (1.21)$$

$$K \equiv \frac{k^2 B^2}{\mu_0 \rho} < -\frac{d\Omega^2}{d\ln r}. \quad (1.22)$$

$K$ and $\kappa$ being both positive, the main condition for the instability is the negative derivative of angular velocity. Angular velocity must decrease with radius:

$$\frac{d\Omega^2}{d\ln r} < 0, \quad (1.23)$$

and then for weak enough fields (1.22) small disturbances will grow exponentially and the flow will become unstable. This is the essence of magnetorotational instability.

The physical meaning of criterion (1.23) is simple. Consider two fluid elements $m_1$ and $m_o$ in the rotating flow threaded by axial magnetic field as if they were connected by a string (see Figure 1.2). Imagine they first rotate at the same angular velocity and radius. Then accidentally the element $m_1$ is displaced inwards to a smaller radius, and element $m_o$ is displaced outwards to a higher radius. The fluid element $m_1$, shifted closer to the centre of rotation, will have slightly higher angular velocity. At the same time, it will experience retarding (negative) torque from the connecting spring. Thus it will lose angular momentum and continue falling inwards. On the contrary, the fluid element $m_o$, which is farther from the center, will experience positive torque which will result in angular momentum gain and drift of the element $m_o$ to a orbit with higher radius. If the spring (magnetic field) is weak enough, the distance between the two elements will grow until the “spring” is “broken” and they are no longer connected. Of course, if the magnetic field is strong, condition (1.22) will not be fulfilled and magnetic tension will not let the fluid elements separate.

### 1.1.4 Analogy of Keplerian and Taylor–Couette flows

Several decades before Balbus and Hawley (1991), Velikhov (1959) and Chandrasekhar (1961) independently derived a similar condition for stability of a viscous, incompressible flow between two rotating cylinders threaded by magnetic fields. This type of flow is called Taylor–Couette flow and is well studied theoretically, numerically and experimentally (Grossmann et al., 2016; Fardin et al., 2014). In the following we will see, that Keplerian flows can be in principle successfully modelled with Taylor–Couette flow.

Let us consider two cylinders of the radii $r_i, r_o$, rotating about the common axis at the angular velocities $\Omega_i, \Omega_o$ (Figure 1.3a). An inviscid fluid is sheared between the cylinders, and the problem in question is the velocity distribution inside the fluid. The geometry of the system suggests to use cylindrical coordinate system...
Fig. 1.2.: Magnetorotational instability acting via magnetic tension. In a vertical magnetic field, two fluid elements $m_i$ and $m_o$ are connected by magnetic tension as if they were connected by springs. Due to the force $F_s$, acting on the slightly disturbed elements, they experience negative (positive) torque $G_i$ ($G_o$), lose (gain) angular momentum and separate one from another.

Fig. 1.3.: (a) Schematic of the Taylor-Couette geometry. A fluid is confined between two coaxial cylinders of radii $r_i$ and $r_o$, which can rotate independently at angular velocities $\Omega_i$ and $\Omega_o$. (b) Stability diagram of the Taylor–Couette flow in the case of co-rotating cylinders.
where \( z \) is aligned with the axis of rotation. Due to the symmetry of the system, radial and axial velocities must be zero \((v_r = v_z = 0)\), and azimuthal velocity \( v_\phi \equiv V(r) \) must be a function of radius only. In this case, Navier–Stokes equations are reduced to the following system:

\[
\frac{dp}{dr} = \frac{\rho V^2}{r}, \quad (1.24)
\]

\[
\frac{d^2V}{dr^2} + \frac{1}{r} \frac{dV}{dr} - \frac{V}{r^2} = 0. \quad (1.25)
\]

The solutions to equation (1.25) can be found in the form of \( r^n \). Substituting \( r^n \) into (1.25), we get a second order equation for \( n \):

\[
n^2 - n = 0.
\]

The roots of this equation are \( n = \pm 1 \), subsequently giving velocity profile of

\[
V(r) = C_1 r + C_2 \frac{1}{r}. \quad (1.26)
\]

The constants \( A, B \) can be found from the boundary conditions \( V_{i,o} = \Omega_{i,o} r_{i,o} \) at the inner \((i)\) and outer \((o)\) cylinder. The exact solution to (1.25) is:

\[
V(r) = \frac{\Omega_o r_o^2 - \Omega_i r_i^2}{r_o^2 - r_i^2} r + \frac{(\Omega_i - \Omega_o) r_i^2 r_o^2}{r_o^2 - r_i^2} \frac{1}{r}. \quad (1.27)
\]

The pressure distribution can be recovered from (1.24).

Calculating angular momentum with the use of (1.27) and \( \Omega = V(r)/r \), and taking its radial derivative, the following stability condition arises:

\[
(\Omega_o r_o^2 - \Omega_i r_i^2) \Omega > 0. \quad (1.28)
\]

If the cylinders are corotating and coordinates are chosen so that \( \Omega \) is positive, the Rayleigh criterion (1.3) for stability is recovered:

\[
\Omega_o r_o^2 > \Omega_i r_i^2. \quad (1.29)
\]

If the radius ratio is fixed (in this thesis it is set to \( r_i/r_o = 0.5 \)), the ratio of angular velocities uniquely defines the stability of the flow. The stability diagram in the case of co-rotating cylinders is schematically sketched in the Figure 1.3b. Three separate regimes are highlighted: region I (Rayleigh–unstable according to (1.29)), region II and region III (both Rayleigh-stable). The so-called Rayleigh line \( \Omega_i r_i^2 = \Omega_o r_o^2 \) (red dashed) separates hydrodynamically stable and unstable regions. According to (1.29), when the inner cylinder rotates and outer cylinder is stationary, flow is unstable no matter what the rotation rate is. In reality, for low rotation speeds, viscosity stabilizes the flow, postponing the onset of instability. When advection is much higher than viscosity (for high Reynolds numbers) the actual stability border (solid violet) approaches the Rayleigh line asymptotically. In the region II, the
angular momentum increases outwards, whereas the angular velocity still decreases, and Keplerian flows with (1.6) belong to this region. Solid-body line (dot-dashed in the Figure 1.3b) refers the uniform rotation $\Omega_i = \Omega_o$, and separate the flows with super rotation, i.e. angular velocity increasing with radius (region III).

The velocity profile (1.26) does not match exactly velocity profiles of the general form $\Omega(r) \sim r^q$, including $q = -1.5$ for Keplerian rotation (1.6). Yet we may note that the constants $C_1$ and $C_2$ in (1.26) depend on the angular velocities $\Omega_i, \Omega_o$ and radii $r_i, r_o$ of the cylinders. By appropriately choosing the rotation-ratio $\mu = \Omega_o / \Omega_i$, one can place the velocity profile in the quasi-Keplerian regime, for which the angular velocity decreases radially, whereas the angular momentum increases, i.e. $\eta^2 < \mu < 1$, like in Figure 1.4. The magnetorotational instability with its requirement of decreasing outward angular velocity (1.23) can in principle operate in the region II in the Figure 1.3b. Of course, for that the fluid sheared between cylinders must be an ionized gas or liquid metal. The question is, how highly conducting does it have to be to allow for the magnetorotational instability?

1.2 From accretion disk to experiment

So far we discussed the magnetorotational instability of ideal fluids, where neither viscosity $\nu$ nor magnetic diffusivity $\lambda$ was taken into account. In fluid flows the key parameter is the ratio of advection to viscous forces, namely the Reynolds number $Re = vd/\nu$. Here $v$ is typical velocity, and $d$ is the largest possible scale of the flow (the gap size $d = r_o - r_i$ in Taylor-Couette flow or height of the accretion disk $H$ for Keplerian flows). By analogy, the ratio of advection of magnetic field to magnetic diffusion can be estimated with the magnetic Reynolds number $Rm = vd/\lambda$. If $Rm$
is large, magnetic filed lines are “frozen” into fluid and are advected with the flow. Here in addition to the imposed or “seed” field, induced magnetic fields can arise. On the contrary, when $Rm$ is small, the induced field will rapidly dissipate. The limit of $Rm \to 0$ is called inductionless approximation. The hydrodynamic and magnetic Reynolds numbers are connected as $Rm = RePm$, where $Pm = \nu/\lambda$ is magnetic Prandtl number and incorporates physical properties of the fluid. Large $Pm$ mean the fluid is highly conducting; this can happen in highly ionized gases (plasmas). Small $Pm \in (10^{-5} - 10^{-6})$ have dense fluids like liquid metals. Their resistivity is large compared to viscosity, but they are convenient for magnetohydrodynamic experiments.

Since in astrophysics length scales are enormous and usually result in huge $Re$, viscosity is considered to be not very important for astrophysical problems. However, various models predict that accretion disks have large areas of quite cold and poorly ionized matter (Gammie, 1996; Dzyurkevich et al., 2013), so that the estimated magnetic Prandtl numbers for the disks can be even smaller than those of liquid metals. Brandenburg and Subramanian (2005) give $Pm \sim 10^{-8}$ for protostellar disks and respective $Rm \sim 10$. Thus, resistivity may be important for the magnetorotational instability. It was noticed soon that resistivity changes significantly the instability properties, namely, turbulent transport triggered by instability (Fleming et al., 2000). Sano and Inutsuka (2001) found that in the presence of axial fields angular momentum transport is inefficient for magnetic Reynolds numbers $Rm < 1$. Kitchatinov and Rüdiger (2010) showed that in the presence of axial magnetic field the instability is fully controlled by magnetic Reynolds number and Lundquist number and requires them to be at least $Rm \sim O(10)$ and $S \sim O(3)$. Physically this means that both rotation period and Alfvén wave crossing time should be shorter than magnetic diffusion timescale; otherwise the MRI modes will not grow.

Ji et al. (2001) and Rüdiger and Zhang (2001) suggested the experimental study of the MRI in an electrically conducting fluid sheared between two concentric cylinders, exactly as considered originally by Velikhov (1959) and Chandrasekhar (1961). However, the critical parameter values $Rm \sim O(10)$, $S \sim O(3)$ are challenging to achieve experimentally. The difficulty is that liquid metals have very small magnetic Prandtl numbers (e.g. $Pm \sim 10^{-6}$ for gallium alloys), leading in this case to very large Reynolds numbers ($Re \gtrsim 10^7$) necessary to observe the MRI with axial magnetic fields (so-called standard MRI, SMRI). In fact, such high Reynolds numbers have never been achieved even for non-magnetic Taylor-Couette flows. A further difficulty of Taylor–Couette experiments in the quasi-Keplerian regime arises because of Ekman vortices that arise adjacent to the endplates. Unless a very specific endplate arrangement is used (Edlund and Ji, 2015; Lopez and Avila, 2017), the Ekman vortices extend deep into the flow and even at moderate Reynolds number the basic Couette flow cannot be obtained experimentally (Edlund and Ji, 2014). The resulting velocity profiles are no longer quasi-Keplerian and hydrodynamic
instabilities render the flow turbulent even in the absence of magnetic fields (Avila, 2012; Nordsiek et al., 2014).

### 1.2.1 Keplerian flows with imposed helical and azimuthal magnetic fields

Accretion disks, in fact, may have both axial and azimuthal magnetic field components. Balbus and Hawley (1992) and Ogilvie and Pringle (1996) showed that a combination of axial and azimuthal magnetic fields can also trigger magnetorotational instability, provided that these are neither too weak nor too strong. But this configuration of magnetic field was not considered important; it was believed that the presence of even a relatively small axial field will cause a breakdown of the instability arising from the azimuthal component (Balbus and Hawley, 1998). Interest in these “helical” magnetic fields increased after Hollerbach and Rüdiger (2005) proposed a new type of MRI experiment.

Hollerbach and Rüdiger (2005) also considered conducting fluid sheared between two corotating cylinders of \( r_i/r_o = 0.5 \), but threaded with a combination of axial and azimuthal magnetic field \( \mathbf{B} = B_0 (\mathbf{e}_z + \beta (r_i/r) \mathbf{e}_\phi) \). In experiments, the approximately uniform axial magnetic field \( B_0 \mathbf{e}_z \) can be created with a solenoid, wired around the outer cylinder, and azimuthal component \( B_0 \beta (r_i/r) \mathbf{e}_\phi \) can be created with a current-carrying wire inside the inner cylinder. The parameter \( \beta \) characterizes the strength of azimuthal magnetic field component compared to axial.

Hollerbach and Rüdiger (2005) linearized the Navier-Stokes and induction equation (1.38)-(1.39) around the basic Taylor–Couette flow (1.27). Additionally, they expressed perturbation velocity and magnetic field as

\[
\mathbf{u} = v \mathbf{e}_\phi + \nabla \times (\psi \mathbf{e}_\phi), \quad \mathbf{b} = b \mathbf{e}_\phi + \nabla \times (\mathbf{a} \mathbf{e}_\phi)
\]  

They searched for axisymmetric disturbances in the form of \( \exp(ikz + \gamma t) \), with linearized equations being

\[
\begin{align*}
\text{Re} \gamma v &= D^2 v + \text{Re} ikr^{-1}(r^2 \Omega)' \psi + Ha^2 ikv, \\
\text{Re} \gamma D^2 \psi &= D^4 \xi - \text{Re} 2ik \Omega v + Ha^2 ik(D^2 a + 2 \beta r^{-2} b), \\
\text{PmRe} \gamma b &= D^2 b - PmRe iky' r a + ikv - 2ik \beta r^{-2} \psi, \\
\text{PmRe} \gamma a &= D^2 a + ik \psi, \\
D^2 &= \nabla^2 - 1/r^2, \quad ' = d/dr.
\end{align*}
\]

Hollerbach and Rüdiger (2005) considered flow at experimentally relevant \( Pm = 10^{-5} \) and \( 10^{-6} \) and found that the presence of the azimuthal magnetic field reduced the minimal Reynolds number, at which instability occurs, by several orders of magnitude: \( Re_c \sim O(10^3) \). The stronger the azimuthal magnetic field was (in comparison to axial field), the lower \( Re_c \) became. They also showed that for \( \beta = 0 \) the critical Reynolds number scales with \( Pm^{-1} \), but for nonzero \( \beta \) close to the Rayleigh line it
becomes independent of $Pm$. The surprising finding was that even when $Pm \to 0$ (inductionless limit), under the influence of helical field the instability was vivid and showed the same $Re_c$ as $Pm = 10^{-5}$ and $10^{-6}$.

The set of linearized equations (1.31) provides us with an explanation to this extraordinary behavior. When only axial magnetic field is present in the system ($\beta = 0$) the azimuthal component of magnetic field $b$ and meridional circulation $\psi$ are not coupled directly and can interact only via $v$ and $a$. On the contrary, when $\beta \neq 0$, $b$ and $\psi$ can interact directly. According to (Hollerbach and Rüdiger, 2005), the most important coupling is the influence of $\psi$ on $b$, or advection of azimuthal field component by the meridional circulation. Indeed, in the inductionless limit $Pm \to 0$ the only possibility to change $b$ is via $-2ik_0 r^{-2} \psi$ (1.31).

This type of inductionless instability is usually called helical MRI (HMRI). Follow-up experiments by Stefani et al. (2006), Stefani et al. (2009) (PROMISE facility) confirmed the existence of HMRI modes in the parameter range predicted by Hollerbach and Rüdiger (2005).

Five years later, Hollerbach et al. (2010) performed similar stability analysis with a purely azimuthal magnetic field $B = B_0 (r_i/r)e_\phi$. Although azimuthal magnetic fields do not yield any unstable axisymmetric modes considered so far, they do yield unstable non-axisymmetric modes. This is particularly interesting for the possibility of self-sustained MRI via dynamo action, since purely axisymmetric flow does not lead to dynamo according to the Cowling’s theorem (Cowling, 1933). Their linear analysis showed, relatively small values of $Ha \sim O(10^2)$ and $Re \sim O(10^3)$ are required to obtain instability for steep velocity profiles close to the Rayleigh line, even in the inductionless limit. This type of MRI was called “azimuthal” MRI (AMRI).

Seilmayer et al. (2014) observed AMRI modes in the form of travelling waves.

### 1.2.2 Overcoming the Liu limit

It was noticed very soon that considering helical or purely azimuthal fields in the inductionless limit has a crucial disadvantage: they do not make Keplerian rotation unstable, but only flows close to the Rayleigh line. Liu et al., 2006 was the first to show that with WKB analysis of infinite (or periodic) cylinders in the narrow gap limit. He considered resistive but inviscid fluid and disturbances in the form of $\exp(ik_z r + ik_z z - \omega t)$, with total wavenumber $k_r^2 + k_z^2$. The roots of the dispersion relation for resistive but inviscid fluid flow are:

$$\omega \approx \pm \omega_{IO} + i\omega_z^{-1} \left[ \pm 2\omega_\phi \omega_z \omega_{IO}^{-1} \alpha_z^2 \Omega(2 + Ro) - (2\alpha_z^2 \omega_z^2 + \omega_z^2) \right]$$

(1.32)

Here $\alpha_z = k_z/\sqrt{k_r^2}$, $\omega_\phi = B_\phi/r$, $\omega_z = k_z B_z$ - Alfvén frequencies, $\omega_z = \lambda k_r^2$ - resistive frequency, $\omega_{IO} = \kappa^2 \alpha_z^2 = 4(1 + Ro)\Omega^2 \alpha_z^2$ - frequency of hydrodynamic inertial oscillations (1.18), (1.9). The dimensionless parameter

$$Ro = \frac{r}{2\Omega} \partial_z \Omega$$

(1.33)
is the Rossby number. If the flow is highly resistive, $\omega_\lambda \to \infty$ and from (1.32) $\omega \approx \pm \omega_{TO}$. Thus, in the inductionless limit $Pm \to 0$ HMRI is in fact a weakly destabilized inertial oscillation. Instability occurs when the expression in square brackets in (1.32) is positive, or

$$2(\alpha z \omega_\phi)^2 \pm \frac{2 + Ro}{\sqrt{1 + Ro}} \omega_z (\alpha z \omega_\phi) + \omega_z^2 < 0. \quad (1.34)$$

If (1.34) is treated as a quadratic equation in $\alpha \omega_\phi$, this is possible only when

$$Ro < 2 \left(1 - \sqrt{2} \right) \approx -0.8284 \quad \text{or} \quad Ro > 2 \left(1 + \sqrt{2} \right) \approx 4.8284 \quad (1.35)$$

Keplerian flows with $\Omega \sim r^{-3/2}$ and $Ro = -3/4$ are excluded by (1.35) and thus can not develop inductionless HMRI. In other words, although HMRI can be observed for steep velocity profile close to Rayleigh line ($\Omega \sim r^{-2}$), its relevance for accretion disk flows was seriously questioned. The same limit seems applicable for purely azimuthal magnetic fields (Kirillov et al., 2012).

However, later Kirillov et al. (2014) showed, that the Liu limit can be easily overcome, if the magnetic field profile is different from $B_0(r_i/r)e_\phi$. Indeed, ideal $B_0(r_i/r)e_\phi$, indicating generation of magnetic field by an insulated current in the center of the system, is unlikely to appear in accretion disks. Kirillov et al. (2014) introduced a measure of magnetic profile steepness, termed magnetic Rossby number

$$Rb = \frac{r}{2B_\phi r^{-1}} \partial_r (B_\phi r^{-1}), \quad (1.36)$$

by analogy to the hydrodynamic Rossby number (1.33). A linear dependence of magnetic field on radius results in $Rb = 0$ (homogeneous axial current), and $B_\phi \propto r^{-1}$ gives $Rb = -1$. They found, that setting

$$Rb \geq -\frac{25}{32} = -0.78125 \quad (1.37)$$

allows to break the Liu limit in (1.35) for Keplerian flows with $Ro = -3/4$. Thus it is possible to extend inductionless HMRI and AMRI to Keplerian rotation and even to flatter velocity profiles just by modifying the shape of the magnetic field.

### 1.3 Motivation for Taylor–Couette setup

In the last two decades there has been a great deal of numerical work concerned with the nonlinear properties of the MRI. Since the work of Balbus and Hawley (1991) nonlinear direct numerical simulations of MRI are usually performed within a local approximation of accretion disk flow - shearing sheet. In this approximation, the equations are solved in a rotating frame in Cartesian geometry, with rotation given by the linearization of the Keplerian law at a radial point in the disk. Periodic boundary conditions are assumed in all three directions, and radial shear is introduced by
means of coordinate transformation. A sketch of a shearing sheet box is presented in Figure 1.5.

In the absence of rigid boundaries, the important parameter of the problem becomes the size of the computational domain. It was shown that boxes with short aspect ratio \( L_z/L_x \) give rise to intermittent “channel” solutions which disappear if the aspect ratio becomes long enough (Bodo et al., 2008). In such boxes, it is also important to know whether the solutions converge to asymptotic form as the computational box size becomes large (Bodo et al., 2011). Otherwise it is impossible to predict reliably turbulent properties, such as momentum transfer and its scaling with dimensionless parameters of the flow. In hydrodynamic simulations of homogeneous shear turbulence (Sekimoto et al., 2016), it was shown that the spanwise box height, \( L_z \) sets the length and velocity scales of the turbulence. Earlier Nauman and Pessah (2016) showed that increasing the axial box direction is equivalent to increasing magnetic Reynolds number \( Rm \) and thus supports development and sustainability of the MRI turbulence.

The main disadvantage of shearing box is that the size of the computational box fixes the modes that appear and determines their nonlinear saturation. On the other hand, global simulations of accretion disks (Armitage, 1998; Hawley, 2000; Fragile et al., 2007) do not suffer from this problem. However, the dimension of (protoplanetary) accretion disk can be of the order \( O(10^{10}) \) meters. It is impossible to resolve these flows from this scale down to Kolmogorov length scale with direct numerical simulations. Regev and Umurhan (2008a) studied the viability of the
shearing box approximation and concluded that shearing box cannot withstand the challenge posed by MHD turbulence in accretion disks. As an alternative, they offered to develop sub-grid turbulence modes on the basis of shearing box and implement them later in global calculations. Such models have not been developed yet and if they were, it would be hard to prove their viability. For these reasons, instead of choosing one of the two approaches described above, here direct numerical simulation in cylindrical Taylor–Couette geometry are performed. The aspect ratio of the domain $L_z/(r_o - r_i)$ is set as large as feasible in order to let the flow freely define the most natural wave number and resulting modes. Although the presence of boundaries in radial direction introduces the major difference between Taylor–Couette profile and Keplerian profile of accretions disks, it is often possible to neglect the effects of the no-slip boundary conditions on the cylinders, as we will see in the next chapters. The presence of boundaries is unavoidable in developing MRI experiments (Ji and Balbus, 2013; Stefani et al., 2017), and the method described in this thesis can provide a useful benchmark for them.

In the axial direction periodicity is employed, although this may seem unnatural both for accretion disks and MRI simulation. Indeed, free-surface boundary conditions are more natural for accretion disks. In experiments, formation of secondary vortices at the cylinders end caps (Ekman pumping) deforms significantly the Keplerian velocity profile, causing early onset of instabilities and obscuring experimental results (Lopez and Avila, 2017). Both in Ji and Balbus (2013), Stefani et al. (2017) the end-plates of the cylinders are split to avoid this effect. Periodic boundary conditions in axial direction let us avoid undesired endplate effects and allow us to focus on the features intrinsic to the MRI and therefore are the best compromise to both experiments and interiors of accretion disks.

The effects of stratification and compressibility are also neglected in this work, as none of them is crucial for MRI action. For simplicity and clarity, only interaction of incompressible viscous flow with dissipative magnetic field will be considered, thereby taking into account both resistivity and viscosity of the fluid.

1.3.1 Equations and dimensionless parameters

We consider an incompressible viscous conducting fluid that is sheared between two independently rotating cylinders of radii $r_i$ (inner) and $r_o$ (outer). The angular velocity of the cylinders are $\Omega_i$ and $\Omega_o$, respectively, and an external azimuthal magnetic field $(r_i/r)B_0e_\phi$, where $r$ is the radial coordinate, is imposed. The relevant fluid properties are the electrical conductivity $\sigma$, the kinematic viscosity $\nu$, the density $\rho$ and the magnetic diffusivity $\lambda$. The velocity field $\mathbf{v}$ is determined by the Navier-Stokes equations (1.38), whereas the magnetic field $\mathbf{B}$ is determined by the induction equation (1.39), which represents a combination of the laws of Ampere, Faraday, and Ohm. The equations were rendered dimensionless by using the gap
between cylinders \( d = r_o - r_i \) for length, \( d^2/\nu \) for time, and \( B_0 \) for the magnetic field. In dimensionless form they read

\[
(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla^2 \mathbf{v} + \frac{Ha^2}{Pm} (\nabla \times \mathbf{B}) \times \mathbf{B},
\]

\( (\partial_t - \frac{1}{Pm} \nabla^2)\mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}), \)

(1.38)

(1.39)
together with \( \nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0 \). Here \( p \) is the pressure, \( Ha \) the Hartmann number, and \( Pm \) the magnetic Prandtl number. The dimensionless parameters of the system are specified in Table 1.1. Following the PROMISE experiment (Seilmayer et al., 2014), we use in this thesis a radius-ratio of \( \eta = 0.5 \). Close to the Rayleigh line we focus on \( \mu = 0.26 \), and for quasi-Keplerian rotation we take \( \mu = 0.35 \), which yield \( q = -1.94 \) and \( q = -1.48 \) respectively, based on the average shear (Figure 1.4). While \( \mu = 0.35 \) represents Keplerian rotation, \( \mu = 0.26 \) is accessible in experiments (Seilmayer et al., 2014).

Tab. 1.1.: Dimensionless parameters of the magnetohydrodynamic Taylor-Couette problem.

<table>
<thead>
<tr>
<th>Abbrev.</th>
<th>Parameter</th>
<th>Definition</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \eta )</td>
<td>Radius ratio</td>
<td>( r_i/r_o )</td>
<td>0.5</td>
</tr>
<tr>
<td>( L_z )</td>
<td>Length of the cylinders</td>
<td>( 2\pi/k_0 )</td>
<td>1.4—4.5</td>
</tr>
<tr>
<td>( \mu )</td>
<td>Angular velocity ratio</td>
<td>( \Omega_o/\Omega_i )</td>
<td>0.26; 0.35</td>
</tr>
<tr>
<td>( Pm )</td>
<td>Magnetic Prandtl number</td>
<td>( \nu/\lambda )</td>
<td>0—10</td>
</tr>
<tr>
<td>( Re )</td>
<td>Reynolds numbers of inner cylinder</td>
<td>( \Omega_i r_i d/\nu )</td>
<td>( 10^2—4 \cdot 10^4 )</td>
</tr>
<tr>
<td>( Rm )</td>
<td>Magnetic Reynolds number</td>
<td>( \Omega_i r_i d/\lambda )</td>
<td>0—10^5</td>
</tr>
<tr>
<td>( Ha )</td>
<td>Hartmann number</td>
<td>( B_0 d/(\sigma \rho \nu)^{1/2} )</td>
<td>0—3 \cdot 10^3</td>
</tr>
</tbody>
</table>

The length of the domain, defined by axial wave number \( k_0 = 2\pi/L_z \), was set in our nonlinear simulations as high as possible to allow the MRI to evolve naturally. For low-\( Re \) flows (chapter 4) relatively long cylinders of \( L_z = 12.6 \) were used. For highly turbulent flows in chapter 5 and chapter 6 a shorter, almost square domain of \( L_z = 1.4 \) is used in order to save computational cost. Several domain length tests showed that such decrease in \( L_z \) results in decrease of integral quantities such as energy or torque less than 10%. In some intermediate cases where the turbulence has not entirely developed yet but the simulations are already costly, a compromise value of \( L_z = 4.2 \) was used. See Appendix A for the list of typical simulations parameters and integration times.

Note that it is easy to provide analogy between the Taylor–Couette system and the conventional shearing sheet box discussed in the beginning of section 1.3. First step is to set the aspect ratio of the box \( L_x : L_y : L_z \) and of the cylinders \( d : 2\pi r : L_z \) be the same. Second, the Reynolds number in shearing box is typically defined by the
shear Reynolds number $Re_{sh} = V_{sh} L_x / \nu$ (see Fig. 1.5). In Taylor–Couette flow shear Reynolds number can be defined as

$$Re_{sh} = \frac{(\Omega_i r_i - \Omega_o r_o) d}{\nu} = Re \left( 1 - \frac{\mu}{\eta} \right).$$  \hspace{1cm} (1.40)

Adjusting $\eta$ and $\mu$ so that the shear Reynolds number is the same in Taylor–Couette and shearing box systems, we obtain similar flow configurations.

### 1.3.2 Boundary conditions

We employ cylindrical coordinates $(r, \phi, z) \in [r_i, r_o] \times [0, 2\pi] \times [0, L_z]$, for which the no-slip velocity boundary conditions at the cylinders read

$$v_\phi(r_i, \phi, z) = Re, \quad v_\phi(r_o, \phi, z) = \mu Re. \quad (1.41)$$

Periodicity in the axial direction is imposed with basic length $L_z$. The background circular Couette flow $V = V(r)e_\phi$ is a solution to the equations and boundary conditions given by

$$V(r) = \frac{Re}{1 + \eta} \left[ \left( \frac{\mu}{\eta} - \eta \right) r + \frac{\eta}{(1 - \eta)^2} \left( 1 - \mu \right) \frac{1}{r} \right]. \quad (1.42)$$

Equation (1.42) is equivalent to (1.27). The magnetic field is also assumed to be periodic in the axial direction. In the radial direction the boundary condition depends on the material of the cylinders. Typically two idealized cases are considered in the MRI problem: insulating and conducting cylinders. These lead to slightly different results, as theoretically demonstrated by Chandrasekhar, 1961. However, the difference is not great, and here we will consider only the case of insulating boundaries. According to Ampere’s law, the current within an insulator must be zero: $J \sim \nabla \times B = 0$ at $r < r_i$ or $r > r_o$. Thus magnetic field is irrotational and can be expressed in terms of a potential, $\psi$: 

$$B = -\nabla \psi, \quad -\nabla \cdot B = \nabla^2 \psi = 0. \quad (1.43)$$

This equation can be solved by separation of variables $\psi(r, \phi, z) = R(r)\Phi(\phi)Z(z)$:

$$\frac{\Phi''(\phi)}{\Phi(\phi)} = -m^2, \quad \frac{Z''(z)}{Z(z)} = -k^2, \quad (1.44)$$

where $m$ and $k$ are constants. The equation for $R(r)$ is a modified Bessel equation,

$$\frac{1}{r} R'(r) + R''(r) - \left( k^2 + \frac{m^2}{r^2} \right) R(r) = 0. \quad (1.45)$$

Solving this equation and recalling that $\mathbf{B} = -\nabla \psi$, one obtains the following boundary conditions for $\mathbf{B}$:

**Case $k = m = 0$:**

$$B_\phi = B_z = 0. \quad (1.46)$$
Case $k = 0, m \neq 0$: \[ R(r) = r^\pm m \]
\[ i B_r = \pm B_\phi, \quad B_z = 0 \quad (+ \text{ on } r_i, - \text{ on } r_o). \quad (1.47) \]

Case $k \neq 0, m \neq 0$: \[ R(r) = \mathcal{B}_m(x) \]
\[ i B_r = \frac{\mathcal{B}'_m(kr)}{\mathcal{B}_m(kr)} B_z, \quad \frac{m}{r} B_z = kB_\phi, \]
\[ (1.48) \]

where $\mathcal{B}_m(x)$ denotes the modified Bessel function $I_m(x)$ for $r = r_i$ and $K_m(x)$ for $r = r_o$, and $\mathcal{B}'_m = \partial_r \mathcal{B}_m$. Using relations (1.46), (1.47) or (1.48) at the points $r = r_i$ and $r = r_o$ together with the appropriate function $R(r)$, we obtain insulating boundary conditions.

1.4 Motivation and plan of the thesis

Magnetic fields can trigger instability in rotating quasi-Keplerian flows, there is no doubt, but despite recent experimental progress in realising magnetorotational instabilities in the laboratory, many questions still remain unclear. How does this instability look like at the onset? How does the flow become turbulent? Is this turbulence able to transport angular momentum efficiently? Can the turbulence become self-sustained without and externally imposed magnetic field via dynamo action?

In this work I will address all these questions by studying the magnetorotational instability caused by an azimuthal magnetic field; this configuration is less studied but yet promising. Azimuthal magnetic fields result in growth of non-axisymmetric disturbances, and non-axisymmetric disturbances may support dynamo (reciprocal amplification of flow field and magnetic field).

The rest of the thesis is structured as follows. Chapter 2 presents linear stability analysis of the flow, performed with the spectral code of (Hollerbach et al., 2010). It shows that AMRI is continuously connected to the hydrodynamic instability of Taylor–Couette flow and presents the scaling of marginal stability curves. These results were partially published in (Guseva et al., 2017a). Then, in chapter 3 the numerical method and its implementation are discussed. The code is spectral in axial and azimuthal (periodic) directions, and in radial direction employs finite difference stencils of 9th order. Time discretization is based on Crank–Nicolson method. Multiple tests against existing data demonstrate the accuracy of the code.

In chapter 4 the behavior of the AMRI at the onset and its transition from periodic waves to turbulence will be revealed. The results are in good qualitative agreement with the PROMISE observations (Seilmayer et al., 2014) and provide comparison between low and high $Pm$. The results of chapter 4 have been published in (Guseva et al., 2015) and (Guseva et al., 2017a). In chapter 5 I will give asymptotic scaling of turbulent transport with $Re$ and $Rm$ and estimate $\alpha$-parameter of the turbulent viscosity and substantially extend the parameter range explored experimentally.
(Seilmayer et al., 2014). The results of chapter 5 have been published in (Guseva et al., 2017c). In chapter 6 I will demonstrate the dynamo in Taylor–Couette flow evolved from MRI turbulence. Part of these results have appeared in (Guseva et al., 2017b).
The linear stability analysis of the flow has been described to a certain extent in chapter 1. We have learned that a purely azimuthal magnetic field yield non-axisymmetric modes that can be potentially achievable in experiments at relatively low flow rotation and magnetic field ($Re \sim 10^3$, $Ha \sim 10^2$) (Hollerbach et al., 2010). The focus in (Hollerbach et al., 2010) was on low-$Pm$ liquids close to the Rayleigh line. Despite of the subsequent results of Kirillov et al. (2012), Rüdiger et al. (2013), no stability maps have been shown for a wider range of $Pm \in [10^{-6}, 1]$ which covers both poorly conductive and highly conductive fluids. The asymptotic behavior of the instability boundaries is also unknown.

In this chapter, we first revisit the linear stability analysis of the AMRI by considering various rotation laws, over a range of magnetic Prandtl, Hartman and Reynolds numbers. The aim is two-fold: first, the stability maps are essential for guiding nonlinear simulations in parameter space. Second, establishing asymptotes of the instability boundaries helps to predict parameters regions where the instability acts even if they are not accessible with numerical methods.

I adopt the approach of Hollerbach et al. (2010) and linearize the nonlinear equations (1.38)-(1.39) about the basic flow (1.42) with imposed magnetic field

$$B_0 = B_0(\eta_i/\eta)\hat{e}_\phi,$$

and consider disturbances in the form of

$$[\mathbf{u}, \mathbf{b}] = x(\eta) \exp(i\omega t + ikz + im\phi).$$

where $\omega = \gamma - i\sigma$ is a complex number. Equations (1.38)-(1.39) will take the form of linear system

$$A\hat{x} = i\omega C\hat{x},$$

where matrices $A$ and $C$ are functions of the azimuthal and axial wave numbers ($m, k$), the radius, and the dimensionless parameters of the flow $Re, Ha, Pm, \eta, \mu$. In simple systems like in the inviscid approximation of (1.16, 1.17), equation (2.3) can be treated analytically, which results in an explicit dispersion relation connecting the eigenvalues $\omega$ of the linear system with this parameters:

$$\omega = \mathcal{F}(m, k, Re, Ha, Pm, \eta, \mu).$$

The imaginary part of the eigenvalue $\sigma = -\Im[\omega]$ can be considered as growth or decay rate of infinitesimal perturbations (2.2). If $\sigma > 0$, perturbations will grow; if $\sigma < 0$ - perturbations will decay.
Equations (1.38)-(1.39) are much more complex and have non-periodic boundary conditions in radial direction so they have no simple solution like (2.4). Even when the radial structure of variables is to a certain extent simplified, the resulting stability condition is nontrivial and can be analyzed analytically only in asymptotic regimes (Kirillov et al., 2014; Maretzke et al., 2014). But if a dispersion relation can not be deduced analytically, there is always a possibility to solve the system (2.3) numerically. The numerical method and code employed in this chapter was described in (Hollerbach et al., 2010). For the curvature considered here, $\eta = 0.5$, the dominant AMRI mode is non-axisymmetric with azimuthal wavenumber $m = 1$. At fixed parameters $(Ha, Re, Pm, \mu)$, the axial wavenumber $k$ was varied to determine the maximum perturbation growth rate. Furthermore, the parameters $Re, Ha, Pm, \mu$ can be varied to construct the full instability picture.

### 2.1 Emergence of instability

Criterion 1.3 for hydrodynamic stability of the flow is valid only for inviscid fluids. For example, it predicts that if in the Taylor-Couette system the outer cylinder is stationary, and only the inner cylinder is rotating, the flow would be unstable for any rotation speed, forming pairs of axisymmetric ($m = 0$) rolls. In reality, the presence of viscosity competes with inertial forces and postpones the onset of instability, as was shown by Taylor (1923). Similarly to Rayleigh-unstable flows, magnetorotational instability results in unstable flow only after $Re$ exceeds a critical threshold. At the beginning, we seek this minimal rotation that leads to instability, namely the critical Reynolds number $Re_c$ and its dependence on $Pm$.

Figure 2.1a illustrates that this dependence changes with the rotation profile. As a reference, onset of hydrodynamic instability is shown for $m = 0$ and $m = 1$ as dashed and solid violet lines. Above the curve with $m = 0$ the flow is hydrodynamically unstable. The Rayleigh limit ($Re_c \to \infty$ for both $m = 0$ and $m = 1$) occurs as $\mu = \Omega_o/\Omega_i \to 0.25$ at $\eta = r_i/r_o = 0.5$. Flow profiles of $\mu > 0.25$ need imposed magnetic fields to be linearly destabilised. For quasi-Keplerian rotation ($\mu = 0.35$) and flatter velocity profiles $Re_c$ is inversely proportional to $Pm$, while for $\mu = 0.26$ it becomes independent of $Pm$, as it was already found in (Hollerbach et al., 2010). This holds, however, only for $Pm \leq 10^{-1}$. For $Pm \geq 1$ $Re_c$ becomes independent of rotation rate. Furthermore, as $Pm$ increases the AMRI becomes progressively more insensitive to the rotation law, and at $Pm = 10$ it dominates over the centrifugal (Rayleigh) instability even in the case of stationary outer cylinder $\mu = 0$ (see Fig. 2.1a). In fact for $Pm \geq 10$ the AMRI occurs at lower $Re$ than the Rayleigh (centrifugal) instability at all $\mu$. It is hence tempting to study the competition between the hydrodynamic and magnetohydrodynamic instability mode.

After $Re_c$ is surpassed, the flow remains unstable only in a certain range of $Ha$, i.e. for neither too weak nor too strong magnetic fields (Hollerbach et al., 2010). Based on that, stability maps were constructed in $(Ha, Re)$-space for the most MRI-
unstable azimuthal wavenumber $m = 1$. Figure 2.1b shows the stability curves for several $\mu$ at $Pm = 1$. Inside the regions enclosed by the curves, the laminar flow is linearly unstable. For $\mu < 0.25$ the flow is hydrodynamically unstable, but it can be destabilized at even lower $Re$ by adding an azimuthal magnetic field. Interestingly, no new instability region appears as $\mu = 0.25$ is crossed, but rather the left boundary opens to touch the $Ha = 0$ axis. Most unstable axial wavenumbers in Figure 2.1c evolve continuously from the hydrodynamic $m = 1$ mode as $Ha$ is increased from zero. Hence the data suggest that the AMRI mode originates from the hydrodynamic mode, and the role of the azimuthal magnetic field is to enhance the destabilizing effect of radially decreasing angular momentum for $\mu < 0.25$ and to continue destabilizing the hydrodynamic instability mode beyond the Rayleigh line. Note that in Figures 2.1b–c the azimuthal wavenumber is $m = 1$. Axisymmetric disturbances with $m = 0$ will be destabilized earlier than $m = 1$ (see Fig. 2.1a).

In the Figure 2.1d snapshots of the dominant eigenmodes are given for three fixed points of $(Ha, Re)$ from Figure 2.1b (blue crosses). Three cases are considered: $(Ha, Re) = (0, 1300)$ - no magnetic field, hydrodynamic instability; $(40, 850)$ - combined action of AMRI and hydrodynamic instability; $(50, 110)$ - pure AMRI; hydrodynamic mode still stable. All three eigenmodes are similar wavy structures with 6 pairs of rolls of azimuthal wavenumber $m = 1$, supporting continuous transition between hydrodynamic instability and AMRI.

### 2.2 Stability maps

Now let us put aside hydrodynamic considerations and focus on the MHD-problem. The instability maps for different magnetic Prandtl numbers from $Pm = 1.4 \cdot 10^{-6}$ (a) to $Pm = 1$ (f) are presented in Fig. 2.2. The black solid curves in Fig. 2.2 show the neutral stability curve of the AMRI for different $Pm$ and $\mu = 0.26$. The curve starts at $Ha_c$ and $Re_c$, which is the minimum Hartmann and Reynolds number at which the instability occurs. In the region above this curve the laminar flow is linearly unstable. The smaller the $Pm$, the more the instability region is shifted upwards, and thus the faster the cylinders have to be rotated to observe instability ($Re_c \sim 10^2$ for high $Pm$ case against $Re_c \sim 10^3$ for low $Pm$). This is well in line with Figure 2.1a.

Not only the instability maps but also their interior are here the subject of interest. The instability domain is divided into parameter regions where different linear modes of instability dominate. On the borders of these regions the most unstable axial wavenumber $k_{max}$ undergoes discontinuous jumps as $Ha$ is varied (red dots in the Fig. 2.3), and, at the same time, the growth rate changes slope (solid black lines in Fig. 2.3).

Recording the parameter values of the jump $(Ha, Re)$ the following secondary instabilities can be distinguished:
Fig. 2.1.: (a) Dependence of the minimum critical Reynolds number on rotation rate $\mu$ for different magnetic Prandtl numbers. The violet curves denote hydrodynamic stability for $m = 0$ and $m = 1$. (b) Connection of the AMRI to the $m = 1$ hydrodynamic instability at $Pm = 1$ as $\mu$ decreases to cross the Rayleigh line ($\mu = 0.25$). (c) Critical axial wavenumber along the neutral stability curves for the same parameters as in (b). (d) Snapshots of the dominant eigenmode at selected points (blue crosses in Fig. 2.1b) in the neutral stability curves. Values of $(Ha, Re)$ from the left to the right: (0, 1300), (40, 850), (50, 110). The velocity values were taken as $v_z = \pm 30\%v_{max}$. 

Chapter 2  Linear stability of the flow
1. $Pm = 1.4 \cdot 10^{-6}, 10^{-4}$ (Fig. 2.2a-b): instability I (‘basic instability’ which appears for every $Pm$, in black), and similar island of instability II, arising at high values of $Re$ (yellow line).

2. $Pm = 10^{-3}$ (Fig. 2.2c): instability I (black), instability II (yellow), and instability III (cyan), appearing close to the left border of I.

3. $Pm = 10^{-2}$ (Fig. 2.2d): same as in Fig. 2.2c but instability III (cyan) moves to the right in the area of higher $Ha$, penetrating closer to the instability onset. Instability II shrinks to a much more narrow domain and the island of instability IV appears (in red).

4. $Pm = 10^{-1}$ (Fig. 2.2e): the instability II disappear, IV (red) moves close to the onset to lower $Re$, instability V (dark blue) manifests itself at the right border almost swallowing the instability III (cyan), which has moved further to the right.

5. $Pm = 1$ (Fig. 2.2d): the instability III disappear, only IV (red) and V (dark blue) remain.

Dashed cyan lines in Figures 2.2c-e correspond to a precursor path to a discontinuity in axial wave number $k$, showing parameters $(Ha, Re)$ where the slope $k(Ha)$ begins to change, leading finally to the discontinuous jump in $k$ (cyan solid lines in Fig.2.2c-e).

There is something interesting about the right border scaling of neutral stability curve in the Figure 2.2. For every large and small $Pm$ the linear scaling of $Re \sim Ha$ appears at the right border, but at $Pm = 1$ it is extended with instability V (blue), with scaling of $Re \sim Ha^{0.9}$. In section 2.3 this phenomenon is discussed in details, studying both left and right instability border scaling in the asymptotic regime.

The green circles on Fig. 2.2 correspond to the maximum growth rate line. Each point of it marks $Ha$ where the real part of the eigenvalue of the fastest growing mode is maximal (at fixed $Re$). The growth rate of the instability is maximized (for $Re=\text{const}$) along the green diamond line. Guseva et al. (2015) found that for $Pm = 1.4 \cdot 10^{-6}$ this point in the parameter space correlates nicely with the maximum in the torque at the cylinders for fixed $Re$. I follow this parameter path later with direct numerical simulations, estimating an upper bound for transport intensity in this way.

In the figure 2.4a stability maps of the mode $m = 1$ for Keplerian profile $\mu = 0.35$ are presented. Stability maps for $Pm < 10^{-2}$ are not computed since the flow becomes unstable for too high $Re_{c}$ (see Fig. 2.2a.), which are very hard to access for asymptotic regimes. The instability islands are somewhat more narrow on the left side than for $\mu = 0.26$ (Fig. 2.4b), however, the slope of the curve bounding the left border is seemingly the same. The right side of the instability islands is identical for the two rotation rates, and lines of the maximum growth rate coincide. Power laws
fitted into the maximum growth rate lines as functions of $Re(Ha)$ can be found in Appendix B.

2.3 Widening of instability region

The critical AMRI mode becomes unstable only after $Re$ exceeds a critical threshold and remains unstable thereafter only in a certain range of $Ha$, i.e. for neither too weak nor too strong magnetic fields (Hollerbach et al., 2010). Figures 2.4, 2.2 show that at high $Re$ this range widens and the stability borders follow asymptotically power law scalings of the form $Re \propto Ha^\delta$, where $\delta$ is a scaling exponent. In fact, $\delta$ defines the steepness of stability curve on the left and on the right and may depend both on $Pm$ and $\mu$. The aim of this section is to define $\delta$ for the instability boundary between stable and unstable flow. In the following subscript $L$ will denote the left border, and subscript $R$ - the right border.

Fig. 2.5a illustrate the features of high and low $Pm$ with stability curves for $\mu = 0.26$ at $Pm = 1.4 \cdot 10^{-6}$ and 1, respectively. Left border of $Pm = 1.4 \cdot 10^{-6}$ is much steeper than that of $Pm = 1$. Overall, the left border scaling exponent (triangles in the Fig. 2.5b) decreases with increasing $Pm$. A cross-over in range $10^{-4} - 10^{-2}$ separates low and high-$Pm$ behavior. Our data indicates that $\delta_L \to 4$ as $Pm \to 0$ and $\delta_L \to 1.5$ as $Pm \to 1$. The right stability border shown as solid black curves in Fig. 2.5a scales as $Re \propto Ha$ for $Pm = 1.4 \cdot 10^{-6}$. This corresponds to linear scaling with $\delta_R = 1$. For $Pm = 1$ the right stability border scales with $\delta_R = 0.9$. Nevertheless, the scaling of $Re \propto Ha$ exists for all $Pm$, but for $Pm \geq 10^{-1}$ another instability mode emerges (see instability V in Fig. 2.2e-f), which extends the unstable region with a lower scaling exponent $\delta_R \approx 0.9$. This mode is characterized in particular by a discontinuous jump of the axial wave number while crossing the line of $Re \propto Ha$, as shown in section 2.2. Figure 2.5b shows that in fact the exponent $\delta = 1$ is independent of $Pm$.

The red dots in Fig. 2.5b represent data for instability maps $\mu = 0.35$ from linear analysis (see Fig. 2.4a). At least for $Pm \geq 0.01$ the scaling exponents are rather insensitive to rotation profile.

Having defined both $\delta_L$ and $\delta_R$, we can define the widening of the instability island as distance between the minimum and maximum $Ha$ number that allows for instability $\Delta Ha = Ha_R - Ha_L$ and study its dependence on $Re$ and $Pm$. Left and right scalings for certain $Ha_L$ and $Ha_R$ result in same $Re$:

$$\begin{align*}
\left\{ \begin{array}{l}
Re \propto Ha_R^{\delta_R} \\
Re \propto Ha_L^{\delta_L}
\end{array} \right.,
\end{align*}$$

which can be re-written as

$$\begin{align*}
\left\{ \begin{array}{l}
Re^{1/\delta_R} \propto Ha_R \\
Re^{1/\delta_L} \propto Ha_L
\end{array} \right..\end{align*}$$

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Fig. 2.2: Stability maps for $\mu = 0.26$ and different $P_m$: (a) $P_m = 1.4 \cdot 10^{-6}$; (b) $P_m = 10^{-4}$; (c) $P_m = 10^{-3}$; (d) $P_m = 10^{-2}$; (e) $P_m = 10^{-1}$; (f) $P_m = 1$. Green circles correspond to the lines of maximum growth rate of perturbations, along colored solid lines axial wave number undergoes discontinuous jump.
Fig. 2.3.: Growth rates and axial wavenumbers taken at \( Re = \text{const} \): (a) \( Re \approx 6.3 \cdot 10^4, Pm = 1.4 \cdot 10^{-6} \), (b) \( Re \approx 3.2 \cdot 10^3, Pm = 1 \). Discontinuous jump in \( k \) is accompanied by the change in the slope of the growth rate. The blue dashed line corresponds to the neutral curve with \( \sigma = 0 \) separating linearly stable and unstable regimes.

Fig. 2.4.: (a) Stability maps for Keplerian rotation \( \mu = 0.35, Pm = 1 \) (black), \( 10^{-1} \) (green), \( 10^{-2} \) (red). (b) Comparison of stability maps for \( \mu = 0.26 \) (solid) and \( \mu = 0.35 \) (dashed) for \( Pm = 1 \).
Fig. 2.5: (a) Instability islands for $\mu = 0.26$ at $Pm = 1$ and $1.4 \cdot 10^{-6}$. The blue line corresponds to a widening of the instability island because of the emergence of a new mode at high $Pm \geq 10^{-1}$ (instability V in Figure 2.4e-f). (b) $Pm$-dependence of the asymptotic scaling exponents $\delta$ of the left and right sides of the stability curves defining the instability islands. The steepness of the left border decreases with $Pm$, while the right border the scaling of $Re \propto Ha$ holds for all $Pm$, with an extension of $\delta = 0.9$ for $Pm \geq 10^{-1}$ because of the new instability mode.

Subtracting the second equation of (2.6) from the first we get

$$\Delta Ha \propto Re^{1/\delta_R} - Re^{1/\delta_L} = Re^{1/\delta_R} \left(1 - Re^{\delta_R - \delta_L/\delta_R\delta_L} \right).$$

(2.7)

This results in

$$\Delta Ha \propto Re(1 - Re^{1/4}) = Re(1 - Re^{-3/4}).$$

(2.8)

for $Pm \to 0$ and

$$\Delta Ha \propto Re^{10/9}(1 - Re^{10/15 - 10/9}) = Re^{10/9}(1 - Re^{-4/9}).$$

(2.9)

for $Pm \to 1$.

In the limit of high Reynolds numbers $Re \to \infty$ equation (2.7) suggests that the widening of the island is defined by the right border scaling as $\Delta Ha \propto Re^{1/\delta_R}$. Therefore, for high $Pm$ with $\delta_R = 0.9$ the islands widens faster than for low $Pm$. However, the weaker scaling of left border $\delta_L = 1.5$ for high $Pm$ means that stronger magnetic fields are required to drive AMRI. A stronger scaling of the left border is preferable since weak magnetic fields are expected to operate in accretion disks.

2.4 Mechanisms of angular momentum transport: linear approximation

The last section of this chapter takes a closer look at instability transport properties in the linear approximation for $\mu = 0.26$ (close to Rayleigh line) and $\mu = 0.35$.
(Keplerian rotation). As a first step, I study the critical Reynolds number $Re_c$ from Fig. 2.1, but already as a function of $Pm$. Figure 2.6 shows the dependence of $Re_c$ as a function of $Pm$ for $\mu = 0.35$ (solid) and $\mu = 0.26$ (dashed curve). At large $Pm > 1$ the two curves collapse, indicating that the AMRI becomes insensitive to rotation profile. In fact for $Pm \gtrsim 10$ the AMRI occurs at lower $Re$ than the Rayleigh (centrifugal) instability at all $\mu$. As $Pm$ decreases the two curves gradually depart from each other. For quasi-Keplerian rotation ($\mu = 0.35$) and $Pm < 0.1$, $Re_c$ grows inversely proportional to $Pm$, and so the onset of instability occurs at a constant magnetic Reynolds number $Rm_c = Re_c Pm \approx 50$. By contrast, for $\mu = 0.26$ the onset of instability occurs at a constant hydrodynamic Reynolds number $Re_c \approx 1250$. Although the flows at the two rotation rates will most likely share similar features for $Pm \gtrsim 10^{-1}$, they can behave very differently when $Pm$ is small.

One of the most important properties for our problem is the angular momentum transport. In turbulent magnetohydrodynamic flows both fluctuation of velocity and magnetic field can contribute to it. The Reynolds stress component averaged over $\phi$ and $z$ is defined as $\langle v_r v_\phi \rangle$ and the Maxwell stress component as $(Ha^2/Pm)\langle B_r B_\phi \rangle$. Both of them contribute to the angular momentum transport in radial direction. For an extended definition and derivation see section 5.1.1.

To shed more light on the difference in behavior in the $Pm \to 0$ limit, I analyzed the Maxwell and Reynolds stresses for the critical eigenmodes, close to the onset of instability at $Re_c$. The radial distributions of stresses for $\mu = 0.26$ and $\mu = 0.35$ at low $Pm$ are shown in the first and second row of Fig. 2.7, respectively. For $\mu = 0.26$ Maxwell stress is much smaller than Reynolds stress at the onset, while for $\mu = 0.35$ Maxwell stress is of the same magnitude with Reynolds stress. The ratio of Maxwell to Reynolds stresses, each integrated over the radius $r$, for $\mu = 0.35$ is about 1.5 for both $Pm = 10^{-2}$ and $10^{-4}$, whereas for $\mu = 0.26$ it decreases from 0.018 to 0.00027 as $Pm$ is reduced from $10^{-2}$ to $10^{-4}$. In fact, while all the four cases have different
Fig. 2.7.: Reynolds (solid) and Maxwell stresses (dashed) of the eigenmodes near the onset of instability for rotation close to the Rayleigh line (first row) and quasi-Keplerian rotation (second row). The Reynolds numbers are (a) $Re = 840$ ($Rm = 8.4$), (b) $Re = 1300$ ($Rm = 0.13$), (c) $Re = 5000$ ($Rm = 50$), (d) $Re = 5 \cdot 10^5$ ($Rm = 50$).

sets of $Re$, $Pm$ and $\mu$, low-Maxwell stress flows in Fig. 2.7a-b possess relatively low $Rm < 10$, and high-Maxwell stress flows in Fig. 2.7c-d have $Rm_c \approx 50$. This suggests that the relative importance of stresses is essentially set by the magnetic Reynolds number.

Overall, the linear analysis of this section highlights the markedly different character of the instability at low $Pm$ for rotation near the Rayleigh line, with $Re_c \approx 1250$ and quasi-Keplerian rotation, with $Rm_c \approx 50$. According to the stress distribution, at low $Pm$ close to Rayleigh line the AMRI behaves as a magnetically destabilized inertial wave with transport via Reynolds stresses, while quasi-Keplerian rotation shows transport supported both by Maxwell and Reynolds stresses resembling more an unstable magnetocoriolis wave, where magnetic and Coriolis forces are in balance.

2.5 Discussion

In this chapter linear stability analysis of magnetorotational instability in the presence of magnetic fields has been revisited. The onset of instability occurs at $Re_c \propto Pm^{-1}$, except for low $Pm \leq 10^{-3}$, where $Re_c$ becomes independent of $Pm$. In the limit of $Pm \to 0$ only a small range of rotation profiles $\mu \in [0.25, 0.3]$ is accessible, otherwise $Re_c$ becomes too large. For $Pm \geq 1$ $Re_c$ becomes independent of rotation rate. The
data suggest that AMRI originates from the hydrodynamic \( m = 1 \) mode, since the transition from hydrodynamic to magnetohydrodynamic mode is continuous and all modes have similar shape (Fig. 2.1b-c). However, the mechanisms of angular momentum transport are different for MRI and hydrodynamic instability. While linear eigenmodes of MRI exhibit strong growth of magnetic field under certain conditions \( (Rm \sim O(50), \text{Fig. 2.7c-d}) \), Reynolds stresses are the only mechanism of transport for hydrodynamically unstable modes. The transition between the MRI and hydrodynamic instability is out of scope for this thesis and remains a subject of future research.

With linear stability analysis I defined the scaling of the boundaries confining the AMRI in the form of \( Re \propto Ha^\delta \). For \( Pm \leq 10^{-3} \) the right border scales linearly with \( Ha \) \( (\delta_R \approx 1) \), while for \( Pm \in (0.1, 1) \) this scaling changes to \( \delta_R \approx 0.9 \). The line \( Re \sim Ha \) remains as a line of discontinuous jump in the axial wavenumber. If \( Re \to \infty \), the right border scaling coefficient defines the widening of the instability island \( \Delta Ha \propto Re^{1/\delta_R} \). The scaling coefficient of the left instability boundary decreases from \( \delta_L \approx 4 \) (low \( Pm \)) to \( \delta_L \approx 1.5 \) (high \( Pm \)). This implies constraints on the applicability of AMRI for high \( Pm \), as much stronger magnetic fields are needed to destabilize the flow.

Finally, the stability maps for AMRI in a wide range of \( Pm \in [10^{-6}, 1] \) were constructed (Fig. 2.2, 2.4). The axial wave number changes nonmonotonously inside the instability islands, and the parameters of its discontinuous jump were recorded. This jump shows the existence of additional instability modes defining the instability domain. The types of modes and their regions of existence are different for low and high \( Pm \). With the parameter paths of maximum growth rate defined, everything is ready for nonlinear simulations.
Periodicity of Taylor-Couette system in axial and azimuthal direction allows us to use a robust pseudo-spectral method for solving partial differential equations (1.38)-(1.39). Spectral methods approximate solution of equations using global base functions that are non-zero in the whole domain. Finite difference and finite volume methods approximate solution with local polynomials that are non-zero only in small sub-domains on stencils of certain size (usually 3 – 9 points). Spectral methods demonstrate faster convergence in comparison to local approach methods. For example, finite difference method of the 4th order results in error decreasing as $N^{-4}$ with grid size, while typical convergence rate for spectral methods is $O(N^{-m})$ for any $m$ (so-called “exponential convergence”). This remarkable behavior of spectral methods is related to the differences in how local and global approximations exchange information between discretization points. Imagine a discrete computational domain with discretized on it variable (for example, velocity field $v$). At some moment the value of $v$ at the point $i$ changes. In local approximation, only the points within the stencil length $[i-4, i+4]$ will receive the information about this change. Therefore, to increase accuracy of the method one needs to increase the width of the stencil or the resolution $N$. This results in noticeable growth of calculation cost. On the contrary, if global approximation is involved, the information is communicated immediately along all points in the domain and convergence is faster even for lower resolutions. Spectral methods are better suited for rapidly changing processes. Note that spectral methods are most efficient in periodic domains. In bounded domains they often suffer stability problems. Further details can be found, for example, in (Canuto et al., 2012).

The algorithm that is used for all nonlinear simulations in this work, can shortly be described in following steps:

- Based on the previous time step calculate nonlinear terms of equations (1.38)-(1.39)
- Predict pressure and velocity field using pressure-Poisson equation
- Predict magnetic field
- Correct ensuring both divergence-free and insulating boundary conditions.

The (pseudo)spectral nature of the method appears in the discretization of variables. In radial direction with solid boundaries variables are discretized using finite differences of high order, and in periodic azimuthal and axial directions variables are transformed to Fourier space and expanded with series of exponents.
exp[i(k_0k'z + m_0m'ϕ)]. All linear operations (Laplacian, gradients, etc.) are performed in Fourier space, and nonlinear terms are calculated in physical space. Fast Fourier Transform allows fast transformation of variables between physical and spectral space, with computational complexity of only \(O(M \star K \ln(M \star K))\), where \(M\) and \(K\) are the numbers of Fourier modes. The method was implemented in a FORTRAN code by Dr. Ashley P. Willis, and validated and extended to include externally imposed azimuthal fields by myself. In the following sections of this chapter the details of the method and its implementation will be revealed.

### 3.1 On the pressure boundary conditions for Navier-Stokes equation

Hydrodynamic Navier–Stokes equations

\[
(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla p + \nabla^2 \mathbf{v}, \quad (3.1)
\]

\[
\nabla \cdot \mathbf{v} = 0 \quad (3.2)
\]

are presented by four equations (three momentum and one mass conservation) with four variables: three components of velocity \(\mathbf{v}\) and pressure \(p\). There is no explicit equation for pressure, such as, for example, equation of state in compressible flow problems. Strictly speaking, \(p\) is not a physical (thermodynamical) variable here and can be defined only up to a constant. However, pressure gradient \(\nabla p\) is physical and represents the forcing term (without pressure gradient there is no movement). Technically, the pressure field should be found so that the condition \(\nabla \cdot \mathbf{v} = 0\) is satisfied in the bulk of the flow and at the boundaries as well. In this sense, divergence-free condition can be considered as the “true” equation for pressure. Thus, equations in the system (3.1)-(3.2) are coupled. This coupling, together with nonlinearity in \(\mathbf{v} \cdot \nabla \mathbf{v}\), is the main difficulty in solving the Navier–Stokes problem. In the following I will briefly discuss two common methods to solve Navier–Stokes equations (3.1): the projection method and the construction of Pressure-Poisson equation (PPE).

#### 3.1.1 Projection method

Projection method gives only approximation to the exact solution of Navier–Stokes equations. Given initial field \(\mathbf{v}_0\), let us decompose one time step \(t^{n+1}\) in two sub-steps. First, omitting pressure gradient, calculate an estimate of velocity \(\mathbf{v}^*\):

\[
\frac{\mathbf{v}^* - \mathbf{v}^n}{\Delta t} = -\mathbf{v}^n \cdot \nabla \mathbf{v}^n + \nabla^2 \mathbf{v}^n, \quad (3.3)
\]

\(\mathbf{v}^*\) is not divergence free. Then correct \(\mathbf{v}^*\). Consider

\[
\frac{\mathbf{v}^{n+1} - \mathbf{v}^*}{\Delta t} = -\nabla p^{n+1}. \quad (3.4)
\]
Equations (3.3) and (3.4) together give an approximation to the momentum equation (3.1). We should find \( p^{n+1} \) such that \( \nabla \cdot \mathbf{v}^{n+1} = 0 \). Applying the divergence operator \( \nabla \cdot \) to equation (3.4) we get
\[
\nabla^2 p^{n+1} = \frac{\nabla \cdot \mathbf{v}^*}{\Delta t},
\]
(3.5)
or reformulating (3.5) with \( \varphi = p^{n+1} \Delta t \),
\[
\nabla^2 \varphi = \nabla \cdot \mathbf{v}^*.
\]
(3.6)
Note that \( \varphi \) is not pressure and depends on the time step size. To obtain boundary conditions for \( \varphi \), let us consider equation (3.4) at the boundary \( \Gamma \):
\[
\mathbf{v}^{n+1}|_{\Gamma} - \mathbf{v}^*|_{\Gamma} = -\nabla (p^{n+1} \Delta t)|_{\Gamma} = -\nabla \varphi|_{\Gamma}.
\]
(3.7)
In the case of no-slip boundary conditions on velocity on the wall we get
\[
\nabla \varphi|_{\Gamma} = \frac{\partial \varphi}{\partial n} = 0.
\]
(3.8)
Summarizing the method:
1. Calculate an estimate of velocity via (3.3)
2. Solve the equation for \( \varphi \) (3.6) with boundary conditions (3.7)
3. Correct velocity as \( \mathbf{v}^{n+1} = \mathbf{v}^* - \nabla \varphi \).

When needed, pressure can be obtained from \( p = \varphi / \Delta t \).

The projection method projects an estimation for velocity on the space of solenoidal fields and thus is an approximate technique for solving Navier–Stokes equations.

3.1.2 Formulation of Pressure–Poisson equation
Instead of separating (3.3) and (3.4), as we did in previous section, we can construct an extra equation for pressure using divergence-free condition. Taking divergence of (3.1) we get
\[
\nabla \cdot \partial_t \mathbf{v} + \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = -\nabla^2 p + \nabla \cdot \nabla^2 \mathbf{v}.
\]
(3.9)
Commuting operators \( \nabla \cdot \), \( \partial_t \) and requiring \( \nabla \cdot \mathbf{v} = 0 \) we get a Poisson equation for pressure:
\[
\nabla^2 p = -\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}):
\]
(3.10)
PPE itself does not guarantee that the solution of Navier–Stokes equations will be divergence free. Indeed, subtracting (3.10) from (3.9) we see that the divergence of the velocity field satisfies a heat equation:
\[
\partial_t(\nabla \cdot \mathbf{v}) = \nabla^2(\nabla \cdot \mathbf{v}).
\]
(3.11)
If our problem is well-posed and initial field is divergence-free ($\nabla v_0 = 0$), divergence of velocity is zero inside the computational domain if and only if it is held zero on the boundaries. We can consider PPE in the following form:

$$\nabla^2 p = -\nabla \cdot (v \cdot \nabla v) + \nabla \cdot \nabla^2 v,$$

which subsequently leads to

$$\partial_t (\nabla \cdot v) = 0.$$  \hfill (3.13)

Now the time derivative of divergence stays constant in time regardless of its value on the boundary. However, if the initial divergence is not equal zero ($\nabla \cdot v_0 \neq 0$), it will be preserved for all times.

One may conclude that starting from a solenoidal velocity field, a divergence-free solution is always obtained. Gresho and Sani (1987) argued that "any divergence-free velocity field induces a pressure field, which, in its turn, assures that the velocity field remains divergence-free". However, Rempfer (2006) showed that in general case its is not true. Improper boundary conditions on PPE may lead to discontinuous $\nabla \cdot v$ for $t > t_0$. If so, from (3.13) we conclude that $\nabla \cdot v$ will stay non-zero and constant in time. Velocity fields with $\nabla \cdot v \neq 0$ for incompressible problems are unphysical, do not satisfy Navier–Stokes equations (3.1)-(3.2) and therefore can not be its solution. Thus imposing proper boundary conditions for PPE (3.10) or (3.12) becomes an important issue in order to eliminate divergence on the boundary and get correct solution.

Gresho and Sani (1987), discussing pressure boundary conditions, suggested that since (3.10) (or (3.12)) is a derived equation, then boundary condition for it should be also derived. They proposed a wall-normal component of momentum conservation equation as a relevant boundary condition for pressure:

$$\frac{\partial P}{\partial n} = \nabla^2 v_n - (\partial_t v_n + v \cdot \nabla v_n).$$  \hfill (3.14)

This boundary condition is usually referred to as Neumann boundary condition for pressure. Rempfer (2006) pointed out the ill-posedness of (3.1)-(3.12)-(3.14) problem. While deriving (3.14) we did not introduce any new information that was not already contained in (3.1), while the required additional information is indeed $(\nabla \cdot v)|_\Gamma = 0$. To obtain the correct boundary condition for pressure, we need to relate it with divergence-free constraint. Rempfer (2006) showed that this relation is linear and argued for the use of influence matrix technique to establish it.
3.2 Numerical method

3.2.1 Velocity field

In the numerical simulations only the deviation from the basic flow \( u = v - V \) is computed. Its governing equations read

\[
(\partial_t - \nabla^2) u = Nl - \nabla p, \quad \nabla \cdot u = 0,
\]

which are supplemented with homogeneous boundary conditions \( u = 0 \). Here \( Nl \) stands for the nonlinear term in the Navier-Stokes equations (1.38), which contains the advective terms and the Lorentz force:

\[
Nl = u \times (\nabla \times u) - (V \cdot \nabla) u - (u \cdot \nabla) V + \frac{Ha^2}{Pm} (\nabla \times B) \times B
\]

Imagine that the equation (3.15) was discretized in time. The known velocity at the previous time step is denoted as \( u^n \), velocity field at the current time step \( u^{n+1} \) is unknown, and nonlinear term \( Nl^{n+\frac{1}{2}} \) is predicted based on the known velocity \( u^n \). The time-discretized equation takes the form

\[
\begin{cases}
Xu^{n+1} = Y u^n + Nl^{n+\frac{1}{2}} - \nabla p, \\
\nabla^2 \bar{p} = \nabla \cdot (Y u^n + Nl^{n+\frac{1}{2}}), \\
\nabla \cdot u^{n+1} = 0, \\
u^{n+1} = 0 \text{ on } r = \{r_i, r_o\}.
\end{cases}
\]

The Poisson equation for pressure was obtained taking the divergence of (3.15). Unfortunately, incompressible flows do not provide explicit boundary condition for pressure. Any approximate boundary condition for pressure (i.e. Neumann boundary conditions \( \partial_r p = 0 \)) will introduce non-zero divergence on the boundaries. To avoid that, we use the influence matrix method developed by Kleiser and Schumann (Kleiser and Schumann, 1980; Canuto et al., 2007). Kleiser and Schumann called the problem 3.17 the \( A\)-problem. The solution of the \( A\)-problem can be expressed as a linear combination of the solution of three auxiliary systems \( B\)-problems: (1) the inhomogeneous differential equation with homogeneous boundary conditions:

\[
\begin{cases}
X \bar{u} = Y u^n + Nl^{n+\frac{1}{2}} - \nabla \bar{p}, \\
\nabla^2 \bar{p} = \nabla \cdot (Y u^n + Nl^{n+\frac{1}{2}}), \\
\bar{u} = 0 \text{ on } r = \{r_i, r_o\}, \\
\partial_r \bar{p} = 0 \text{ on } r = \{r_i, r_o\}.
\end{cases}
\]
(2) the homogeneous differential equation with unit pressure gradient on the radial boundaries:

\[
\begin{cases}
Xu' = -\nabla p', \\
\nabla^2 p' = 0, \\
u' = 0 \text{ on } r = \{r_i, r_o\}, \\
\partial_r p' = \{0, 1\} \text{ on } r = \{r_i, r_o\},
\end{cases}
\]

(3.19)

and (3) the homogeneous differential equation with unit velocity on the respective radial boundaries:

\[
\begin{cases}
Xu' = 0, \\
u'_+ = \{0, 1\}, \quad u'_- = \{0, 1\}, \quad u'_z = \{0, i\} \text{ on } \{r_i, r_o\}.
\end{cases}
\]

(3.20)

The boundary conditions were chosen so that $u'_\pm$ are purely real and $u'_z$ is purely imaginary. The desired divergence-free velocity field is constructed as a linear combination of the solutions to these linear systems:

\[
u^{n+1} = \bar{u} + \sum_{j=1}^{8} h_j u'_j
\]

(3.21)

The expression (3.21) is substituted in the original set of boundary conditions $\nabla \cdot u^{n+1} = 0, \quad u^{n+1} = 0 \text{ on } r = \{r_i, r_o\}$. Let $g$ be the generalized form of the boundary conditions:

\[
\begin{bmatrix}
\nabla \cdot u^{n+1}|_{r_i,o} = 0 \\
u_+^{n+1}|_{r_i,o} = 0 \\
u_-^{n+1}|_{r_i,o} = 0 \\
u_z^{n+1}|_{r_i,o} = 0
\end{bmatrix}
\]

(3.22)

then after substitution

\[
g(u^{n+1}) = g(\bar{u}) + \sum_{j=1}^{8} h_j g(u'_j) = 0,
\]

(3.23)

or in a matrix form

\[
I h = -g(\bar{u}),
\]

(3.24)

where $I = I(g(u'))$ is an $8 \times 8$ influence matrix. This matrix and its inverse are real due to the choice of boundary conditions for auxiliary systems and can be precomputed during the pre-processing step. Then at each time step the following algorithm is executed:

1. The first $B$-problem (3.18) is solved with homogeneous boundary conditions and an estimation $\bar{u}$ to the desired divergence-free field is obtained

2. The correction coefficients $h$ are recovered from (3.24) as $h = -I^{-1}g(\bar{u})$
3. The “true” velocity is calculated from (3.21).

For each time step, the influence matrix technique requires only evaluation of the deviation from the boundary condition, multiplication by an $8 \times 8$ real matrix, and addition of eight purely real or imaginary functions to each component of $u$. The computational overhead is negligible if compared to evaluation of nonlinear terms. The advantage of this technique is that it keeps the error in the boundary conditions $g(u^{n+1})$ at the level of machine precision.

3.2.2 Magnetic field

In analogy to the Navier–Stokes equations (3.17), the discretized induction equations read as

$$
\begin{align*}
X B^{n+1} &= Y B^n + N I^{n+\frac{1}{2}}, \\
\nabla \cdot B^{n+1} &= 0.
\end{align*}
$$

(3.25)

Unlike in the equations for velocity field (3.17), for magnetic field there is no pressure term and thus the zero divergence in the interior of the computational domain is not imposed by construction. This can lead to accumulation of divergence from artificial internal sources, i.e. discretization error. To prevent this, a divergence-cleaning step is applied via introducing an artificial pressure $\Pi$ (Brackbill and Barnes, 1980):

$$
\nabla^2 \Pi = \nabla \cdot B.
$$

(3.26)

The system (3.25) can be rewritten as

$$
\begin{align*}
X \bar{B} &= Y B^n + N I^{n+\frac{1}{2}} - \nabla \Pi, \\
\nabla^2 \Pi &= \nabla \cdot (Y B^n + N I^{n+\frac{1}{2}}), \\
\bar{B} &= B^n \text{ on } r = \{r_i, r_o\}, \\
\partial_r \Pi &= 0 \text{ on } r = \{r_i, r_o\},
\end{align*}
$$

(3.27)

which gives us the first auxiliary $B$-problem for the system (3.25). The boundary condition for $\Pi$ is any choice such that for $\nabla \cdot B = 0$ the function $\Pi$ is a constant; $B$ is then unaltered by the projection (Ramshaw, 1983). On the boundaries $r_{i,o}$ the values of magnetic field from the previous time step are imposed as approximation to the real boundary conditions.

Boundary conditions for magnetic field (1.47)–(1.48) are given in the form of relations between different magnetic field components and can not be applied explicitly. At the same time it is also necessary to keep zero divergence of magnetic field at the boundaries. This problem can be solved again with the use of influence matrix method. The boundary conditions (1.47)–(1.48) give two conditions for magnetic field to be satisfied, the requirement $\nabla \cdot B^{n+1} = 0$ gives another one,
which on the two radii \( r_i \) and \( r_o \) results in the need of six additional functions to correct magnetic field given by (3.27). Consider auxiliary \( B \)-problem:

\[
\begin{align*}
\chi \mathbf{B}' &= 0, \\
B'_+ &= \{0, 1\}, \quad B'_- = \{0, 1\}, \quad B'_z = \{0, i\} \text{ on } \{r_i, r_o\}.
\end{align*}
\] (3.28)

Solution to this problem with unit boundary conditions gives required additional functions. Thus magnetic filed, similarly to velocity field, can be represented as

\[
g(\mathbf{B}^{q+1}) = g(\bar{\mathbf{B}}) + \sum_{j=1}^{6} h_j g(\mathbf{B}'_j) = 0.
\] (3.29)

The procedure is similar to the one applied to the velocity field. Again, the influence matrix technique ensures that the error in boundary condition stays at the level of machine precision.

### 3.2.3 Decoupling the equations

The governing equations (1.38), (1.39) include Laplacian of a vector \( \nabla^2 \mathbf{A} \). Its separate components read as

\[
\begin{align*}
(\nabla^2 \mathbf{A})_r &= \nabla^2 A_r - \frac{2}{r^2} \partial_\phi A_\phi - \frac{A_r}{r^2}, \\
(\nabla^2 \mathbf{A})_\phi &= \nabla^2 A_\phi + \frac{2}{r^2} \partial_\phi A_r - \frac{A_\phi}{r^2}, \\
(\nabla^2 \mathbf{A})_z &= \nabla^2 A_z,
\end{align*}
\] (3.30)

where \( \nabla^2 \) is Laplacian of a scalar function in cylindrical coordinates.

\[
\nabla^2 = \left( \frac{1}{r} \partial_r + \partial_{rr} \right) + \frac{1}{r^2} \partial_{\phi\phi} + \partial_{zz}.
\] (3.31)

This means that equations for \( r \)- and \( \phi \)-components of velocity and magnetic fields are coupled in the Laplacian. To decouple the equations, the following transformation is performed:

\[
A_\pm = A_r \pm i A_\phi,
\] (3.32)

for which the \( \pm \) are considered respectively. Original variables are easily recovered:

\[
A_r = \frac{1}{2}(A_+ + A_-), \quad A_\phi = -\frac{i}{2}(A_+ - A_-).
\] (3.33)

Governing equations for the velocity field are then

\[
\begin{align*}
(\partial_t - \nabla^2) u_\pm &= Nl_\pm - (\nabla p)_\pm, \\
(\partial_t - \nabla^2) u_z &= Nl_z - (\nabla p)_z,
\end{align*}
\] (3.34) (3.35)
where
\[ \nabla_\pm^2 = \nabla^2 - \frac{1}{r^2} \pm \frac{i}{r^2} \partial_r. \]  
(3.36)

The induction equation can be decoupled in the same way.

### 3.2.4 Temporal discretization

Time discretization is based on predictor-corrector algorithm. Consider the model equation:
\[ (\partial_t - \nabla^2) f = N_p, \]  
(3.37)

where pressure gradient and nonlinear terms were absorbed in \(N_p\). The predictor at the time \(t_n\), with Euler nonlinear terms and implicitness \(c\) is
\[ \frac{f_{n+1} - f^n}{\Delta t} - \left( c\nabla^2 f_{n+1}^1 + (1 - c)\nabla^2 f^n \right) = N_p^n, \]  
(3.38)

\[ \left( \frac{1}{\Delta t} - c\nabla^2 \right) f_{n+1}^1 = \left( \frac{1}{\Delta t} + (1 - c)\nabla^2 \right) f^n + N_p^n. \]  
(3.39)

Corrector iterations are based on Crank-Nicolson method:
\[ \left( \frac{1}{\Delta t} - c\nabla^2 \right) f_{n+1}^j = \left( \frac{1}{\Delta t} + (1 - c)\nabla^2 \right) f^n + c N_p^{n+1} + (1 - c) N_p^n, \]  
(3.40)

or equivalently, for the correction \(f_{corr} = f_{n+1}^j - f_{n+1}^j\)
\[ \left( \frac{1}{\Delta t} - c\nabla^2 \right) f_{corr} = c N_j^{n+1} - c N_{j-1}^{n+1}, \]  
(3.41)

where \(j = 1, 2, \ldots\) is the number of correction iteration and zero iteration is given by the predictor \(N_{p0}^{n+1} = N_p^n\). The size of the correction \(\|f_{corr}\|\) must reduce at each iteration. For \(c = \frac{1}{2}\) the scheme is second order such that \(\|f_{corr}\| \sim \Delta t^2\).

The time step size is controlled by the Courant number \(C\)
\[ \Delta t = C \min(\Delta / |u|), \quad 0 < C < 1 \]  
(3.42)

and can be adjusted automatically during the simulation.

### 3.2.5 Spatial representation

The system has two periodic directions: azimuthal and axial. In these directions the Fourier expansion for each variable is performed:
\[ A = \sum_{|k'|<K/2} \sum_{|m'|<M/2} A_{k',m'}(r) \exp[i(k_0 k' z + m_0 m' \phi)]. \]  
(3.43)

Since \(A\) represent velocities, or magnetic field components, it needs to be kept real. This implies that coefficients \(A_{k',m'}\) should be conjugate-symmetric: \(A_{k',m'} = \)
$A_{k',-m'}$. It is sufficient to keep only $m' \geq 0$, and for $m' = 0$ keep $k' \geq 0$. The real wave numbers therefore are

$$k = k_0 k', \quad m = m_0 m',$$

and each mode $A_{k'm'}$ can also be denoted as $A_{km}$. Parameters $k_0, m_0$ define the minimum wave number and hence the size of computational domain in $z$ and $\phi$ direction.

In the radial direction finite difference method was used. Consider Taylor expansions about central point $x_0$, $i$ neighbouring points each side:

$$f(x_{-k}) = f(x_0) + (x_{-k} - x_0) f'(x_0) + \frac{(x_{-k} - x_0)^2}{2!} f''(x_0) + \ldots$$

$$f(x_k) = f(x_0) + (x_k - x_0) f'(x_0) + \frac{(x_k - x_0)^2}{2!} f''(x_0) + \ldots$$

The expansions (3.45) can be written

$$f = D \text{df}, \quad \text{df} = D^{-1} f, \quad f = \begin{bmatrix} f(x_{-k}) \\ \vdots \\ f(x_k) \end{bmatrix}, \quad \text{df} = \begin{bmatrix} f(x_0) \\ f'(x_0) \\ f''(x_0) \\ \vdots \end{bmatrix}$$

Derivatives are calculated using weights from the appropriate row of $D^{-1}$. A 9-point stencil was typically used. The full resolution of each simulation is represented by $N \times K \times M$, where $N$ is the number of radial points, $K$ and $M$ - number of axial and azimuthal modes, respectively.

### 3.2.6 Implementation

The Taylor-Couette flow code was written in Fortran90. Nonlinear terms are evaluated using the pseudo-spectral method and are de-aliased using the 3/2 rule. The Fourier transforms are performed with the FFTW3 library (Frigo and Johnson, 2005) and matrix and vector operations are performed with BLAS (Lawson et al., 1979). Each predictor-corrector iteration involves the solution of banded linear systems with forward-backward substitution using banded LU-factorizations that are precomputed prior to time-stepping. These operations are performed with LAPACK (Anderson et al., 1999). The code was parallelized so that data is split over the Fourier harmonics for the linear parts of the code: evaluating curls, gradients and matrix inversions for the time-stepping (these linear operations do not couple modes). Here all radial points for a particular mode are located on the same processor; separate modes may be located on separate processors. Data is split radially when calculating Fourier transforms and when evaluating products in real space (nonlinear term of the equations). The bulk of communication between processors occurs during the data transposes.
\section*{3.3 Numerical validation}

The code was validated against several published linear stability results, as well as three-dimensional nonlinear simulations of the coupled induction and Navier-Stokes equations. We tested the inductionless limit $Pm = 0$ and finite $Pm$, obtaining excellent agreement in all cases (see below).

\subsection*{3.3.1 Linear stability of the flow}

Linear instabilities were detected in the calculations by monitoring the kinetic energy of the deviation from circular Couette flow after introduction of a small disturbance. In the linear regime we write

$$u \sim \exp(i \omega t + i k z + i m \phi) , \quad E_{\text{kin}} \sim |u|^2 \sim \exp(2 \sigma t),$$

where $\omega = \gamma - i \sigma$ is a complex number; the real part $\gamma$ is the oscillation frequency and the imaginary part $\sigma$ the growth rate of the dominant perturbation. The latter is readily extracted from the relationship for kinetic energy of perturbed flow $\log(E_{\text{kin}}) \sim 2 \sigma t$. If $\sigma$ is positive, perturbation $u$ grows and flow is (linearly) unstable; if $\sigma$ is negative, perturbation $u$ decays and the flow is stable.

To extract the averaged energy of perturbation from velocity or magnetic field, we should take an integral over the whole domain. Since periodicity imposed in axial and azimuthal direction, a sum of Fourier modes amplitudes $A_{km}$ replaces the integral over $z$ and $\phi$, the integration is only in $r$ direction:

$$E = \frac{1}{2} \int A \cdot A \, dV = \frac{2\pi^2}{k_0} \sum_{km} \int |A_{km}|^2 r \, dr \quad (3.47)$$

At first I reproduced the classical results of Roberts (1964), who considered the inductionless limit $Pm = 0$ for narrow gap geometry $\eta = 0.95$ and stationary outer cylinder. For a Hartmann number of $Ha = 5.477$ he obtained a critical Reynolds number of $Re_c = 281.05$ with associated critical axial wavenumber of $k_c = 2.69$ and $m = 0$.

In the simulations I fixed $k_0 = k_c$ and obtained $Re_c = 281.055$ using $N = 33$ radial points. Flow parameters used for this test are: $Pm = 0$, only axial field imposed $B = B_0 e_z$, a narrow gap between cylinders $\eta = 0.95$, Hartmann number $Ha^2 = 30$, axial wavenumber $k_0 = 2\pi / L_z = 2.69$, which corresponds to relatively short domain length. The outer cylinder is stationary $\Omega_o = 0$; number of modes: axial $K = 16$, azimuthal $M = 16$; number of radial points $N = 33$. In the Table 3.1 and Fig. 3.1 the results of this test are presented, they are in agreement with Roberts (1964). Eigenvalues were extracted from Fig. 3.1.

In order to test the azimuthal magnetic field I reproduced the results of Hollerbach et al. (2010) for the AMRI. The results for several $Re$ and $Ha$ values are compared. Other parameters were fixed as follows: $\mu = 0.26$, $\eta = 0.5$ and a pure azimuthal mag-
Fig. 3.1.: Results for basic linear stability test at $Pm = 0$. Kinetic energy of the flow shows: (a) laminar flow at $Re = 100$ and instability at $Re = 300$; (b) zero growth rate at critical $Re_c \sim 281$. Energy and time are given in viscous units, as defined in 1.3.1.

Tab. 3.1.: Results of the linear stability test. Type of flow and eigenvalue depends on Reynolds number $Re$. Growth rates $\sigma$ are in viscous units.

<table>
<thead>
<tr>
<th>$Re$</th>
<th>100</th>
<th>281</th>
<th>300</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>below critical</td>
<td>slightly supercritical</td>
<td>supercritical</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>-22.848</td>
<td>0.022</td>
<td>0.13</td>
</tr>
<tr>
<td></td>
<td>Laminar flow, energy decreases with time</td>
<td>Near the critical point, eigenvalue is close to zero</td>
<td>Instability: instead of TC flow - vortices, energy grows. After $t=7$ nonlinear saturation begins</td>
</tr>
</tbody>
</table>

Tab. 3.2.: Results of the AMRI test. Comparison of the growth rates to Hollerbach et al. (2010).

<table>
<thead>
<tr>
<th>log(Ha)</th>
<th>log(Re)</th>
<th>Ha</th>
<th>$Re$</th>
<th>$k$</th>
<th>$\sigma$ from (Hollerbach et al., 2010)</th>
<th>$\sigma$ (our code)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.10</td>
<td>3.40</td>
<td>125</td>
<td>2511</td>
<td>3.51</td>
<td>13.1</td>
<td>12.6</td>
</tr>
<tr>
<td>2.50</td>
<td>3.00</td>
<td>316</td>
<td>$10^3$</td>
<td>7.17</td>
<td>-78.6</td>
<td>-78.7</td>
</tr>
<tr>
<td>2.60</td>
<td>4.00</td>
<td>398</td>
<td>$10^4$</td>
<td>5.14</td>
<td>269.2</td>
<td>267.7</td>
</tr>
<tr>
<td>3.00</td>
<td>4.00</td>
<td>$10^3$</td>
<td>$10^4$</td>
<td>7.17</td>
<td>-110.3</td>
<td>-110.9</td>
</tr>
</tbody>
</table>
For finite magnetic Prandtl number $Pm = 1$ I reproduced the results of Willis and Barenghi (2002b) for wide gap $\eta = 0.5$ and stationary outer cylinder. For $Ha = 39$, and $k = 2.4$ and $m = 0$ they found $Re_c = 60.5$. I took $N = 33$ radial points, $K = 16$ axial and $M = 16$ azimuthal modes and tried to reproduce this result with our code. The results are in good agreement with (Willis and Barenghi, 2002b): almost zero growth rate ($\sim 0.055$) is observed in Fig. 3.2.

![Fig. 3.2: Results of the test for finite $Pm$. Velocity field energy at the critical point $Re = 60.5$.](image)

### 3.3.2 Nonlinear simulations

Willis and Barenghi (2002a) explored dynamo action in Taylor-Couette flow. They first solved the Navier-Stokes equations in the absence of magnetic field and subsequently applied a small magnetic disturbance to test whether it grew into a dynamo. In the axisymmetric Taylor-vortex regime axisymmetric magnetic fields were found to decay, in accordance with Cowling’s anti-dynamo theorem. Non-axisymmetric magnetic fields may be excited. For $Re = 136.4$, $\eta = 0.5$, $Pm = 2$, $k = 1.57$ and stationary outer cylinder they observed that the magnetic disturbance grows for $m = 1$ ($\sigma_{B,m=1} \approx 0.2$, leading to dynamo action), whereas it decays for $m = 2$ ($\sigma_{B,m=2} \approx -1.4$). I reproduced this setting using $N = 41$, $K = 32$ and $M = 24$ and obtained $\sigma_{B,m=1} \approx 0.16$ and $\sigma_{B,m=1} \approx -1.42$, in good agreement with Willis and Barenghi (2002a).

Finally, I compared results of the axisymmetric HMRI (helical field $B = B_0(e_z + \beta e_\phi)$) obtained with the axisymmetric spectral code of Hollerbach (2008). A typical diagnostic quantity is the torque at the cylinders

$$G \sim -2\pi r^3 \frac{\partial}{\partial r} \left[ \frac{v_\phi}{r} \right] \sim 2\pi r^2 \left[ \frac{v_\phi}{r} - \partial_r v_\phi \right].$$ (3.48)

The laminar flow torque will be used as a scale, so that the dimensionless ratio $G/G_{\text{lam}}$ measures the intensity of angular momentum transfer relative to laminar flow. I choose the parameters $Re = 300$, $Ha = 10$, $\eta = 0.5$, $\gamma = 2$, $k_0 = 0.314$, which are well into the nonlinear regime. After nonlinear saturation the dimensionless
torque on the cylinders obtained with our code for $N = 81$ and $K = 192$ was $G/G_{\text{lam}} = 1.4122$, which is in excellent agreement with the code of Hollerbach ($G/G_{\text{lam}} = 1.4123$).
Onset of instability

Both Taylor–Couette geometry (Fig. 1.3a) and basic circular Couette flow (1.42) possess a high degree of symmetry, which inevitably influences resulting flow patterns. The basic circular Couette flow (1.42) has \( \text{SO}(2) \times \text{O}(2) \) symmetry, where \( \text{SO}(2) \) represents the rotational symmetry in the azimuthal direction. In the axial direction the group \( \text{O}(2) \) may be written as \( \text{O}(2) = Z_2 \rtimes \text{SO}(2) \), where \( Z_2 \) is a reflection (up-down symmetry) and \( \text{SO}(2) \) the translational symmetry in the \( z \) direction. The presence of purely axial or purely azimuthal imposed magnetic field does not change the symmetry group of the system. Hence the resulting states following a Hopf bifurcation can be either standing or traveling waves in the axial direction (Crawford and Knobloch, 1991). By contrast, a combined helical magnetic field breaks the reflection symmetry and only traveling waves (TW) can be observed (Knobloch, 1996). Finally, if the bifurcating solution is non-axisymmetric, as in the AMRI, this will generically be a rotating wave in the azimuthal direction.

From linear stability theory we know that the dominant eigenmode for AMRI is a spiral wave with azimuthal wavenumber \( m = 1 \) (Hollerbach et al., 2010). Indeed, in simulations of periodic cylinders, AMRI was found to be a \( m = \pm 1 \) spiral mode, which drifts together with the outer cylinder (Rüdiger et al., 2013). In experiments, HMRI was identified as an upward travelling wave (direction depending on the direction of magnetic field and cylinders rotation) (Stefani et al., 2006), whereas AMRI appeared as a superposition of waves \( m = \pm 1 \), travelling in the opposite direction (Seilmayer et al., 2014). Even close to onset it appeared that the waves featured clearly non-periodic defective vortices. The analysis of the bifurcation scenario leading to the experimentally observed waves has never been performed, and the transition to chaos in a system subject to MRI has never been traced in detail. This chapter addresses these two questions, focusing first on low \( Pm = 1.4 \cdot 10^{-6} \), and then on relatively high \( Pm = 1 \), and thus representing both flow in liquid metal experiments and astrophysical plasmas. By means of direct numerical simulations of the coupled induction and Navier–Stokes equations described in chapter 3 we avoid undesired endplate effects and focus on the features intrinsic to the AMRI.

In the experiments of Seilmayer et al., 2014 the AMRI was explored near the onset of instability for two different Reynolds numbers \( Re = 1480 \text{ and } 2960 \), and Hartmann numbers in the range \( Ha \in [0, 160] \). The experiments have an aspect-ratio of 10. Here a periodic domain of length \( L_z = 12.6 \) was selected, and the simulations were initialised by disturbing all Fourier modes with the same amplitude, thus allowing the axial wavenumber to be naturally selected. Because of the symmetries, two different Hopf-bifurcation scenarios are possible (Knobloch, 1996). In the first one, the \( z \)-reflection symmetry is broken and depending on the initial conditions either
upward traveling waves (with $k > 0$ modes) or downward traveling waves (with $k < 0$ modes) may be observed. In the second scenario, the $z$-reflection symmetry is preserved and a standing wave emerges. This is a combination of upward and downward traveling waves for which positive and negative $k$ modes are in phase and have exactly the same amplitude. In both scenarios waves rotate in the azimuthal direction.

4.1 Low magnetic Prandtl numbers

$$P_m = 1.4 \cdot 10^{-6}$$

4.1.1 Primary instability: standing waves

At first we follow the parameter region explored in (Seilmayer et al., 2014). The Reynolds number is fixed to $Re = 1480$, and magnetic field strength is varied. At $Re = 1480$ the circular Couette flow (1.42) becomes unstable at $Ha_c = 107$. The emerging pattern is a standing wave (SW) with dominant mode $(k, m) = (\pm 8, 1)$, so that 8 pairs of vortices fit in the domain. Figure 4.1b shows the square of the amplitude of the complex Fourier coefficient $A_{8,1}$ for increasing $Ha$. As expected in a Hopf bifurcation, $A_{k,m} \propto Ha - Ha_c$ near the onset of instability, and this relationship holds up to $Ha \approx 112$. The vortex arrangement of the standing wave at $Ha = 150$ is shown in the flow snapshot of figure 4.1a. In this case the mode $k = 9$ was naturally selected. Thus the dominant axial wavenumber depends on $Ha$ because of the Eckhaus instability, as also observed in hydrodynamic Taylor-Couette flow (Riecke and Paap, 1986).

The torque changes respectively with axial wavenumber (black curve on the Fig. 4.2a), so at the same parameter value states with different wavenumber and torque can be realised, depending on the initial conditions. Further increasing $Ha$ the instability is gradually damped until it disappears at $Ha \approx 175$. Over the whole Hartmann range the additional torque due to the SW never exceeds 1% of the laminar flow (see figure 4.2a), indicating very weak transport of angular momentum. The maximum in torque correlates well with the maximum growth rate from the linear stability analysis shown in Fig. 4.2b.

4.1.2 Onset of spatio-temporal chaos

At $Re = 2960$ a Hopf bifurcation occurs at $Ha_c = 120$ and the emerging SW remains stable until $Ha = 160$. Increasing $Ha$ beyond this point a catastrophic transition to spatio-temporal chaos is observed: the vortex structure is damaged and the up-down symmetry is broken (Fig. 4.3a). Between $Ha = 130$ and 160 there is a hysteresis region in which both SW and spatio-temporal chaos (defects) are locally stable (see Fig. 4.2a). In this $Ha$-range, if the initial condition is a SW from another run with slightly different $Ha$, this remains stable. However, when starting, for example, from a randomly disturbed Couette profile the flow evolves directly to defects.
Fig. 4.1.: Primary instability: a standing wave arises through a supercritical Hopf bifurcation. (a) Standing wave with $k = 9$ (corresponding to 9 vortex-pairs in the axial direction) at $Re = 1480$, $Ha = 150$ and $L_z = 12.6$ (long domain). From left to right: isosurfaces of axial velocity $v_z = \pm 0.005$ (normalized with the velocity of the inner cylinder $\Omega_i r_i$), contours of axial and radial velocity. The aspect ratio of the colormaps has been stretched by a factor of 0.6. (b) Onset of instability at $Re = 1480$. The critical Hartmann number is $Ha_c \approx 107$ with critical axial mode $k = 8$. The square of the amplitude of the Fourier coefficient $A_{8,1}^2$ depends linearly on $Ha - Ha_c$ close to the critical point as expected in a Hopf bifurcation. The coefficient $A_{-8,1}$ has the same amplitude as $A_{8,1}$, confirming that the axial reflection symmetry is preserved (standing wave).

Fig. 4.2.: Onset of spatio-temporal chaos. (a) Dimensionless torque for AMRI versus $Ha$ for $Re = 1480$ and $Re = 2960$. Eckhaus instability at $Re = 1480$: the branches of the black curve belong to different axial wavenumbers ($k=8, 9$ and $10$) of the standing wave. Bistability at $Re = 2960$: in the yellow-shaded region standing waves (green) and defects (red) coexist; between them there is an unstable branch or edge state (blue). (b) Perturbation growth rates $\sigma$ (normalised with $\Omega_i$) as a function of $Ha$ for $Re = 1480$ and $Re = 2960$. Positive values of $\sigma$ correspond to instability.
This catastrophic transition suggests a subcritical bifurcation. I investigated this hypothesis by computing the unstable branch separating defects and SW. For this purpose I combined time-stepping with a bisection strategy as follows. If the SW is slightly disturbed, then the flow should rapidly converge to the SW because it is locally stable. The same applies to defects. For intermediate initial conditions the flow should take a long time before asymptotically reaching either the SW or the defects. Such initial conditions were generated here by performing a linear combination between two selected flow snapshots of SW and defects. This combination was parametrised with a variable $\epsilon$, for which $\epsilon = 0$ corresponds to SW and $\epsilon = 1$ to defects. With the bisection procedure, refining $\epsilon$ results in an initial condition successively closer to the manifold (or edge) delimiting the two basin boundaries. The edge is comprised of those initial conditions that tend neither to defects nor to SW, and the attractor in this manifold is referred to as an edge state (Skufca et al., 2006).

Figure 4.4 shows that as initial conditions are taken closer to the edge, the temporal dynamics become simple as the edge state is approached. At $Ha = 155$, which is very close to the destabilisation of the SW, the dynamics appear to exhibit a damped oscillation (see Fig. 4.4b). Unfortunately, it is difficult to establish whether the oscillation finally decays or saturates at a tiny amplitude, as expected close to the bifurcation point. At $Ha = 140$, however, which is further from the bifurcation point, the oscillation saturates at non-zero amplitude (see Fig. 4.4a). This suggests that the edge state is a relative periodic orbit (or modulated wave) emerging at a subcritical Hopf bifurcation of the SW. Despite this simple temporal behaviour, the
Fig. 4.4.: Edge tracking procedure at $Re = 2960$. (a) $Ha = 140$ and (b) $Ha = 155$. Green curves evolve toward the standing wave state and red curves toward defects, although all of them start very close to the edge state. Oscillations at $Ha = 155$, which is close to the destabilisation point of the standing wave, appear to decay, while at $Ha = 140$ they saturate. Time is normalised using the inner cylinder rotation frequency $1/\Omega_i$, i.e. $t = t \cdot Re$.

The qualitative difference between standing wave, defects and turbulent flow is apparent in time series of the radial velocity $v_r$ taken at the mid-gap between the cylinders $(r, \phi, z) = (1.5, 0, 0)$. Figure 4.6a shows that the radial velocity of the standing wave oscillates periodically around zero. The edge state features a slow temporal frequency modulating the oscillation of the SW (Fig. 4.6b). For defects at $Re = 2960$ the time series is mildly chaotic. As $Re$ increases toward turbulence the velocity pulsates in a very chaotic manner (Fig. 4.6d). However, the main frequency associated with the AMRI can still be discerned. By comparing all panels it becomes apparent that this frequency scales with the rotation-rate of the inner cylinder. This is consistent with the linear stability analysis of Hollerbach et al. (2010), and with the studies (Kirillov and Stefani, 2010; Kirillov et al., 2012), where it is shown that in the low $Pm$ limit the AMRI is an inertial wave. As the Reynolds number is further increased, defects accumulate and are expected to grow gradually into turbulence. At $Re = 4000$ the flow is already quite chaotic and hardly resembles periodic waves; turbulent vortices are small at the inner cylinder and remain quite large at the outer cylinder (Fig. 4.3b), and at $Re = 9333$ this tendency develops into rapidly drifting small vortices at the inner cylinder and slow large vortices at the outer cylinder (Fig. 4.7). There is no preferred direction in the system; vortices can travel up
or down, both at the inner and outer cylinders. On average the same amount of kinetic energy is conserved in negative and positive axial harmonics, and the axial symmetry remains unbroken. This turbulence contributes to the increase of torque at the cylinders, which is proportional to $Re$ and rather slow. Guseva et al., 2015 reports about $G/G_{\text{lam}} \sim Re^{0.15}$ in the range of $Re \in [10^3, 10^4]$, which is surprisingly slow compared to hydrodynamic experiments (Lathrop et al., 1992). However, at these values of $Re$ the flow is very close to the onset and thus this scaling can not be asymptotic.

4.1.3 Comparison to experiment

In order to compare to experimental observations (Seilmayer et al., 2014) I computed the angular drift frequency of the wave. This is shown in figure 4.8 after being normalised with the rotation frequency of the inner cylinder. The wave rotates at approximately the outer cylinder frequency (dashed line) and slows down as the Hartmann number increases, which is in qualitative agreement with the experimental data. Note, however, that in the experiment two frequencies are simultaneously measured, corresponding to the up- and down-traveling spiral waves, respectively. Although in the standing wave the two frequencies are identical, in the experiment the asymmetric wiring creates $B_r$ and $B_z$ components of magnetic field which break the reflectional symmetry. As a result, up and down spirals travel with different frequencies, similar to co-rotating Taylor-Couette flow in which the reflection symmetry is broken by an imposed axial flow (Avila et al., 2006). Another difference is that in
Fig. 4.6: Transition to turbulence. Evolution of radial velocity perturbation $u_r$ at the point $(r, \phi, z) = (1.5, 0, 0)$ with time (a) at $Re = 1480$ and $Ha = 150$ (standing wave); (b) - the same at $Re = 2960$ and $Ha = 140$ (edge state); (c) - at $Re = 2960$ and $Ha = 190$ (defects); (d) - at $Re = 9333$ and $Ha = 456.7$ (turbulence). Time is scaled using the inner cylinder rotation frequency $1/\Omega_i$, i.e. $t = t \cdot Re$; velocities are normalised with the velocity of the inner cylinder $\Omega_i r_i$.

Fig. 4.7: Turbulent flow in a short domain at $Re = 9333$, $Ha = 456.7$ and $L_z = 1.4$. Axial velocity isosurfaces $v_z = \pm 0.011 [\Omega_i r_i]$, contours of axial and radial velocity.
Fig. 4.8.: Comparison to PROMISE experiment: angular drift frequencies of the waves at (a) Re 1480 and (b) Re 2960. Blue and red lines correspond to experimental results, black to our nonlinear simulations; the green line denotes outer cylinder rotation \( \Omega_o/\Omega_i = 0.26 \). The waves rotate at approximately the outer cylinder frequency and slow down with increasing \( Ha \).

the experiment the flow becomes unstable at lower \( Ha \), which may be explained by the different boundary conditions in the experiment from our simulations. In the experiment copper cylinders are used, so perfectly conducting walls would be a closer boundary condition for the magnetic field. More significantly, in the experiments the cylinders are of finite length, so to reproduce their results exactly a no-slip condition on end-plates should be used. Here periodic boundary conditions were applied in the axial direction. They more accurately model the accretion disc problem and allow to compute high Reynolds number flows more efficiently.

4.2 High magnetic Prandtl numbers

\[ Pm = 1 \]

4.2.1 Saturated states of AMRI

The dynamics for \( Pm = 1 \), on which we focus in this section, seems to be much more rich and diverse than for the small \( Pm \) case. The stability maps for both \( \mu = 0.26 \) and \( \mu = 0.35 \) are presented in Fig. 4.9. The simulations in vertical direction (vertical dashed lines) follow the maximum growth rate line, and horizontal dashed lines focus on the constant \( Re = 250 \) with \( Ha \) varied. The secondary instabilities from section 2.2 are given as a reference.

For rotation close to Rayleigh line \( (\mu = 0.26, \text{Figure 4.9a}) \), a traveling wave (TW) arises at the onset of instability at \( Re \approx 100 \), followed by 2-frequency solution as \( Re \) increases up to 250. The 2-frequency oscillating flow is characterized by periodic oscillations of the torque on the cylinders (see Fig. 4.10a) modulated by a lower frequency. The two frequencies seen in the torque are also observed in velocity time series, which also has an additional modulation related to rotation of the pattern (Fig. 4.11a). The latter is not present in the torque because it is
Fig. 4.9: (a) Different flow states found with DNS at $Pm = 1$ ($\mu = 0.26$): standing waves, traveling waves, 2- and 1-frequency oscillations (of the torque), chaotic solutions (defects). Oscillating in time solutions are stable. The different instability regions are shown as a reference. Vertical dashed line represents the maximum growth rate line, horizontal dashed line - line of $Re = 250$. (b) The same for $Pm = 1$, $\mu = 0.35$. 

if $Re$ is increased further, 2-frequency torque oscillation loses the slow modulation component and transforms to 1-frequency solution (4.10b). The velocity becomes only 2-frequency time-periodic (4.11b). This is surprising because more organized flow is more favorable for the system, despite the increase in $Re$. The 1- and 2-frequency time-periodic solutions also drift axially, similarly to TW. However, if we continue increasing $Re$, the flow finally becomes chaotic (Fig. 4.10c, 4.11c). At $Re = 500$ the flow is at the onset of turbulence: the vortices of different size are clustered at the inner cylinder and travel up and down with no preferred direction. The snapshots of the flow in Fig. 4.12 show that the spatial pattern is very similar for 2-frequency oscillations at $Re = 250$ and 1-frequency oscillations at $Re = 350$, while spatio-temporally chaotic flow at $Re = 500$ is more complex. All three cases are characterized by presence of defects which develop on top of the symmetric vortex pattern of the TW at the instability onset. A different scenario is found when $Ha$ is increased and $Re$ is kept constant. The horizontal dashed line of Fig. 4.9 denotes the simulations that were performed at $Re = 250$. Similar to $Pm = 1.4 \cdot 10^{-6}$, close to stability boundary the flow is a spatially periodic standing wave (SW), and becomes chaotic in the center of the instability island via a 2-frequency state. While $Ha$ number is increased, defects are suppressed gradually to 2-frequency oscillations, then to travelling wave, then to standing wave manifesting itself at the right instability border before the flow becomes linearly stable. Quasi-Keplerian flows in Fig. 4.9b have a similar family of flow states. The main difference with $\mu = 0.26$ is the prevalence of standing wave along maximum growth rate line. Still at the onset AMRI emerges as a traveling wave. The two-frequency oscillations were not observed. However, they may still be encountered along the
maximum growth rate line between standing waves and 1-frequency oscillations if parameters are refined enough. A more detailed study of bifurcation scenarios for high $Pm$ is out of the scope of this work and we leave it as future work.

### 4.2.2 Subcritical turbulence

The linear stability analysis in section 2.3 shows that for high $Pm$ the left instability boundary widens as $Re \sim Ha^{1.5}$. Nonetheless, the nonlinear simulations reveal that for $Pm = 1$ the AMRI pattern survives outside the left linear stability border, if the flow is disturbed strongly enough. Approaching the left stability boundary from the right (decreasing $Ha$), we observe that the turbulence triggered by AMRI does not vanish at once with the linear instability, being found far in the nonlinear region (Figure 4.13a). Figure 4.13b gives a snapshot of such a turbulent flow at $Re = 10^4$, $Ha = 432$. As $Ha$ is kept decreasing, turbulent flows calm down to less chaotic 1-frequency oscillations or periodic traveling waves. Right after that the flow becomes stable again and neither small, nor strong perturbations survive. The boundary where the nonlinear instability ceases to exist is denoted by indigo
Fig. 4.11.: Time series of radial velocity at the point \((r, \phi, z) = (1.5, 0, 0)\), \(P_m = 1\): (a) 2-frequency oscillation at \(Re = 250, Ha = 150\); (b) 1-frequency oscillation at \(Re = 350, Ha = 203\); (c) chaotic solution at \(Re = 500, Ha = 295\). Velocity is normalized with the rotation speed of the inner cylinder \(\Omega_i r_i\), time is scaled with rotation period of the inner cylinder \(1/\Omega_i\).
Fig. 4.12: Isosurfaces of axial velocity $\pm u_z$ (left) and contours of axial $u_z$ and radial $u_r$ velocity (right), $Pm = 1$: (a) 2-frequency oscillation at $Re = 250$, $Ha = 150$; (b) 1-frequency oscillation at $Re = 350$, $Ha = 203$; (c) chaotic solution at $Re = 500$, $Ha = 295$.

dashed line. On the contrary, if this nonlinear border is approached from the small disturbances as initial condition, they inevitably decay unless parameters of the flow are inside of the instability island. These results indicate the subcritical nature of the bifurcation at the left border for $Pm = 1$, which is different from $Pm = 1.4 \cdot 10^{-6}$, where the primary bifurcation is supercritical (Hopf), as seen in section 4.1.1. This is valid for both $\mu = 0.26$ and $\mu = 0.35$. For $\mu = 0.35$ the nonlinear scaling exponent is $\delta_L = 2.6$, which implies stronger left border scaling, compared to the linear estimate $\delta_L = 1.5$. This predicts a faster widening of the instability region for $Pm = 1$ and suggests the turbulence might not die out even if the field strength decays, provided a strong magnetic field existed initially.

4.3 Discussion

In this chapter I studied the magnetorotational instability with applied azimuthal magnetic field in the Taylor-Couette setup (AMRI). The focus was made on the comparison between two magnetic Prandtl numbers: small $Pm = 1.4 \cdot 10^{-6}$ and large $Pm = 1$. The linear stability analysis does not give any information about the final, saturated state of a system. Fortunately, the critical parameters of $Re \sim 10^3$ and $Ha \sim 10^3$ are accessible in liquid metal experiments. Such experiments indeed have been performed with InGaSn alloy ($Pm = 1.4 \cdot 10^{-6}$) (Seilmayer et al., 2014). The AMRI was observed as a superposition of two waves, traveling in the opposite direction at slightly different speeds. This was surprising because the azimuthal magnetic field does not break the axial reflection symmetry of a Taylor-Couette system. It was shown numerically in section 4.1, that with axially periodic boundary conditions at $Pm = 1.4 \cdot 10^{-6}$ the AMRI arises as a standing wave (i.e. stationary
in axial direction), which at the same time rotates azimuthally approximately at the outer cylinder frequency. This wave arises via a supercritical Hopf bifurcation from the laminar flow, where \((Ha - Ha_c)\) acts as a bifurcation parameter. However, the standing waves (SW) are stable only close to the onset of instability. At higher \(Re\) a subsequent subcritical Hopf bifurcation destabilizes SW and spatial defects accumulate in the system. More information on symmetries and bifurcation in fluid dynamics in general and Taylor-Couette flow in particular can be found in (Crawford and Knobloch, 1991).

At \(Pm = 1\) the AMRI arises first as a spatially periodic standing or travelling wave, and as the Reynolds number increases the spatially periodic flow turns into turbulence. While for low \(Pm = 1.4 \times 10^{-6}\) this transition happens as a sequence of super- and subcritical Hopf bifurcations, for \(Pm = 1\) the transition scenario is much more complicated and involves several spatially complex flow patterns exhibiting 1- or 2-frequency oscillations of integral quantities such as torque or kinetic energy. This phenomenon does not seem to be connected to the existence of several types of modes found in the linear stability analysis of the flow, since it happens at low \(Re\) close to the onset of instability and does not touch the lines of discontinuous jump in axial wavenumber \(k\), predicted by linear analysis. The competition between several linear modes occurs in the parameter region where the flow is already too turbulent to observe remnants of the linear modes.

At large \(Re\) the flow becomes fully turbulent and the turbulent structures do not resemble low-\(Re\) SW or TW. Nevertheless, transport mechanisms may still be different, as is clear from section 2.4. The differences in small-\(Pm\) and large-\(Pm\)
flow dynamics demonstrate the need for a study of intermediate $P_m$ for a better understanding of the transition between low- and high-$P_m$ transport properties.
Angular momentum transport

The mechanisms of angular momentum transport remained the main question of accretion disk theory for years. The action of poloidal fields, which can be generated by the accreting object in the center of the disk or advected from outside, is well-studied. Weak poloidal magnetic fields lead to the amplification of axisymmetric disturbances and give rise to self-sustained turbulence in nonlinear simulations (Balbus and Hawley, 1991; Hawley et al., 1995; Stone et al., 1996). This MRI turbulence was found to significantly enhance angular momentum transport via Maxwell stresses, which were several times larger than Reynolds stresses. However, these early works did not take account of viscosity and magnetic resistivity (ideal MHD). Lesur and Longaretti (2007) considered non-ideal fluids and showed a power-law dependence of the transport coefficient of the form $\alpha \sim Pm^a$, with $a \in (0.25, 0.5)$, for the explored range of magnetic Prandtl numbers $Pm \in [0.12, 8]$ and Reynolds numbers $Re \in [200, 6400]$.

For an imposed axial magnetic field, Gellert et al. (2012) found the transport coefficient $\alpha$ to be independent of magnetic Reynolds $Rm$ and magnetic Prandtl $Pm$ numbers, only scaling linearly with the Lundquist number $S$ of the axial magnetic field. However, the critical parameter values $Rm \sim O(10)$, $S \sim O(3)$ are challenging to achieve experimentally because of the low values of $Pm \in [10^{-6}, 10^{-5}]$ for liquid metals and therefore extremely high critical Reynolds numbers $Re = Rm/Pm \geq O(10^6)$. While signatures of this standard MRI were detected experimentally in the form of damped magnetocoriolis waves (Nornberg et al., 2010), experimental observations of unstable magnetocoriolis waves (giving rise to the MRI) have not been reported in the literature so far.

The MRI can also be triggered by toroidal magnetic fields provided that these are neither too weak nor too strong (Balbus and Hawley, 1992; Ogilvie and Pringle, 1996). For $Pm \rightarrow 0$ this inductionless azimuthal version of MRI continues to exist even for $S, Rm \rightarrow 0$, as long as $Re \sim O(10^3)$ and $Ha \sim O(10^2)$. It takes the form of an inertial wave destabilized by the magnetic field via the Lorentz force, and hence is a magnetohydrodynamic rather than hydrodynamic instability (see Kirillov and Stefani, 2010, for an extended discussion and connection to the standard MRI via helical fields).

Seilmayer et al. (2014) reported the experimental observation of the predicted non-axisymmetric AMRI modes in Taylor–Couette flow with $q = -1.94$. Because of the strong currents of nearly 20kA needed to generate the required azimuthal field, measurements could only be conducted close to the stability boundary. The direct numerical simulations of chapter 4 probed deep into the nonlinear regime and computed the angular momentum transport of the AMRI. Despite the highly turbulent
nature of the flow, for $Pm = 1.4 \cdot 10^{-6}$ (InGaSn alloy) the angular momentum transport was found to be barely faster than in laminar flow. Recently, Rüdiger et al. (2015) examined the effective viscosity $\nu_t$ for the three relevant rotation rates: close to the Rayleigh line ($q \sim -2$), quasi-Keplerian ($q \sim -1.5$) and galactic ($q \sim -1$), in the range of $Pm \in [10^{-1}, 1]$ and $Re \in [2 \cdot 10^2, 2 \cdot 10^3]$. They suggested a scaling of the dimensionless effective viscosity as $\nu_t/\nu = \sqrt{PmRe}$, with Maxwell stresses dominating for large $Pm \geq 0.5$. However, in the range of parameters investigated in Rüdiger et al. (2015), AMRI turbulence does not yet clearly exhibit asymptotic scaling and the transition between low-$Pm$ and high-$Pm$ instability and transport properties at low and moderate $Pm$ remain unclear.

In this chapter, using direct numerical simulations I compute angular momentum transport in the system for $Re$ up to $4 \cdot 10^4$ and $Pm \in [0, 1]$. Although it would be very interesting to perform a two-parameter study of the dynamics in $Ha$ and $Re$, this is computationally expensive and beyond the scope of the current work. Simulations of this chapter follow a parameter path of the form

$$\text{Re} = x \text{Ha}^y,$$

where $x$ and $y$ can be found in Appendix B. This paths are shown as a green circles in Fig. 2.2, 2.4 and provide a very good approximation to the curve of maximum growth rate of the linear stability analysis. They go deep into the instability region and so the instability is expected to fully develop as $Re$ increases with $Ha$ subject to (5.1). The results of this chapter give a comprehensive picture of turbulent transport via the AMRI.

5.1 Theoretical framework

5.1.1 Angular velocity current

Frequently a net loss (gain) of angular momentum on the inner (outer) cylinder is measured in Taylor-Couette experiments as torque (Paoletti and Lathrop, 2011; Wendt, 1933). Dimensionless laminar torque $G_{\text{lam}}$ can be explicitly calculated from the radial derivative of azimuthal velocity profile,

$$\Omega_{\text{lam}} = \Omega_i \frac{1}{1 - \eta^2} \left[ \left( \mu - \eta^2 \right) + r_i^2 \left( 1 - \mu \right) \frac{1}{r^2} \right],$$

at the wall

$$G_{\text{lam}} = \frac{2\pi}{\nu} r^3 \partial_r (\Omega_{\text{lam}}),$$

or equivalently,

$$G_{\text{lam}} = \frac{4\pi \eta |\mu - 1|}{(1 - \eta)^2(1 + \eta)} Re.$$
In a statistically steady state, the time-averaged torques on the inner and outer cylinders are equal in magnitude.

Eckhardt et al. (2007) derived a conservation equation for the current \( J^\omega \) of the angular velocity \( \Omega = \frac{v_\phi}{r} \) in hydrodynamic Taylor–Couette flow. In this work I extend this to the magnetohydrodynamic case as follows.

Taking into account the following identity of vector calculus
\[
\frac{1}{2} \nabla (B \cdot B) = B \times (\nabla \times B) + (B \cdot \nabla)B
\]
and defining magnetic pressure
\[
p' = p + \frac{H a^2 B^2}{Pm} \tag{5.6}
\]
one can rewrite the Navier–Stokes equation in the form
\[
(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla p' + \nabla^2 \mathbf{v} + \frac{H a^2}{Pm} (B \cdot \nabla)B. \tag{5.7}
\]

The \( \phi \)-component of the Navier–Stokes equation (5.7) becomes
\[
\partial_t v_\phi = -(\mathbf{v} \cdot \nabla)v_\phi - \frac{v_r v_\phi}{r} - \frac{1}{r} \partial_\phi (p') + \left( \nabla^2 v_\phi - \frac{v_\phi}{r^2} + \frac{2}{r^2} \partial_\phi v_r \right) + \frac{H a^2}{Pm} \left\{ (B \cdot \nabla)B_\phi + \frac{B_r B_\phi}{r} \right\} \tag{5.8}
\]
with the differential operators above having the following meaning:
\[
(\mathbf{v} \cdot \nabla) f = \left( v_r \partial_r + v_\phi \frac{1}{r} \partial_\phi + v_z \partial_z \right) f,
\]
\[
\nabla^2 f = \frac{1}{r} \partial_r (r \partial_r f) + \frac{1}{r^2} \partial_\phi^2 f + \partial_z^2 f. \tag{5.9}
\]

We apply these operators to the equation for \( v_\phi \) component:
\[
\partial_t v_\phi = -v_r \partial_r v_\phi - \frac{1}{r} \partial_\phi v_\phi - v_z \partial_z v_\phi - \frac{v_r v_\phi}{r} - \frac{1}{r} \partial_\phi p' + \left( \frac{1}{r} \partial_r (r \partial_r v_\phi) + \frac{1}{r^2} \partial_\phi^2 v_\phi + \frac{v_\phi}{r^2} + \frac{2}{r^2} \partial_\phi v_r \right) + \frac{H a^2}{Pm} \left( B_r \partial_r B_\phi + B_\phi \frac{1}{r} \partial_\phi B_\phi + B_z \partial_z B_\phi + \frac{B_r B_\phi}{r} \right). \tag{5.10}
\]

To derive the corresponding current, we average equation (5.10). The average over a cylindrical surface of area \( A(r) = 2\pi rl \), co-axial to the rotating cylinders, is \( r \)-independent:
\[
\langle \cdots \rangle_A = \int \frac{r d\phi dz}{2\pi rl} \cdots = \int \frac{dz}{l} \int \frac{d\phi}{2\pi} \cdots \tag{5.11}
\]
Averaging over surface $A$ and also taking time average, we find

$$0 = \langle -v_r \partial_r v_\phi - v_z \partial_z v_\phi - \frac{v_r v_\phi}{r} + \frac{1}{r} \partial_r (r \partial_r v_\phi) - \frac{v_\phi}{r^2} \rangle + \frac{H \alpha^2}{P_m} \bigg( B_r \partial_r B_\phi + B_z \partial_z B_\phi + \frac{B_r B_\phi}{r} \bigg).$$

(5.12)

Making use of the continuity equation and the divergence-free condition for magnetic field, we can rewrite $u_z \partial_z u_\phi$ and $B_z \partial_z B_\phi$ in terms of $r$- and $\phi$-components:

$$0 = \langle -v_r \partial_r v_\phi - v_\phi \partial_r v_r - \frac{2v_r v_\phi}{r} + \frac{1}{r} \partial_r (r \partial_r v_\phi) - \frac{v_\phi}{r^2} \rangle + \frac{H \alpha^2}{P_m} \bigg( B_r \partial_r B_\phi + B_\phi \partial_r B_r + \frac{2B_r B_\phi}{r} \bigg).$$

(5.13)

The first three term sum up to

$$-v_r \partial_r v_\phi - v_\phi \partial_r v_r - \frac{2v_r v_\phi}{r} = -r^{-2} \partial_r (r^2 v_r v_\phi),$$

(5.14)

the same with Lorentz force:

$$B_r \partial_r B_\phi + B_\phi \partial_r B_r + \frac{2B_r B_\phi}{r} = r^{-2} \partial_r (r^2 B_r B_\phi).$$

(5.15)

Multiplying the equation with $r^2$ and rewriting the viscous term as an $r$-derivative, we obtain

$$0 = \partial_r \left( r^2 \left[ \langle v_r v_\phi \rangle_{A,t} - r \partial_r \langle \frac{v_\phi}{r} \rangle_{A,t} - \frac{H \alpha^2}{P_m} \langle B_r B_\phi \rangle_{A,t} \right] \right) = \partial_r (J^\omega).$$

(5.16)

The first term in equation (5.16) represents a Reynolds stress, whereas the second and third terms represent viscous and Maxwell stresses, respectively. For the steady rotation $J^\omega$ is conserved $[\partial_t (J^\omega) = 0$, see equation (5.16)]. The constant $J^\omega$ can be interpreted as the conserved transverse current of azimuthal motion transporting $\Omega(r, \phi, z, t)$ in the radial direction. Its unit is $[J^\omega] = m^4 s^{-2} = [\nu]^2$. The angular velocity current $J^\omega$ is closely related to the dimensionless torque on the cylinders (Eckhardt et al., 2007):

$$G = 2\pi \nu^{-2} J^\omega.$$  

(5.17)

In the case of laminar flow $\langle \Omega \rangle_{A,t} = \Omega_{\text{lam}}$ the first and the third term in Equation (5.16) are zero and the laminar angular velocity current is defined by

$$J^\omega_{\text{lam}} = -\nu r^3 \partial_r \Omega_{\text{lam}},$$

(5.18)

so that (5.17) and (5.3) coincide.
5.1.2 Effective viscosity
For the case of turbulent flow, in analogy to (5.18) we model angular velocity current with the mean angular velocity:

\[ J_\omega = -\nu_{\text{eff}} r^3 \partial_r \langle \Omega \rangle, \quad (5.19) \]

where the effective viscosity \( \nu_{\text{eff}} \) is parameterized with the mean angular velocity and the size of the gap between cylinders

\[ \nu_{\text{eff}} = \alpha_{\text{eff}} \langle \Omega \rangle d^2. \quad (5.20) \]

Substituting (5.20) into (5.19) results in the following for the parameter \( \alpha_{\text{eff}} \):

\[ \alpha_{\text{eff}} = -\frac{J_\omega}{\langle \Omega \rangle d^2 r^3 \partial_r \langle \Omega \rangle} = \frac{J_\omega}{\langle \Omega \rangle^2 d^2 r^2 q \nu^2} = \frac{J_\omega}{q |q^2 Re^2_a|}. \quad (5.21) \]

Here \( q = \partial \ln \langle \Omega \rangle / \partial \ln r \) and \( Re_a \) is a Reynolds number based on the average angular velocity. Estimation of \( Re_a \) based on \( \langle \Omega \rangle \approx (\Omega_i + \Omega_o)/2 \) and the midgap radius \( r = (r_i + r_o)/2 \) gives \( Re_a^2 \approx 1.026 Re^2 \) for both \( \mu = 0.26 \) and 0.35. Recalling (5.17) we finally obtain:

\[ \alpha_{\text{eff}} = \frac{1}{2\pi |q|} \frac{G}{Re^2}. \quad (5.22) \]

In the traditional Shakura-Sunyaev \( \alpha \)-prescription \( \nu_t = \alpha_c s H \), with sound speed \( c_s \) and disc height \( H \). For TCF, sound speed and disc height is replaced by local angular velocity \( \Omega \) and the gap size between cylinders \( (r_o - r_i) \), because the size of turbulent eddies is determined by the smallest dimension of the flow. Protostellar-disk accretion rates indicate \( \alpha_{\text{eff}} \geq 10^{-3} \) (Hartmann et al., 1998).

5.2 Angular momentum transport
I performed DNS spanning a wide range in \( Pm \in [0, 1] \) and \( Re \) up to \( 4 \cdot 10^4 \) and computed the torque \( G \) in order to quantify the scaling of angular momentum transport via the AMRI. Each filled symbol in Fig. 5.1a marks a simulation in the parameter space \((Re, Ha)\), with \( \mu = 0.26 \). The simulations with \( \mu = 0.35 \) follow the same path as for \( \mu = 0.26 \) and are shown as empty symbols of the same color. Because of the cost of varying both \( Ha \) and \( Re \), I followed one-dimensional paths in parameter space (dashed-dotted lines connecting the symbols) that correspond to the maximum growth rate lines of the linear analysis (depicted by the dashed-dotted line in Fig. 2.2 for different \( Pm \) and \( \mu = 0.26 \)). Note that in section 4.1.1 it was shown that the maximum growth rate of the linear analysis correlates very well with the maximum of the transport for \( \mu = 0.26 \) at low and high \( Pm \). Mamatsashvili et al., 2017 have observed the same correlation for the helical MRI. Hence the results presented in the following can be seen as an upper bound on the angular momentum transport.
The value of $Re_c$ depends strongly on $\mu$ and $Pm$ as shown in Fig. 2.6. Hence comparing the scaling of $G$ with $Re$ at different $Pm$ and $\mu$ is not straightforward. Moreover, Guseva et al. (2015) and Guseva et al. (2017a) found that at low $Pm$, close to the Rayleigh line, the turbulence arising from the AMRI is not efficient in transporting momentum. At $Pm = 10^{-6}$ and $Re \lesssim 3 \cdot 10^4$, molecular viscosity is responsible for a significant portion of the momentum transport. As a consequence, the laminar contribution to the torque can obscure the scaling of the turbulent contribution, which will obviously dominate at the asymptotically large $Re$ of interest.

To enable a representation more useful for extrapolations toward large $Re$, in this section we quantify transport by showing the reduced torque ($G/G_{lam} - 1$), which is proportional to the turbulent viscosity, as a function of the relative Reynolds number $Re' = Re - Re_c$.

### 5.2.1 Close to the Rayleigh line

Figure 5.1b shows that for $\mu = 0.26$ the reduced torque scales linearly with $Re'$, whereas the dependence on $Pm$ is not straightforward. Lines of $(G/G_{lam} - 1) = a Re'$, where the pre-factor $a$ is a function of $Pm$, provide good fits to all data sets. The pre-factors $a$ and respective errors were calculated based on the average of local fits to the data using stencils of 3 up to 5 points. The dependence of the pre-factor $a$ on $Pm$ is shown in Fig 5.1d. For $Pm \geq 10^{-2}$, $a \propto Pm^{0.53}$, supporting the $\sqrt{Pm} Re$ scaling proposed by Rüdiger et al. (2015), whereas for $Pm \leq 10^{-3}$, $a$ saturates to a small but constant value independent of $Pm$. I verified this by performing DNS in the inductionless limit ($Pm = 0$, violet empty circles in 5.1b) and the results are in fact indistinguishable from $Pm = 1.4 \cdot 10^{-6}$ ($a(Pm = 0) = 2.3 \pm 0.6 \cdot 10^{-5}$, $a(Pm = 1.4 \cdot 10^{-6}) = 2.7 \pm 0.8 \cdot 10^{-5}$). This supports the hypothesis that in the limit of very small magnetic Prandtl numbers $Pm \to 0$ the turbulent angular momentum transport triggered by AMRI turbulence depends only on $Re$, and this dependence is linear.

The case of $Pm = 10^{-2}$ requires special attention. Close to the onset of instability the scaling factor $a$ tends to the low-$Pm$ values and the $\sqrt{Pm} Re$-scaling is approached only at high $Re$. This is most likely connected to the change in the dominant mode at $Re \approx 5 \cdot 10^3$ ($Rm \approx 50$), illustrated by the change of slope of the maximum growth rate in Fig. 5.1a. Thus, the quality of the linear fit for $Pm = 10^{-2}$ is not very good, and the error bar for $a$ on Figure 5.1d is the largest. Still the average value of the pre-factor $a$ approximates well the local fits of the high-$Re$ part of the curve, and can be taken as an estimate.

### 5.2.2 Quasi-Keplerian rotation

Although the case of $\mu = 0.26$ allows us to span a broad range of relevant $Pm$, the astrophysically most relevant profile is the quasi-Keplerian one ($\mu = 0.35$). To compare the torque of the two rotation rates, I performed simulations at $\mu = 0.35$ and $Pm = 1, 10^{-1}$ and $10^{-2}$ where $Re_c$ is low enough for DNS to be feasible (Fig.
Fig. 5.1: (a) Filled symbols show the parameter values at which our DNS were performed ($\mu = 0.26$). They follow one-dimensional curves in $(Re, Ha)$-space, corresponding to the maximum growth rate lines of the linear stability analysis (see Fig. 2.5a). Data for quasi-Keplerian rotation $\mu = 0.35$ at $Pm = 1$, $10^{-1}$ and $10^{-2}$ are shown as empty symbols of the same color. The violet empty circles correspond to DNS of the inductionless limit ($Pm = 0$) at $\mu = 0.26$. (b) Normalized turbulent torque $(G/G_{lam} - 1)$ for $\mu = 0.26$ as a function of modified Reynolds number ($Re' = Re - Re_c$). $Pm = 1$ - black triangles, $Pm = 10^{-1}$ - green squares, $Pm = 10^{-2}$ - red diamonds, $Pm = 10^{-3}$ - blue triangles, and $Pm = 1.4 \cdot 10^{-6}$ - cyan circles, $Pm = 0$ - violet empty circles. For each $Pm$ a line $(G/G_{lam} - 1) = aRe'$ is fitted. (c) Comparison of the quasi-Keplerian rotation $\mu = 0.35$ (empty symbols) to $\mu = 0.26$ (filled symbols). Same color code as in the Fig. 5.1a. (d) Average scaling factor $a$ as a function of $Pm$. Dark green - $\mu = 0.26$, magenta - $\mu = 0.35$.
2.6). The results of this comparison are presented in Fig. 5.1c. Unlike in Rüdiger et al. (2015), we do not observe that the torque is lower for the $\mu = 0.35$ profile. The lower values of torque at low $Re$ are because of the later onset of instability at $\mu = 0.35$ and converge toward the values for the $\mu = 0.26$ case as the turbulence develops further. Hence, once the dependence of $Re_c$ on $\mu$ is taken into account, the torque scales identically for both rotation profiles, as demonstrated in Figure 5.1d. Because for $\mu = 0.35$ the onset of instability occurs at $Rm_c \approx 50$, studying $Pm < 10^{-2}$ becomes numerically unfeasible. Hence it cannot be directly tested whether a transition from the $\sqrt{Pm} Re$-scaling to the pure $Re$-scaling, as observed close to the Rayleigh line, occurs also in the quasi-Keplerian case.

5.3 Analysis of transport mechanisms

The analysis of the Maxwell and Reynolds stresses of the linear eigenmodes at low $Pm$ shown in section 2.4 suggests that for $\mu = 0.35$ Reynolds and Maxwell stresses are relevant, whereas for $\mu = 0.26$ only Reynolds stresses play a role. In this section I analyze the dependence of the stress contributions for the data from the nonlinear simulations shown in Fig. 5.1. In nonlinear simulations of the fully coupled Navier–Stokes and induction equations, the total transport of momentum expressed by the conserved angular velocity current $J^\omega$ is the sum of the contribution of Reynolds, Maxwell and viscous stresses (5.16). At sufficiently large $Re$, the viscous contribution is confined to thin boundary layers attached to the cylinders. Because of the no-slip and insulating boundary conditions, at the cylinders there is only viscous transport (quantified by the torque $G$). As we are interested in the high $Re$ limit, in this section only the Maxwell and Reynolds contributions are analyzed, which is consistent with the analysis of $G/G_{lam} - 1$ presented in the previous section, and the characterization of the linear eigenmodes shown in Fig. 2.7.

Figure 5.2 presents the Reynolds stresses (solid) and Maxwell stresses (dashed) as functions of radius for three representative points in the parameter space. The stresses were normalized with the full azimuthal motion current $J^\omega$, which does not depend on $r$ according to (5.16). Because of conservation of $J^\omega$, the viscous part can be obtained by subtracting the Maxwell and Reynolds contributions from 1. At $Pm = 1.4 \cdot 10^{-6}, Re = 2 \cdot 10^4 (Rm = 2.8 \cdot 10^{-2}, cyan)$ Maxwell stresses are negligible and up to 40% of the angular momentum is transported by Reynolds stresses. Despite the large $Re$, viscous transport is still prevalent and amounts to 60% of the total. At $Pm = 10^{-2}, Re = 10^4 (Rm = 10^2, red)$ the Maxwell stresses amount to 20% of the total transport in the middle of the gap, and Reynolds stresses contribute 40%. For $Pm = 1, Re = 6 \cdot 10^3 (Rm = 6 \cdot 10^3, black)$ the Maxwell stresses dominate the transport in the center part of the domain, while Reynolds and viscous stresses are very small. Interestingly, the Reynolds stresses are negative in the center of the domain, corresponding to inward momentum transport due to velocity fluctuations. Here Maxwell stresses are larger than the total current in the
middle part of the gap so that \( J^\omega \) remains conserved at each radial position. Note that the negative contribution of the Reynolds stress is also present in the linear eigenmode at \( Pm = 10^{-4} \) and \( Rm = 50 \) shown in Fig. 2.7d.

The magnetic Reynolds number \( Rm = RePm \) increases in passing through the described points: \( Rm = [2.8 \cdot 10^{-2}; 10^2; 6 \cdot 10^3] \), suggesting that the relative contribution of the stresses to the total current depends strongly on magnetic Reynolds number. This is again in line with the behavior of the linear eigenmodes discussed in section 2.4. Figure 5.3a shows the contributions of Maxwell and Reynolds stresses, normalized by their sum, at the mid-gap as a function of magnetic Reynolds number. At low \( Rm \lesssim 10 \) the contribution of Maxwell stresses is marginal, but thereafter it begins to noticeably grow until it becomes equal to the Reynolds-stress contribution at \( Rm \approx 100 \). If \( Rm \) is increased further, Maxwell stresses dominate the turbulent angular momentum transport, and for \( Rm \gtrsim 10^3 \) Reynolds stresses become negative and act as to counteract the outward transport by Maxwell stresses. Thus, \( Rm = O(100) \) marks the border between inertial-wave turbulence (excited by the imposed magnetic field) and turbulence arising from magnetocoriolis waves, i.e. the usual MRI for which magnetic stresses prevail. This result is in agreement with our eigenmode analysis and the work of Gellert et al. (2016), who compared magnetic and kinetic energies and found them equal at \( Rm \sim 200 \). They considered so-called Chandrasekhar states, where magnetic and velocity fields have the same radial profiles (unlike here) in the similar range of \( Rm \in [10^{-3}, 5 \cdot 10^4] \).

The evolution of stresses with \( Rm \) for the quasi-Keplerian case (\( \mu = 0.35 \)) is shown in Fig. 5.3b. Because the flow is only unstable for \( Rm \gtrsim 50 \), here Maxwell stresses dominate directly from onset. Thereafter, the behavior is identical to that for rotation close to the Rayleigh line. Returning to the torque scaling of the form
Fig. 5.3.: $Rm$-dependence of Maxwell and Reynolds stresses at the mid-gap for $\mu = 0.26$ (a) and $\mu = 0.35$ (b). Different colors corresponds to data with different magnetic Prandtl number: $Pm = 10^{-3}$ (blue), $Pm = 10^{-2}$ (red), $Pm = 10^{-1}$ (green) and $Pm = 1$ (black), as in Fig. 5.1. Here solid and dashed lines denote the Reynolds and Maxwell stress contributions, respectively, normalized by their sum (i.e. excluding the viscous contribution).
\( (G/G_{\text{lam}} - 1) = a \, R e' \) shown in Fig. 5.1d, one observation can be made. All data with pre-factor \( a \propto \sqrt{Pm} \) is for \( Rm > 50 \) (magnetocoriolis wave), whereas the scaling with constant pre-factor \( a \) is in the regime \( Rm < 50 \) (magnetically excited inertial wave). Further, the data for \( \mu = 0.26 \) and \( Pm = 10^{-2} \), spanning \( 10 < Rm < 400 \), appears to feature a transition from one type of scaling to the other as \( Rm \) increases (see Fig. 5.1b), which also corresponds to the change in dominant eigenmode (see Fig. 5.1a). Hence the crossover in transport scaling occurs at \( Rm = \mathcal{O}(100) \).

### 5.4 Estimation of \( \alpha_{\text{eff}} \) from torque

In Taylor–Couette flow, angular momentum transport is exactly quantified by the torque, yet in the context of accretion-disk theory it is customary to use the \( \alpha_{\text{eff}} \) parameter to quantify transport. In this section, we estimate \( \alpha_{\text{eff}} \) from the torque data. The simulations show that the turbulent part of the torque is proportional to modified Reynolds number:

\[
G/G_{\text{lam}} - 1 \approx a \, R e',
\]

where

\[
a \sim \begin{cases} 
\text{const} & Rm < \mathcal{O}(100), \\
Pm^{0.5} & Rm > \mathcal{O}(100).
\end{cases}
\]

(5.24)

After inserting (5.23) in equation (5.22) the expression for \( \alpha_{\text{eff}} \) reads as

\[
\alpha_{\text{eff}} = \frac{1}{2\pi|q|} \frac{(a R e' + 1) G_{\text{lam}}}{R e^2}
\]

(5.25)

Considering \( G_{\text{lam}} \propto R e \) from (5.4) and \( R e' \approx R e \) we find that \( \alpha_{\text{eff}} \) scales as:

\[
\alpha_{\text{eff}} \propto a + 1/R e',
\]

(5.26)

and when \( R e \to \infty \):

\[
\alpha_{\text{eff}} \propto a.
\]

(5.27)

Thus, effective viscosity is independent of \( Pm \) for \( Pm \leq 10^{-3} \) and scales as \( Pm^{0.5} \) for \( Pm \geq 10^{-2} \).

For near Rayleigh-line rotation at \( Pm = 1.4 \cdot 10^{-6} \) the turbulent torque scales with \( a = 2.7 \cdot 10^{-5} \). Inserting \( a, q = -1.94 \) and \( G_{\text{lam}} \approx 12 R e' \) from (5.4) into (5.25) we get a lower bound for \( \alpha_{\text{eff}} \) in the limit of \( R e \to \infty \):

\[
\alpha_{\text{eff}} \approx 2.6 \cdot 10^{-5}.
\]

(5.28)

corresponding to the inductionless case. At \( Pm = 1 \) the highest value for \( \alpha_{\text{eff}} \approx 3.4 \cdot 10^{-3} \) is attained. In the case of quasi-Keplerian rotation, \( q = -1.48 \), equation (5.4) gives \( G_{\text{lam}} \approx 11 R e' \) and \( \alpha_{\text{eff}} \) increases by a factor of 1.2 when compared to near Rayleigh-line rotation.
Here a caution note must be done. In Taylor–Couette flow, because of the presence of the solid cylinders bounding the fluid, turbulence modifies the mean velocity profile. Thus, the parameter $q$ in equation (5.25), estimated from the mean turbulent velocity, will depart from ideal values and at first will typically grow with $Re$. In our simulations, $q$ calculated in the middle of the gap between cylinders changes, but always remains negative. As turbulence becomes fully developed, $q$ appears to saturate at around $-1/2$, which is a (more) hydrodynamically stable velocity profile, unstable to MRI. This saturation implies independence of $\alpha_{eff}$ in $Re \to \infty$ regime. The key feature of the flow here is the proportionality to $Re$ of the outward turbulent transport of momentum; using the turbulent estimate $q = -1/2$ simply results in values of $\alpha_{eff}$ a factor of 3 higher, as seen from equation (5.22). Finally, we speculate that even if the mean flow profile were forced to remain unchanged (quasi-Keplerian), this scaling may remain unaffected, as later we will observe in DNS of self-sustained quasi-Keplerian dynamos in Taylor–Couette flow (see chapter 6).

5.5 Discussion

Following maximum growth rate lines from linear stability analysis, I estimated the upper bound of angular momentum transport by AMRI in the wide range of $Pm \in [0, 1]$ up to $Re = 4 \cdot 10^4$. At these $Re$ the flow becomes fully turbulent. The type of turbulence is defined by magnetic Reynolds number $Rm$, as was already suggested in (Gellert et al., 2016). Reynolds stresses entirely dominate Maxwell stresses for $Rm < 10^3$. The opposite situation happens at $Rm > 10^3$. The transition occurs in the interval of $10 < Rm < 10^3$ with a crossover at $Rm_c \approx 100$. It was found that normalized torque $G/G_{lam} - 1$ scales linearly with $Re$ for all $Pm$. Our data confirm $\sqrt{PmRe}$ scaling of (Rüdiger et al., 2015) for $Pm \geq 10^{-2}$. For $Pm \leq 10^{-3}$ the scaling becomes independent of $Pm$ and in the limit $Pm \to 0$ it remains proportional to $Re$. Note that the non-normalized torque $G$ scales with $Re^2$ since $G_{lam} \propto Re$. This means that the $\alpha_{eff}$-parameter in turbulent viscosity $\nu_t$ is independent of $Re$ and is a function of $\eta$ and $\mu$ only for $Pm \leq 10^{-3}$. For $Pm \geq 10^{-2}$ $\beta$ depends also on $\sqrt{Pm}$. The lower and the upper bounds on $\alpha_{eff}$ in our simulations were estimated as $2.6 \cdot 10^{-5}$ for $Pm = 1.4 \cdot 10^{-6}$ and $3.4 \cdot 10^{-3}$ for $Pm = 1$. While $\alpha_{eff} \approx 10^{-5}$ is comparable to hydrodynamic turbulence (Dubrulle et al., 2005; Richard and Zahn, 1999; Paoletti and Lathrop, 2011), it might be too small to provide effective accretion in a low-ionized protostellar disk ($Pm = 10^{-8}$, from Brandenburg and Subramanian, 2005). In highly ionized disks or disk regions, AMRI can still be a vivid source of angular momentum transport, since $\alpha_{eff} \propto \sqrt{Pm} \geq 10^{-3}$ for $Pm \geq 1$.

The relation between $Rm_c = 100$, defining the type of turbulence, and the torque scaling is unclear. From the data for $Pm = 10^{-2}$ (Fig. 5.1) one could argue that torque scaling changes from $Re^2$ to $\sqrt{PmRe^2}$ when passing $Rm_c$. Yet the curve for $Pm = 10^{-1}$ also passes $Rm_c$, but without change in scaling. The uncertainty of
scaling for $Pm = 10^{-2}$ may be also related to the change of slope of the chosen parameter path (see Figure 5.1a).
In the previous chapters we have seen that MRI can yield turbulence which is effective in angular momentum transport provided a magnetic field of finite amplitude and specific shape is imposed. This is unlikely the case in accretion disks. Explicit evidence for magnetic field in disks is sparse. It is often believed that magnetic fields arise due to the dynamo action: motion of electrically conducting fluid stretches and amplifies some original seed field.

Dynamo theory was first developed as an explanation to the magnetic field of the Earth. The large scale, mostly dipole magnetic fields in planets or stars would have dissipated with time if they were not amplified by the flow field. Differential rotation of the liquid metal core of the Earth (plasma in convection zone of the Sun) spirals an azimuthal field $B_\phi$ out of the dipole field $B_p$. This is called $\Omega$-effect. In its turn, small-scale, nonaxisymmetric helical disturbances lift and twist azimuthal magnetic field lines, which creates a local electromagnetic force (EMF). The EMF induces currents which support a large-scale dipole magnetic field if local EMFs are aligned on average in the same direction: $\langle \mathbf{v} \times \mathbf{b} \rangle = \alpha D \langle \mathbf{B} \rangle$. The latter is called $\alpha$-effect (Parker, 1955). The flow must possess non-zero mean helicity $\langle \mathbf{v} \cdot \nabla \times \mathbf{v} \rangle \neq 0$ for the $\alpha$-effect to be effective. The flow in planets and stars is governed by the balance between differential rotation and convection, whereas the influence of magnetic field (Lorentz force) is believed to be weak. The onset of this *kinematic* dynamo becomes a linear instability problem: the initial seed field will be exponentially amplified if advection of magnetic field by the flow overweights magnetic field dissipation. This process is successful if conductivity of the fluid is large enough. Therefore, the magnetic Reynolds number $R_m$, ratio of advection of magnetic field to magnetic diffusion, should be sufficiently large: $R_m \sim O(10)$ (Davidson, 2001). The main question to be answered in planetary dynamo theory is how a statistically steady velocity field can lead to magnetic fields of simple, steady, large structure. This velocity field is sustained by other forces, for example, can be driven by convection. Yet outside of a planetary or solar context, a second question becomes important: can this velocity field be maintained by the induced magnetic field (dynamically consistent dynamo)? This concerns dynamo action in magnetohydrodynamic turbulence. Turbulent flows often have no mean helicity since the flow is disorganised. Magnetic field is then of the same scale as turbulent motion, and the measure of magnetic field is rather magnetic energy $\mathbf{B}^2$ than mean magnetic field $\langle \mathbf{B} \rangle$. Batchelor (1950) considered the competition between the magnetic flux tubes stretching by $\mathbf{v}$ and Ohmic dissipation of the small-scale flux tubes and suggested that a seed field will grow if viscosity is larger than magnetic diffusivity $\nu > \lambda$ and decay if $\nu < \lambda$. Recent numerical results of Schekochihin et al. (2007) showed that although high $R_m$ is
vital for magnetohydrodynamic turbulence to be sustained, at low \( Pm \) even higher \( Rm \) are needed. Partially the explanation is given by considering the lengthscales of dynamo: it grows on the viscous scale at \( Pm > 1 \) and on the resistive scale for \( Pm < 1 \). In reality, this problem is highly nonlinear: infinitesimally small magnetic fields would be damped, but an initial magnetic field of sufficient amplitude could yield a configuration which maintains both the turbulence and the magnetic field (as shown by Riols et al., 2013, in shearing sheet simulations). This dynamo was found to be highly intermittent and its life times are very sensitive to the amplitude of initial condition. Later Riols et al. (2015) found that in shearing box of the size \( L_x \times L_y \times L_z = (0.7 \times 20 \times 2) \) it is hard to sustain dynamo at \( Pm < 1 \), even for relatively large \( Rm \). Nauman and Pessah (2016) found dynamo for \( Pm < 1 \), provided that the vertical length of the box is large enough \( L_z/L_x \geq 8 \). They suggested that increasing \( L_z \) helps the separation between the outer scale of the flow (which is proportional to \( L_z \)) and allows for a more complex flow pattern. Increasing \( L_z \) in this sense is equivalent to increasing \( Rm \).

An important condition for dynamo is that the flow is sufficiently complicated. Axisymmetric flows like Keplerian \( \Omega \sim r^{-3/2} \) do not yield dynamo action, according to Cowling’s theorem (Davidson, 2001). Small turbulence created via MRI-action is a flow of sufficient complexity to make dynamo possible. Yet we arrive at the "chicken-and-egg" problem: where does the initial field that triggers MRI come from? Instead of a linear process, a nonlinear finite amplitude dynamo is a probable solution. A sufficiently strong initial field can yield a configuration that maintains both magnetic field and turbulence. This type of dynamo has been explored in shearing box simulations (Brandenburg et al., 1995; Hawley et al., 1996; Fromang et al., 2007; Yousef et al., 2008; Riols et al., 2013; Bhat et al., 2017), however global calculations are lacking. On the other hand, existing dynamos in Taylor–Couette geometry rely on velocity profile subject to hydrodynamic instability (Willis and Barenghi, 2002a), which is unlikely the case in Keplerian flows, or depend on uniform axial magnetic fields of zero net flux (Ebrahimi and Blackman, 2016). In this chapter, I will show that fluid turbulence and magnetic field can be mutually generated in a nonlinear fashion given \( Rm \) is high enough.

6.1 Emergence of the dynamo

6.1.1 Setup and initial conditions

As initial condition to trigger dynamo action, I took a turbulent flow state, that arose due to magnetorotational instability at \( Re = 10^4, Pm = 1 \). The turbulence was caused by external azimuthal magnetic field of the strength \( Ha = 432 \) and shape \( B = B_0(r_i/r)e_\phi \). The external azimuthal magnetic field was then switched off and two runs were performed: the first one without change in parameters (i.e. at \( Pm = 1, Rm = 10^4 \)) and the second one with 10 times larger \( Pm = 10 \) (\( Rm = 10^5 \)). The resolution for the first run was \( N \times K \times M = 280 \times 400 \times 600 \) and was decreased
Further as the kinetic energy of the flow decayed. The resolution of the $Rm = 10^5$ run reached $N \times K \times M = 480 \times 512 \times 1024$. Here $N$ - number of radial points, $K, M$ - number of axial and azimuthal modes, respectively. The domain length remained unchanged ($L_z = 1.4$), and so did the radius ratio ($r_i/r_o = 0.5$).

### 6.1.2 Primary results

After the external magnetic field is switched off, at first kinetic energy of the flow for both $Rm$ begins to decay (Fig. 6.1a). The kinetic energy of laminar Couette flow was subtracted for clarity. On the contrary, the magnetic energy for $Pm = 10$ ($Rm = 10^5$) case increases immediately (Fig. 6.1b). Magnetic field at first feeds on the energy on the flow, so kinetic energy decays even more rapidly than for $Pm = 1$ ($Rm = 10^4$). Nevertheless, after about 200 inner cylinder rotations magnetic field grows strong enough to give feedback on the flow field. Kinetic energy begins to grow, and after 400 inner cylinder rotations both velocity and magnetic field are close to saturation. This does not happen at $Rm = 10^4$ (blue line on the Fig. 6.1), which shows more or less smooth decay both in kinetic and magnetic energy far beyond 1500 inner cylinder rotations (see Figure 6.1b). The latter case is clearly not a dynamo.

However, the growth in magnetic field and its subsequent saturation indicates that turbulence at $Pm = 10$ ($Rm = 10^5$) becomes self-sustained. To visualize this turbulence, instantaneous isosurfaces and contours of velocity and magnetic fields were plotted (Fig. 6.2). Both velocity and magnetic field have small-scale structure in all three directions, with magnetic field structures (Fig. 6.2b) being smaller than velocity field pattern (Fig. 6.2a). Meridional slices show some clustering of vortices at the inner cylinder but otherwise no clear boundary layer structure. Neither
velocity nor magnetic field exhibit periodicity or large-scale vortices comparable with the system's dimensions.

As one would expect from the snapshots in Figure 6.2, the energy in energy spectra is not concentrated around large scales but instead great amount of energy is distributed along small wave numbers. Energy spectra in Fig. 6.3a confirm this idea.
The spectra both in $z$ (axial wave number $k$) and $\phi$ (azimuthal wave number $m$) are almost flat over about two orders of magnitude in wave number before they begin to decay. The spectra are computed at mid-gap ($r = 1.5$) only, but spectra averaged over the entire volume are quite similar. Closely related to these spectra are the magnetic and velocity lengthscales

$$l_B^2 = \frac{\int_{V_d} B^2 \, dV_d}{\int_{V_d} (\nabla \times B)^2 \, dV_d}, \quad (6.1)$$

$$l_u^2 = \frac{\int_{V_d} u^2 \, dV_d}{\int_{V_d} (\nabla \times u)^2 \, dV_d}, \quad (6.2)$$

integrated over the whole volume of the domain $V_d$. The instantaneous values from the end of the run give $l_u = 4.5 \cdot 10^{-2}$ and $l_B = 9.8 \cdot 10^{-3}$, consistent with energy spectra drop off in Figure 6.3, and also with $O(Rm^{-1/2})$ lengthscale at which small-scale dynamo is expected to operate (Dormy and Soward, 2007). The related time scales, corresponding to these lengthscales, $Re \cdot l_u^2 \approx 20$ and $Rm \cdot l_B^2 \approx 10$, are both short compared to the integration time $t = 1500$ in Figure 6.1, indicating that we observe a permanent, persistent dynamo and not transient turbulence.

In accretion disc context a particularly interesting quantity is the angular momentum transport. As done in the previous chapters, in Taylor-Couette system it can be measured easily as torque at the cylinders, equal on average for steady rotation. Figure 6.3b shows torque normalized by laminar torque of the basic state $G_{\text{lam}}$. The run at $Pm = 1$ ($Rm = 10^4$) results in decaying turbulence and values of torque approaching laminar $G/G_{\text{lam}} = 1$. On the contrary, $Pm = 10$ ($Rm = 10^5$) after substantial transients settles to the torque value of order $O(10)$ of the laminar torque.
Fig. 6.4: Mean magnetic field $B_\phi$ and $B_z$ at $P_m = 10$ ($Rm = 10^5$). Zero mean field $B_r$ is ensured by insulating boundary conditions for magnetic field. Components of magnetic field are normalized with $\sqrt{2E_{mag}}$ and averaged over more than 1000 inner cylinder rotations.

It is worth mention that, although kinetic energy in Figure 6.1 saturates at the value similar to initial condition with imposed field, torque is about twice as large. This can be explained by the contribution to angular momentum from Maxwell stresses from magnetic field. Second, if the values of kinetic energy and torque are normalized by their maximum, they exhibit fluctuations of similar amplitude.

6.2 Properties of the dynamo

6.2.1 Mean magnetic field and magnetic flux

Highly turbulent and small-scale nature of both flow and magnetic field in the Figure 6.2 raises the question whether the flow possesses a non-vanishing mean-field. This is essential to distinguish between mean-field dynamo described by $\alpha$- and $\Omega$-effects and small-scale dynamo that is local and is supported by the interaction of small scales. In the Figure 6.4 the mean profiles of $B_\phi(r)$ and $B_z(r)$ are plotted. They were averaged over $z$, $\phi$, and time. Both $B_\phi$ and $B_z$ are surprisingly uniform in radius in the bulk of the flow if compared to instantaneous snapshots (Fig. 6.2); the strong gradients at the walls are probably induced by the insulating boundary conditions. The mean magnetic field is two or three orders in magnitude smaller than the square root of mean magnetic energy from the Figure 6.1b:

$$|B| \sim \sqrt{2E_{mag}} = 1.587 \cdot 10^3.$$  \hspace{1cm} (6.3)

Moreover, integrating also over radius we can estimate mean magnetic flux through the system. Mean axial component of magnetic field $B_z$ creates magnetic flux in $z$-direction, which can be calculated as integral over circle slice $rdrd\phi$:

$$F_z = \int_0^{2\pi} \int_{r_i}^{r_o} B_z(r) \, dr \, d\phi = 2\pi \int_r rB_z(r) \, dr.$$  \hspace{1cm} (6.4)
Mean azimuthal component of magnetic field $B_\phi$ creates magnetic flux in $\phi$-direction, which can be calculated as integral over cylinder slice $dzdr$:

$$F_\phi = \int_{0}^{2\pi/k_0} \int_{r_i}^{r_o} B_\phi(r) \, dr \, dz = 2\pi/k_0 \int_r B_\phi(r) \, dr.$$  \hfill (6.5)

Numerical integral over radius $r$ of the previously averaged over $z$ and $\phi$ mean magnetic fields give axial magnetic flux $\langle F_z \rangle = -5.2886$ and azimuthal magnetic flux $\langle F_\phi \rangle = 1.5639$. These values are very small compared with (6.3) and they likely come from statistical error from taking an average. Therefore, in addition to the absence of an imposed mean-field component, the dynamo has also zero mean flux.

### 6.2.2 Balance of Lorenz and Coriolis forces

Dormy (2016) distinguished weak-field and strong-field dynamos in the context of Earth magnetohydrodynamic flows. Weak-field dynamos are characterized by weaker dipole magnetic fields and balance of pressure gradient and Coriolis force, Lorentz force being smaller. On the contrary, strong-field dynamos are multi-polar and result from the balance of Coriolis and Lorenz force. The ratio of Lorentz to Coriolis forces was defined by Dormy (2016) as Elsasser number:

$$\Lambda = \frac{(\mu \rho)^{-1} \nabla \times \mathbf{B} \times \mathbf{B}}{2\Omega \times \mathbf{v}'}.$$  \hfill (6.6)

Dormy, 2016 considered velocity $\mathbf{v}'$ in the rotating frame, and all simulations in this chapter were performed in inertial frame. The transition to inertial frame $\mathbf{v}$ is straightforward: $\mathbf{v}' = \mathbf{v} - \Omega_f \times \mathbf{r}$. The angular velocity profile between the two cylinders is

$$\Omega(r) = \frac{1}{1 + \eta} \left[ \left( \frac{\mu}{\eta} Re - \eta Re \right) + \frac{\eta}{(1 - \eta)^2} (Re - \mu Re) \frac{1}{r^2} \right] = C_1 + \frac{C_2}{r^2}. \hfill (6.7)$$

Given $C_2/r^2$ is relatively small, the speed of rotation frame can be defined as $\Omega_f \approx C_1$. A rough estimate for Elsasser number can be obtained from the premultiplied magnetic spectra from Fig. 6.5b,d:

$$\Lambda = \frac{(\mu \rho)^{-1} \nabla \times \mathbf{B} \times \mathbf{B}}{2\Omega \times (\mathbf{v} - \Omega_f \times \mathbf{r})} \sim \frac{\sum_k k E_B}{C_1^2} \sim \frac{\sum_m m E_B}{C_1^2}.$$  \hfill (6.8)

Integration over the data from Fig. 6.5b,d gives $\Lambda_k = 4.4433$ for $k$-based estimate and $\Lambda_m = 1.1386$ for $m$-based estimate. Note that these estimates are valid only at the mid gap $r = 1.5$.

More precise estimates based on the average angular velocity of the cylinders as the frame rotation $\Omega_f = (\Omega_i + \Omega_o)/2$ and also direct integration of the expression (6.6) over $r, \phi, z$ give $\Lambda = 1.131$. This indicates that the dynamo is in the strong field regime and Lorenz force is in balance with Coriolis force.
Fig. 6.5.: Premultiplied spectra of kinetic energy (left) and magnetic energy (right) as a function of axial (up) azimuthal (down) wave number and radius at $P_m = 10$ ($R_m = 10^5$). Kinetic and magnetic energy are normalized with $\Omega_i^2 r_i^2$.

### 6.2.3 Premultiplied spectra

A typical feature of wall-bounded turbulence is that the size of energy containing eddies increases with the distance from the wall (Jiménez, 2012). Accretion disk flows lack solid boundaries and are unlikely to share the same properties with wall-bounded flows. 2D energy spectra as function of wave number and distance from the wall help to prove this point. In the Figure 6.5 contours of premultiplied kinetic (left) and magnetic (right) energy spectra are given: (a) - $kE_{\text{kin}}(k, r)$, (b) - $kE_{\text{mag}}(k, r)$, (c) $m_r E_{\text{kin}}(m_r, r)$, (d) $m_r E_{\text{mag}}(m_r, r)$. The x-axis is represented by real axial $k = k_0 k'\Omega_i$ or the real azimuthal wavenumber $m_r = m/r_{\text{mid}}$, $r_{\text{mid}} = 1.5$ are used; the y-axis is wall-normal distance from the wall, given by the radial coordinate. The most energetic structures are contained near the inner cylinder with $r = 1$, which is not surprising since inner cylinder rotates faster $Re_i > Re_o$. The clustering of intense fluctuations of velocity and magnetic field is also observed in Figure 6.2 as small but bright vortices near the inner cylinder. However, the axial or azimuthal wave number of most energetic structures does not change with radius. For axial spectra in Fig. 6.5(a-b) this value is around $k \approx 40$, for azimuthal spectra in Fig.
Fig. 6.6: (a) Mean velocity profile, $Pm = 10 \ (Rm = 10^5)$: black dashed - quasi-Keplerian, red solid - modified by dynamo turbulence. (b) Mean angular velocity profile, corresponding to (a).

6.5(c-d) - $m_r \in [5, 10]$. From these figures we can conclude that the dynamo is not strongly affected by what happens near the wall, as the size of the turbulent structures does not change with radius (no attached eddies observed).

6.2.4 Mean velocity profile

The presence of turbulence modifies the mean velocity profile, presented in the Figure 6.6a. Velocity was averaged spatially over $z, \phi$, and also in time over the steady period of dynamo saturation (Fig. 6.1). The boundary layers at $r \lesssim 1.05$ and $r \gtrsim 1.95$ show high velocity gradients at the boundary. Since $G/G_{\text{lamin}} = O(10)$, the gradient of velocity at the boundary must be of the same order. As a comparison, the laminar quasi-Keplerian profile is given. Surprisingly, the respective change in angular velocity in Figure 6.6b is relatively small. Angular velocity even matches Keplerian profile at $r = 1.5$. Both angular velocity and velocity profiles are even more hydrodynamically stable than basic Taylor–Couette profile. It is not entirely clear why the solution adjusts to have so flat profile of $U_\phi$ in the interior and $\Omega$ remains similar to laminar flow. For a discussion in non-magnetized flows see (Brauckmann and Eckhardt, 2017).

6.2.5 Forcing Keplerian profile

Taylor–Couette flow between two cylinders and Keplerian flows of accretion discs are similar in many aspects. Nevertheless, the presence of walls in the Taylor-Couette setup is a crucial difference between the two geometries and may influence some aspects of nonlinear equilibrium. Particularly, the average angular velocity of accretion disks can not deviate much from Keplerian. If gravity from the central object is the dominant force, it will always enforce $\Omega \sim r^{-3/2}$ no matter how intense the turbulence is.

The resulting mean profile of the Taylor-Couette dynamo is very different from quasi-Keplerian (Fig. 6.6a). I performed an additional simulation, where the flow is
forced to be quasi-Keplerian. This can be done by setting in the code the mode $u_{00}$, contributing to the mean velocity profile to zero and is equivalent to introducing a body force of $F_b = -\lambda(t)u_{00}(t, r)e_\phi$ that drives components of $u$ back to quasi-Keplerian profile. The energy plot on the Figure 6.7 show the flow evolution after the forcing term was introduced. The dynamo continues to exist, and magnetic energy even increases. Note that the magnetohydrodynamic turbulence is statistically stationary in axial direction; travelling waves are not observed since the energy containing in positive axial modes is on average equal to energy in negative axial modes $E_{k>0} \approx E_{k<0}$. This is true from both the forced and the unforced case. At the same time, the dynamo exhibits similar stress distribution in radius for both initial dynamo simulation (Fig. 6.8a) and forced simulation (Fig. 6.8b). The peaks in Reynolds stresses close to the walls disappear in comparison to the initial simulation. However, the ratio between Maxwell and Reynolds stresses remains basically unchanged. Maxwell stress dominates angular momentum transport throughout the bulk of the flow, and Reynolds stresses account only to 10% of the total transport. Viscous stresses are small and constant in the bulk of the flow. In the forced run (Fig. 6.8b) there is of course a stress contribution from the forcing term $F_b$ but it does not affect much the Maxwell and Reynolds stress distributions.

6.2.6 Critical magnetic Reynolds number
It is important to know at which parameter values the dynamo disappear, in order to probe the possibility of its operation in accretion disks. Since $Pm = 10 (Rm = 10^5)$ results in dynamo and $Pm = 1 (Rm = 10^4)$ relaminarises, there is likely a critical $Rm$ (or $Pm$), where the dynamo ceases to exist. I approach this problem fixing $Re = 10^4$ and varying $Pm$ (and therefore $Rm$). Several simulations show that the
Fig. 6.8.: Stresses at $Pm = 10$ ($Rm = 10^5$) (a) initial dynamo run, Fig. 6.1, (b) after enforcing quasi-Keplerian profile by setting $u_{00} = 0$.

Fig. 6.9.: Critical $Rm$ for the dynamo. Reynolds number is fixed to $Re = 10^4$, $Pm$ varied.

dynamo dies out between $Rm = 4.5 \cdot 10^4$ and $5 \cdot 10^4$ (Figure 6.9). A more precise definition of $Rm_c$ is a subject for future work.

6.3 Discussion

We have seen that magnetorotational turbulence is able to sustain itself even in the absence of imposed large scale magnetic fields. The resulting magnetic field is small-scale and shows no decay over more than one thousand inner cylinder rotations. The respective mean magnetic field is essentially zero. With Lorentz force being larger than Coriolis force, this dynamo belongs to the so-called strong field dynamos (Dormy, 2016). Interestingly, although turbulence modifies significantly the mean profile of azimuthal velocity, the mean angular velocity profile remains very close to initial quasi-Keplerian profile and almost exactly matches it in the center of the domain (Figure 6.6). The dynamo also does not die out even if Keplerian rotation is imposed, showing similar stress distribution. The primary role in angular
momentum transport is played by Maxwell stress, being 10 times more efficient than Reynolds in both forced and non-forced setups. This forced–flow results suggest that the dynamo presented here is generic to any Rayleigh-stable differential rotation flows, including flows in accretion disks, and is not specific only to Taylor–Couette flows. This argument is also confirmed by the absence of attached eddies at the wall, implying the near-wall turbulence does not define the flow structure. The generic existence of such a dynamo suggests that accretion disks can operate with self-sustained magnetic fields, without relying on magnetic field from the central object. When the magnetic Reynolds number is varied, the dynamo eventually decays between $4.5 \cdot 10^4 < Rm < 5 \cdot 10^4$. These values are large but not unrealistic in the context of accretion discs. However, here $Rm$ and $Pm$ were varied simultaneously, while in fact in shearing box simulations both $Rm$ and $Pm$ appear to be important (Riols et al., 2015; Nauman and Pessah, 2016). In fact, in those works dynamo has been characterised by its life times, and thus the amplitude of initial condition becomes important. Further work should shed more light on the sensitivity of the dynamo to the initial condition and magnetic Prandtl number $Pm$. 
Overview and discussion

Since the seminal work of Balbus and Hawley explaining the emergence of turbulence in Keplerian flows by interaction of these flows with magnetic fields, there has been considerable interest in the magnetorotational instability (MRI) from the theoretical, numerical and experimental points of view. While observations of the flow structures in accretion disk remain an outstanding challenge, experimental studies have focused on whether nonlinear hydrodynamic turbulence is possible in quasi-Keplerian flow confined between two corotating cylinders (Ji et al., 2006; Paoletti and Lathrop, 2011; Edlund and Ji, 2014). By appropriately choosing the rotation-ratio of the cylinders, velocity profiles of the general form $\Omega(r) \sim r^q$, including $q = -1.5$ for Keplerian rotation, can be well approximated experimentally at very large Reynolds numbers $Re \sim O(10^6)$ (Edlund and Ji, 2015; Lopez and Avila, 2017). The current understanding is that if end-plates of the cylinders are designed to avoid Ekman vortices penetrating deep into the domain and modifying the velocity profile, hydrodynamic turbulence is ruled out for $Re$ as high as $10^6$ (Lopez and Avila, 2017).

At the same time, PROMISE experiment in Helmholtz Center Dresden-Rossendorf provided the first experimental evidence of MRI triggered by a combination of poloidal (axial) and toroidal (azimuthal) magnetic fields (Stefani et al., 2009) and later by a purely azimuthal magnetic field (Seilmayer et al., 2014). Although the parameters of this experiment are far from a real accretion disk (the velocity profile is far from Keplerian rotation and $Re \approx 10^3$), the striking similarity of flow structures in experiments and nonlinear simulations make these experiments a valuable framework for validation of any numerical model (Mamatsashvili et al., 2017, see chapter 4 and also). The future development of a new PROMISE facility aiming at $Rm > O(10)$ may provide data for the instability with a purely axial magnetic field as well.

Theoretical studies of MRI are usually based on the linearized Navier-Stokes equations (Velikhov, 1959; Chandrasekhar, 1961; Balbus and Hawley, 1991; Hollerbach et al., 2010) and provide a good basis for understanding the physical principles of the MRI, with tension of magnetic field lines acting on fluid elements as if they were connected by strings. The most comprehensive linear stability analysis of MRI was done by Kirillov et al. (2014), who defined in the most general form a dispersion relation for MRI and proved that not only axial, but also azimuthal magnetic fields can destabilize Keplerian flows if the profile steepness of azimuthal magnetic field component is allowed to depart from the experimental $B_\phi \sim r^{-1}$, even for zero magnetic Prandtl number $Pm \rightarrow 0$. However, the initial motivation to estimate
angular momentum transport outward, supporting the infall of the matter to the center of accretion disk (accretion), requires nonlinear simulations of MRI.

Most nonlinear simulations of the MRI have been performed using the shearing sheet approximation, which consists of a local model of an accretion disk (Hawley et al., 1995). In this approximation, the equations are solved in a rotating frame in Cartesian geometry, with the rotation given by the linearization of the Keplerian law at a radial point in the disk. Periodic boundary conditions are assumed in all three directions, and radial shear is introduced by means of a coordinate transformation. These boundary conditions determine the geometry of the modes observed in the simulations and their saturation in the nonlinear regime (Regev and Umurhan, 2008b). This makes it hard to predict angular momentum transport. In addition, most simulations of shearing boxes neither resolve all flow scales nor implement subgrid models that capture the impact of small flow scales on the larger scales.

Two long-standing observational problems are extremely important for assessing nonlinear simulations. First, little is known about the shape and intensity of magnetic fields in accretion disks. While axial magnetic fields can be imposed by the central object of the disk (white dwarfs, neutron stars and black holes have extremely strong magnetic fields), azimuthal magnetic fields can arise due to the stretching of axial magnetic field lines by differential rotation of disk fluid flow. Donati et al. (2005) reported a surface magnetic field of approximately 1 G with a significant azimuthal component close to the center of the protostellar accretion disk FU Orionis. This is about 2000 times larger than magnetic field on the Earth’s surface. Second, even less is known about the ratio of viscosity to magnetic diffusivity in accretion disks (the magnetic Prandtl number $Pm$). The available estimates of this important parameter in magnetohydrodynamic turbulence are based on electron-ion collisions in plasmas and rely primarily on the temperature. Temperature can vary a lot locally in accretion disks (central regions being hotter, peripheral regions colder), and thus both low-$Pm$ ($\sim O(10^{-8})$) and high-$Pm$ (up to $10^4$) regions may occur (Brandenburg and Subramanian, 2005).

These two problems result in significant difficulties in constructing a realistic Taylor-Couette experiment, in the lab or numerically. The standard version of MRI with axial magnetic field requires magnetic Reynolds numbers $Rm \sim O(10)$. Liquid metals, being most used material in magnetohydrodynamic experiments, possess small $Pm \in [10^{-6}, 10^{-5}]$. This implies rotation rates to be as high as $Re \geq O(10^6)$ (the two parameters are related by $Rm = Pm Re$) and is extremely hard to achieve in Taylor-Couette geometry. Then, if an azimuthal magnetic field is involved, MRI arises already at $Re \sim [10^2, 10^3]$ for steep velocity profiles close to Rayleigh line. It was found by Hollerbach et al., 2010 that for steep velocity profiles close to the Rayleigh line the AMRI is governed by the Reynolds $Re$ number, instead of $Rm$. 

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However, if flatter profiles like Keplerian are considered the MRI scales with $Rm$ again and the constraint of Reynolds number holds again.

In this work, the focus was on bringing together the results of experimental and astrophysical MRI simulations. The azimuthal magnetic fields $B \sim r^{-1}$ considered here have the same shape as in PROMISE, and allow for non-axisymmetric disturbances to dominate. This is favorable for dynamo action and self-sustained turbulence. The uncertainty in $Pm$ of accretion disks requires a study of a broad range of $Pm$ in order to make any prediction of transport properties, and here a wide range of $Pm \in [0, 1]$ has been covered - from liquid metals to highly conductive plasmas. Both experimentally relevant angular velocity profile of $\Omega \sim r^q$, $q = -1.94$, and $q = -1.5$, approximating Keplerian profile, have been explored. Our simulations were performed with a powerful spectral DNS method of Dr. Ashley P. Willis, which I have further developed and validated against published results to excellent agreement. The method allowed to compute flows up to $Re = 4 \cdot 10^4$. This allows for asymptotic predictions of transport scaling to be made that can be extrapolated to accretion disk flows. In the following, I will shortly summarize the main findings of this thesis.

Applying linear stability analysis I have determined the scaling of the boundaries that confine the AMRI in the form of $Re \sim Ha^\delta$. The right border of instability scales as $Re \sim Ha$ both for high and low $Pm$. For large $Pm$ case the instability is widened up to $Re \sim Ha^{0.9}$. The scaling coefficient of the left instability boundary, which sets the minimum strength of the magnetic field $Ha$ necessary to drive the instability, decreases from $\delta_L \approx 4$ (low $Pm$) to $\delta_R \approx 1.5$ (high $Pm$). This implies constraints on the applicability of AMRI for high $Pm$ as strong magnetic fields are needed to destabilize the flow. However, for $Pm = 1$ the analysis of chapter 4 shows a faster widening of the instability region with $\delta_L \approx 2.6$ because the instability becomes subcritical. The analysis of Reynolds and Maxwell stresses of linear eigenmodes suggests the relevance of the magnetic Reynolds number $Rm$ in determining the radial transport of angular momentum.

At $Pm = 1.4 \cdot 10^{-6}$ the AMRI in Taylor–Couette flow manifests itself as a wave rotating in the azimuthal direction and standing in the axial direction, thereby preserving the reflection symmetry in the latter. As Re increases, a catastrophic transition to spatio-temporal chaos occurs directly from the SW. In a range of parameters SW and chaos are both locally stable and can be realised depending on the initial conditions. The first step in this transition process is a subcritical Hopf bifurcation giving rise to an unstable relative periodic orbit, which has been computed using an analogue of the edge-tracking algorithm introduced by Skufca et al. (2006) in a shear flow model. This unstable relative periodic orbit consists of a long-wave modulation of the axially periodic pattern of the standing wave and destroys the homogeneity of
the vortical pattern. It can thus be seen as a temporally simple defect precursor of the ensuing spatio-temporal chaos. Because of the computational cost, I could not track further instabilities on the unstable branch, which may result in chaotic flow before the dynamics stabilize at a turning point \((Ha = 130 \; \text{at} \; Re = 2960)\). After the turning point defects are stable and can be computed simply by time-stepping. Such long-wave instabilities are ubiquitous in fluid flows. In linearly stable shear flows, such instabilities of traveling waves were found to be responsible for spatial localisation (Melnikov et al., 2014; Chantry et al., 2014). In fact, in pipe flow the ensuing localised solutions, which are also relative periodic orbits, suffer a bifurcation cascade leading to chaos (Avila et al., 2013). One difference is that in pipe flow the traveling waves are disconnected from laminar flow, where the standing wave of the AMRI is connected to the circular Couette flow. At \(Pm = 1\) the AMRI arises as either standing or traveling wave, with the latter being a spontaneous breaking of system symmetry. The bifurcation scenario at this \(Pm\) is more complex and involves spatially chaotic but periodic in time stable structures. In both cases, accumulation of spatial defects causes turbulence.

Angular momentum transport by AMRI turbulence is analysed in Chapter 5. Two distinct velocity profiles were considered, namely quasi-Keplerian \((\mu = 0.35)\) and almost-constant specific angular momentum \((\mu = 0.26)\). For \(Rm > O(100)\), regardless of rotation profile, the flow is unstable to the usual MRI. Transport is governed by Maxwell stresses and scales as \(\sqrt{Pm Re^2}\), so that \(\alpha_{\text{eff}} \propto \sqrt{Pm}\), consistent with Rüdiger et al. (2015). At \(Pm = 1\) we found \(\alpha_{\text{eff}} > 10^{-3}\). Hence in highly ionized disks or disk regions, the AMRI may be a vigorous source of angular momentum transport. At low \(Rm < O(100)\), instability is found only for steep profiles very close to the Rayleigh-line. Here the flow is unstable to the inductionless MRI for hydrodynamic Reynolds number \(Re \gtrsim 1000\), transport is governed by Reynolds stresses, scales as \(Re^2\) and is weak with \(\alpha_{\text{eff}} = O(10^{-5})\). Note that \(\alpha_{\text{eff}}\) is constant only if \(G \sim Re^2\), like friction in rough pipe flow (Nikuradse, 1933), but quite different from hydrodynamic turbulence in Taylor-Couette flow (Grossmann et al., 2016).

The ratio of Maxwell to Reynolds stresses is solely determined by \(Rm\) and increases from zero to one as \(Rm\) is increased. This is in agreement with Meheut et al. (2015), who performed shearing-box simulations of MRI turbulence at two magnetic Reynolds numbers \(Rm = 400\) and 2600, while varying either \(Pm\). At each \(Rm\) they found a constant, but different, ratio of Maxwell to Reynolds stress both for azimuthal and axial magnetic fields, with Maxwell stresses growing with \(Rm\).

The nonlinear results are in line with the linear analysis of Kirillov and Stefani, 2010 for the MRI with imposed helical magnetic fields. They identified the inductionless instability close to the Rayleigh line as a magnetically destabilized inertial wave, whereas the usual MRI can be interpreted as arising from an unstable magnetocori-
olis wave (Nornberg et al., 2010). In addition, Kirillov and Stefani, 2010 showed that the transfer of instability between the inertial and magnetocoriolis modes was continuous. Our data support also a similar scenario for the AMRI: for $\mu = 0.26$ and $Pm = 0.01$ a continuous transition between the two types of turbulent flow can be observed as $Rm$ increases. Thus we suggest that the dependence of scaling type on $Rm$ shall also apply to MRI in the presence of helical magnetic fields. Future liquid metal experiments planned by Stefani et al. (2017), aiming at $Rm > O(10)$, should confirm the crossover between the two flavors of MRI and transport scalings shown here.

Chapter 6 explores the possibility of MRI-turbulence to be self-sustained in the absence of imposed magnetic fields. A small-scale dynamo operating in Rayleigh-stable flows is observed for $Pm > 1$ and $Rm > 4.5 \cdot 10^4$. The existence of such dynamo suggests that accretion disks can generate self-sustained magnetic fields without relying on large-scale fields from the central object. It is a matter for future research to map out the full range of parameters where the dynamo operates. The important question here is whether the dynamo is possible for low $Pm$ as well. The physical mechanisms supporting dynamo in low-$Pm$ and high-$Pm$ regime are qualitatively different. For $Pm \geq 1$ the resistive scale of the flow (where the Ohmic dissipation takes place) is much smaller than viscous scale $l_\eta/l_\nu \sim Pm^{-1/2} \ll 1$, and dynamo is supported by random stretching of the magnetic field by the fluid motion on the viscous scale (Batchelor, 1950; Schekochihin et al., 2007). At $Pm \ll 1$, on the contrary, the resistive scale is much larger than the viscous scale $l_\eta/l_\nu \sim Pm^{-3/4} \gg 1$, and magnetic fields stretched by flow field on the viscous scale will inevitably dissipate. Although it is not entirely clear whether vortices of inertial range or outer-scale motions are responsible for dynamo at low $Pm$, it is known that higher critical $Rm$ are required (Schekochihin et al., 2007).

The MRI is a rigorous mechanism of triggering turbulence in such flows, but it is not entirely clear whether it is the main mechanism of the dynamo. Dynamo in Keplerian flows is an intermittent nonlinear process, and before was observed to have sufficiently long lifetimes only at $Pm > 1$ (Riols et al., 2015). Recently, Nauman and Pessah (2016) suggested that increasing the vertical size $L_z$ of the computational domain is equivalent to increasing effective $Rm$ of the flow and allows to observe self-sustained turbulence in quasi-Keplerian flows at $Pm < 1$. This important result may become an important framework to study turbulence in Keplerian flows in the future.

The research presented in this thesis has resolved a few issues that arose in previous studies of magnetorotational instability. Yet it also poses problems that are relevant for the field. For example, the spontaneous symmetry breaking at large $Pm$, where traveling waves arise despite the reflection and translational symmetry of the flow and magnetic field in $z$ direction, is itself an interesting physical phenomenon. An-
other question is a comprehensive study of MRI in the presence of azimuthal and axial magnetic fields (helical MRI), as performed here for the AMRI. Helical MRI with its transformation from inductionless instability of inertial modes to magneto-terioliis waves can feature transport scaling laws in the form of $G \sim \text{Re}^7 P_m^a$ analogous to AMRI. If the magnetic field is varied as $B = \beta B_0 e_\phi + (1 - \beta) B_0 e_z$, $\beta \in [0, 1]$, transition from azimuthal to axial MRI via helical fields can be observed. Such a study, however, requires considerable amount of computing power since asymptotic scalings for various $P_m$ would be the desired outcome. One more direction of research is related to implementing a realistic geometry similar to PROMISE experiment by using conducting boundary conditions on magnetic field instead of insulating, and, more importantly, considering end-walls. Finally, a comparison of flows in the shearing sheet box to global simulations in Taylor–Couette geometry is essential for better understanding of turbulence in Keplerian flows.

Taylor–Couette flow is a physical system and allows to obtain physically meaningful results for a variety of problems. I believe that Taylor–Couette flow can be used as a promising model for studies of rotating astrophysical flows, and I hope that this thesis makes a step in this direction.
Bibliography


Bibliography


Appendices
List of numerical simulations for transport analysis

Comments. Time is given in viscous time units \((d^2/\nu)\). It can be easily scaled with rotation period of the inner cylinder \(1/\Omega_i\). Consider \(r_i = d = 1\) in the simulations:

\[
T_{\text{visc}} = \frac{t_{\text{real}}}{d^2/\nu}, \\
T_{\text{in. cyl.}} = T_{\text{visc}} \frac{\Omega_i r_i d}{\nu} = \frac{t_{\text{real}}}{d} \Omega_i r_i = \frac{t_{\text{real}}}{1/\Omega_i},
\]

Conversion of time from viscous units to resistive units \((d^2/\lambda)\) is also easy:

\[
T_{\text{res}} = T_{\text{visc}}/Pm = \frac{t_{\text{real}}}{(d^2/\nu)} \frac{1}{(\nu/\lambda)} = \frac{t_{\text{real}}}{d^2/\lambda}.
\]

The resolution of each run is \(N \times K \times M\).

Tab. A.1.: List of simulations for transport analysis from chapter 5

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\[ Pm = 0, \mu = 0.26 \]

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<td>954</td>
<td>480 × 200 × 70</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Tab. B.1.: Maximum growth rate lines

<table>
<thead>
<tr>
<th>$P_m$</th>
<th>Maximum growth rate lines ( \mu = 0.26 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Re &lt; 700 ) ( Re = 2.705 Ha^{0.918} )  ( Re &gt; 700 ) ( Re = 1.916 Ha^{1.005} )</td>
</tr>
<tr>
<td>( 10^{-1} )</td>
<td>( Re &lt; 2700 ) ( Re = 6.208 Ha^{0.966} )  ( Re &gt; 2700 ) ( Re = 6.027 Ha^{1.006} )</td>
</tr>
<tr>
<td>( 10^{-2} )</td>
<td>( Re &lt; 24000 ) ( Re = 4.100 Ha^{1.199} )  ( Re &gt; 24000 ) ( Re = 15.677 Ha^{1.028} )</td>
</tr>
<tr>
<td>( 10^{-3} )</td>
<td>( Re &lt; 42000 ) ( Re = 0.452 Ha^{1.649} )</td>
</tr>
<tr>
<td>( 1.4 \cdot 10^{-6} )</td>
<td>( Re &lt; 70000 ) ( Re = 0.586 Ha^{1.581} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$P_m$</th>
<th>Maximum growth rate lines ( \mu = 0.35 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Re &lt; 760 ) ( Re = 2.763 Ha^{0.907} )  ( Re &gt; 760 ) ( Re = 1.876 Ha^{1.010} )</td>
</tr>
<tr>
<td>( 10^{-1} )</td>
<td>( Re &lt; 4200 ) ( Re = 7.166 Ha^{0.941} )  ( Re &gt; 4200 ) ( Re = 5.776 Ha^{1.012} )</td>
</tr>
<tr>
<td>( 10^{-2} )</td>
<td>( Re &lt; 39000 ) ( Re = 21.507 Ha^{0.954} )  ( Re &gt; 39000 ) ( Re = 15.924 Ha^{1.027} )</td>
</tr>
</tbody>
</table>
Parameters and abbreviations

Constants and fluid properties:

\begin{align*}
\nu & \quad \text{kinematic viscosity} \\
\rho & \quad \text{density} \\
\lambda & \quad \text{magnetic diffusivity} \\
\mu_0 & = 4\pi \cdot 10^{-7} \text{ Hm}^{-1} \\
G_{gr} & = 6.674 \cdot 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \text{s}^{-2} \\
\end{align*}

Basic parameters and variables:

\begin{align*}
\begin{array}{ll}
\rho & \quad \text{mass of a fluid element} \\
M_c & \quad \text{mass of the central object} \\
\xi & = (\xi_r, \xi_\phi) \\
v & = (v_r, v_\phi, v_z) \\
u & = v - V e_\phi \\
B & = (B_r, B_\phi, B_z) \\
b & = B - B_0 \\
p & \quad \text{pressure} \\
\Omega & \sim r^q \\
q & = \ln \Omega / \ln r \\
L & \sim r^2 \Omega \\
\omega & = \gamma - i\sigma \\
\end{array}
\end{align*}

\begin{align*}
\sigma & \quad \text{growth (decay) rate of perturbations} \\
E & \quad \text{energy} \\
E_{\text{kin}} & \quad \text{kinetic energy} \\
E_{\text{mag}} & \quad \text{magnetic energy} \\
F & \quad \text{magnetic flux} \\
\mathbf{v}' & = \mathbf{v} - \mathbf{\Omega_f} \times \mathbf{r} \\
v_A & = \frac{B}{\sqrt{\mu_0 \rho}} \\
\end{align*}
Parameters of Taylor-Couette setup:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r_i(o) )</td>
<td>radius of inner (outer) cylinder</td>
</tr>
<tr>
<td>( d = r_o - r_i )</td>
<td>gap between cylinders</td>
</tr>
<tr>
<td>( L_z = 2\pi/k_0 )</td>
<td>minimum length of the cylinders</td>
</tr>
<tr>
<td>( k_0 )</td>
<td>minimum natural axial wave number (geometrical parameter)</td>
</tr>
<tr>
<td>( m_0 )</td>
<td>minimum natural azimuthal wave number (geometrical parameter), here ( m_0 = 1 )</td>
</tr>
</tbody>
</table>

\[ V_d = [r_i, r_o] \times [0, 2\pi] \times [0, L_z] \]

\( \Omega_{i(o)} \)

\( \eta = r_i/r_o \)

\( \mu = \Omega_o/\Omega_i \)

\( V(r) = \Omega(r) = C_1 r + C_2/r \)

\( B_0 = B_0 e_z \)

\( B_0 = B_0 (r/r_i) e_\phi \)

\( B_0 = B_0 (e_z + \beta (r/r_i) e_\phi) \)

\( G_{i(o)} = G = \partial L/\partial t \)

\( \Omega_{\text{lam}} \)

\( J^\omega \)

\( J_{\text{lam}}^\omega \)

Computational parameters:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N \times K \times M )</td>
<td>number of radial points, axial and azimuthal Fourier modes</td>
</tr>
<tr>
<td>( k = k_0 k' )</td>
<td>axial wave number</td>
</tr>
<tr>
<td>( m = m_0 m' )</td>
<td>azimuthal wave number</td>
</tr>
<tr>
<td>( k', m' )</td>
<td>number of a Fourier mode</td>
</tr>
<tr>
<td>( A_{k,m}(r) )</td>
<td>Fourier coefficient of the mode with axial wave number ( k ) and azimuthal wave number ( m )</td>
</tr>
<tr>
<td>( Nl )</td>
<td>nonlinear term of equation</td>
</tr>
<tr>
<td>( Np )</td>
<td>nonlinear term of equation with pressure gradient absorbed</td>
</tr>
<tr>
<td>( I )</td>
<td>influence matrix</td>
</tr>
</tbody>
</table>
Units and dimensionless parameters:

\[ d, \frac{d^2}{\nu}, B_0 \]

\[ (\mu_0 \rho)^{0.5} \nu / d \]

\[ Re = \Omega_i r_i d / \nu \]

\[ Rm = \Omega_i r_i d / \lambda \]

\[ Pm = \nu / \lambda \]

\[ Ha = B_0 d / \sqrt{\mu_0 \rho \nu \lambda} \]

\[ Ro = (r / 2 \Omega) \partial_r \Omega \]

\[ Rb = (r / 2 B_\phi r^{-1}) \partial_r (B_\phi r^{-1}) \]

\[ \Lambda = (\mu_0 \rho)^{-1} \nabla \times B \times B / (2 \Omega \times \nu') \]

\[ S = v_\lambda d / \lambda \]

\[ Re_c, Ha_c \]

\[ Re' = Re - Re_c \]

\[ a \]

Abbreviations:

MRI magnetorotational instability

AMRI azimuthal magnetorotational instability (azimuthal magnetic field)

HMRI helical magnetorotational instability (axial and azimuthal magnetic field)

SMRI standard magnetorotational instability (axial magnetic field)

SW standing wave

TW traveling wave

DNS direct numerical simulations

PPE Pressure-Poisson equation

Other:

\[ \delta \]

scaling of instability boundaries \( Re \sim Ha^\delta \)

\[ \delta_{L(R)} \]

scaling of left (right) instability boundary
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta Ha$</td>
<td>instability widening</td>
</tr>
<tr>
<td>$\nu_t$</td>
<td>turbulent viscosity (in Shakura-Sunyaev theory parameterized with sound speed and disk height $\nu_t = \alpha c_s H$)</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\alpha$-parameter of Shakura-Sunyaev theory</td>
</tr>
<tr>
<td>$\nu_{\text{eff}}$</td>
<td>effective (turbulent and laminar) viscosity</td>
</tr>
<tr>
<td>$\alpha_{\text{eff}}$</td>
<td>$\alpha$-parameter of the effective velocity (here $\nu_{\text{eff}} = \alpha_{\text{eff}} \langle \Omega \rangle d^2$)</td>
</tr>
<tr>
<td>$\alpha_D$</td>
<td>$\alpha$-effect from dynamo theory</td>
</tr>
<tr>
<td>$l_u, l_B$</td>
<td>velocity and magnetic lengthscales of the dynamo</td>
</tr>
</tbody>
</table>
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4.11 Time series of radial velocity at the point \((r, \phi, z) = (1.5, 0, 0)\), \(Pm = 1\): (a) 2-frequency oscillation at \(Re = 250\), \(Ha = 150\); (b) 1-frequency oscillation at \(Re = 350\), \(Ha = 203\); (c) chaotic solution at \(Re = 500\), \(Ha = 295\). Velocity is normalized with the rotation speed of the inner cylinder \(\Omega_i r_i\), time is scaled with rotation period of the inner cylinder \(1/\Omega_i\).

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5.2 Maxwell (dashed lines) and Reynolds (solid lines) stresses along the radius, normalized by the total angular velocity current \(J^\omega\) at \(\mu = 0.26\). Three cases are shown: \([Pm, Re] = [1.4 \cdot 10^{-6}, 2 \cdot 10^4] \ (Rm = 2.8 \cdot 10^{-2}, \ cyan), [10^{-2}, 10^{4}] \ (Rm = 10^2, \ red)\) and \([1, 6 \cdot 10^{3}] \ (Rm = 6 \cdot 10^3, \ black)\). Stresses have a prefactor of \(r^2\), as in (5.16). Viscous contribution can be obtained by subtracting Maxwell and Reynolds from 1.

5.3 \(Rm\)-dependence of Maxwell and Reynolds stresses at the mid-gap for \(\mu = 0.26\) (a) and \(\mu = 0.35\) (b). Different colors corresponds to data with different magnetic Prandtl number: \(Pm = 10^{-3}\) (blue), \(Pm = 10^{-2}\) (red), \(Pm = 10^{-1}\) (green) and \(Pm = 1\) (black), as in Fig. 5.1. Here solid and dashed lines denote the Reynolds and Maxwell stress contributions, respectively, normalized by their sum (i.e. excluding the viscous contribution).

6.1 (a) Kinetic energy of the flow. (b) Magnetic energy. The short segment of black line at \(t \leq 50\) demonstrates the level of energy of the initial condition, the blue dashed line refers to \(Pm = 1\) \((Rm = 10^4)\), the red solid line - to \(Pm = 10\) \((Rm = 10^5)\). Time is normalized with the inner cylinder rotation frequency \(1/\Omega_i\). Kinetic and magnetic energy are normalized with \(\Omega_i^2 r_i^2\).

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