GÖDEL'S INCOMPLETENESS THEOREMS
WITH CONCATENATION INSTEAD OF ADDITION AND MULTIPLICATION
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In [6] we have replaced the axiom system of Gödel with its use of 0,1,+,<,x by a simple system which uses as its only free variable the following predicate E:

xEy ↔ y has the digit 1 in the xth position in the binary representation

(Counting the positions starts with position "0" from the righthand side.)

Nearer at hand than E is the use of the concatenation νx = λy xy (for the usual binary representations) and the two binary successor functions

f0* = λx x0 and f1* = λx x1.

We exclude 0 from the natural numbers because otherwise the concatenation function would not be associative. For example (10)0 would be 100, but 1(00) would be 10, if we identify 0...0 as usual with 0. The set of natural numbers without 0 is called B. Like in [6] we do not use an induction scheme in the axiom system.

Notice, we use the usual binary representation, not the representation of [4].

"∨" means "either...or". If we use it several times like in the first axiom, we mean that exactly one case is true.

Because of optical reasons we write x0 instead of f0(x) in the axiom system. But in fact 0 is not a variable. The free variables of the axiom systems are 1, ν and f0, where 1 is a individual variable, ν is a two-place function variable and f0 is a one-place function variable. Therefore, the cases in A2,A3,A4 that appear to be special cases for 0 are necessary.

AXIOM SYSTEM A:

A1 ∀x (x=1 ∨ ∃u x=u0 ∨ ∃u x=u1)
A2 ∀xyz (xz = yz ∨ xz = zy ∨ x0 = y0 → x = y)
A3 ∀xy (xy ≠ x ∧ x0 ≠ x)
A4 ∀xyz x(yz) = (xy)z ∧ ∀xy x(y0) = (xy)0
A5 ∀xyab (xy = ab → yENDb ∨ bENDy)

xENDy is meaning x=y ∨ ∃z y=zx.

In this case we call x an end of y. Please notice that the single end of 1000 (in binary representation) is 1000 itself and the only ends of 1001 are 1 and 1001.

DEFINITION.

1. A first order formula B is called C-formula (concatenation formula) iff it contains as free variables only individual variables and ν and f0.

We write xy sometimes instead of ν(x,y) and f1(x) instead of ν(x,1).

If we talk about variables we mean from now on only individual variables different from 1. As variables we use I, II, III,.... The number of strokes is called index. A C-formula is called n-place iff it contains exactly n different variables free.

2. If the n-place C-formula B has as distinct free variables x1,...,xn (ordered according to growing index), we write B[x1,...,xn] instead of B. The denotation B(T1,...,Tn) is used for the formula arising from B[x1,...,xn] by substituting Ti for xi (1≤i≤n) simultaneously (for any terms T1,...,Tn).

3. A term which can be generated from 1 by a finite number of transitions from T
to \( f_1(T) \) \((i=0,1)\) is called binary representation. For any \( n \) of \( \mathbb{N}_1 \) we have exactly one binary representation \( n^* \).

4. In any model \( I \) on \( D \) the interpretations of 1, \( f_0, v \) (in bold letters) are denoted by 1, \( f_0, v \) (in usual letters). Instead of \( v(x,1) \) we write sometimes \( f_1(x) \). Let \( I^* \) be the standard model on \( \mathbb{N}_1 \) and \( 1^*, f_0^*, v^* \) the interpretations of 1, \( f_0, v \) by \( I^* \). We write also \( f_1^*(x) \) instead of \( v^*(x,1^*) \), and \( xy \) for \( v^*(x,y) \) resp. \( v(x,y) \). For interpreted formulas we use usual print instead of bold print.

5. The \( n \)-place predicate \( P \) on \( \mathbb{N}_1 \) is called semirepresented in \( \mathbb{N}^* \) by the C-formula \( A \) iff \( A \) is \( n \)-place and for all natural numbers \( i_1, \ldots, i_n \)

(a) in case of \( P_{i_1} \ldots i_n \) the formula \( A(i_1^*, \ldots , i_n^*) \) can be proved in \( \mathbb{N}^* \),

(b) the standard interpretation \( I^* \) on \( \mathbb{N}_1 \) is a model of \( A(i_1^*, \ldots , i_n^*) \) iff \( P_{i_1} \ldots i_n \).

6. A C-formula is called \( V \)-bounded iff it is generated from equations and negated equations by a finite number of the following steps:

a) from \( A, B \) to \( (A \land B) \) or \( (A \lor B) \).

b) from \( A \) to \( \exists x \, A \).

c) from \( A \) to \( \forall x \, \neg x \text{END} y \, A \), also written \( \forall x \, x \text{END} y \, A \).

**Lemma 1.** In \( \mathbb{N}^* \) the following formulas are provable:

1. \( \forall x y \, xy \neq 1 \)
2. \( \forall x y \, xy \neq y \)
3. \( \forall x y z \, (xy = f_0(z) \rightarrow \exists u \, (y = f_0(u) \land xu = z)) \)
4. \( \forall x y z \, (xy = z1 \rightarrow y = 1 \land x = z \lor \exists u \, (y = u1 \land xu = z)) \)

**Proof.**
We prove the validity of the formulas in any model \( I \) on \( D \).

1. According to \( A_1 \) we have

\[ y=1 \land \exists u \, y = f_0(u) \land \exists u \, y = f_1(u). \]

With \( A_4 \) we get

\[ xy = f_1(u) \] for a certain \( u \) and \( i=0 \) or \( i=1 \), hence with \( A_1 \) the proposition.

2. Let be \( xy = y \), thus

\[ xy = x(xy), \text{ thus with } A_4 \]

\[ xy = (xx)y, \text{ thus with } A_2 \]

\[ x = xx, \text{ contradiction to } A_3. \]

3. Let be \( xy = f_0(z) \). With \( A_1 \) we get

\[ y=1 \land \exists u \, y = f_0(u) \land \exists u \, y = f_1(u), \text{ thus with } A_4 \text{ and } A_1 \]

\[ y = f_0(u) \] for some \( u \), thus with \( A_4 \)

\[ xy = f_0(xu), \text{ thus with } A_2 \]

\[ xu = z, \]

4. Let be \( xy = z1 \).

In case of \( y=1 \) we have with \( A_2 \)

\[ x = z. \]

In case of \( y = u1 \) we have with \( A_4 \)

\[ xy = (xu)1, \text{ thus with } A_2 \]

\[ xu = z. \]

In case of \( y = f_0(u) \) we have with \( A_4 \)

\[ xy = f_0(xu). \]
Contradiction to $A_1$.

**Lemma 2.**
1. For any $x, y$ of $\mathbb{N}_1$ with $x=y$ resp. $\neg x=y$ the formulas $x^* = y^*$ resp. $\neg x^* = y^*$ are provable in $\mathcal{A}_1$.
2. In any model $I$ on $D$ of the axioms we write $D_0$ for the intersection of all subdomains of $D$ containing 1 and for any $x$ also $f_0(x)$ and $f_1(x)$.
   Then all axioms are valid on $D_0$. The restriction of $I$ to $D_0$ is isomorphic to the standard model $I^*$ on $\mathbb{N}_1$. A closed $\forall \exists$-bounded formula is valid for $I$ on $D$, if it is valid for the restriction of $I$ to $D_0$.

**Proof.**
1. We have to prove the validity of the formulas in any model $I$ on $D$. For $x=y$ the proposition is evident. For $x\neq y$ the assumption $x^* = y^*$ leads to a contradiction because of the first two axioms.
2. For the restriction of $I$ to $D_0$ the following second order axiom is valid:
   \[
   A_6 \quad \forall M \ (M_1 \land \forall x (Mx \rightarrow Mf_0(x) \land Mf_1(x)) \rightarrow \forall x Mx).
   \]
   Furthermore we use the following axiom instead of $A_2$:
   \[
   A_2^* \quad \forall x \forall y \ (f_1(x) = f_1(y) \rightarrow x = y)
   \]
   We show first that the axiom system \{\$A_1, A_2^*, A_6\$\} is monomorphic by defining an isomorphism $\phi$ relative to $1, f_0, f_1$ from the standard interpretation $I^*$ on $\mathbb{N}_1$ to the interpretation $I$ restricted to $D_0$. Let
   \[
   \phi(1^*) = 1
   \]
   \[
   \phi(f_0^*(x)) = f_0(\phi(x))
   \]
   \[
   \phi(f_1^*(x)) = f_1(\phi(x)).
   \]
   Because of the definition we have just to show that $\phi$ is a one-to-one mapping onto $D_0$.
   1) By induction on $x$ we show $x\neq y \rightarrow \phi(x) \neq \phi(y)$.
      a) For $x=1^*$ and $y = f_i^*(u)$ for some $u$ and $0 \leq i \leq 1$ we have $\phi(x) = 1$ and $\phi(y) = f_i(\phi(u))$, thus $\phi(y) \neq 1$ using $A_1$.
      b) Let $x = f_i^*(u)$ for some $u$ and $x \neq y$. For $y=1^*$ compare a). Now let $y = f_k^*(v)$ for some $v$ and some $k$ with $0 \leq k \leq 1$. Because of $x \neq y$ we have $i \neq k$ or $i = k \land u \neq v$.
   2) Because of $A_6$ the set of values of $\phi$ is $D_0$.
   3) Because of $A_4$ the values of the function $\nu$ are in $D_0$ if the arguments belong to $D_0$. $\phi$ is also an isomorphism from $I^*$ on $\mathbb{N}_1$ to $I$ restricted to $D_0$ relative to $\nu$, that means: For any $x, y$ of $\mathbb{N}_1$
   \[
   \phi(\nu^*(x, y)) = \nu(\phi(x), \phi(y)).
   \]
   This is easily shown by induction on $y$.
   Therefore the axioms are also valid for the restriction of $I$ to $D_0$.
   4) We prove for any $x$ of $D$ by induction on $z$:
   \[
   z \in D_0 \land x \in D z \rightarrow x \in D_0.
   \]
   a) For $z=1$ we get the proposition with Lemma 1.1.
b) Induction step from $z$ to $f_i(z)$ for $i=0,1$.
For $x\text{END}f_1(z)$ we get $x=f_1(z)$ (thus $x\epsilon D_0$) or
$yx = f_1(z)$ for some $y$.
For $i=0$ we have $x=f_0(u)$ for some $u$ according to Lemma 1.3 and
$yu = z$, because of the induction hypothesis $u\epsilon D_0$ and therefore $x\epsilon D_0$.
For $i=1$ we get with Lemma 1.4
$x=1 \land y=z$, thus $x\epsilon D_0$ or
$x = u1$ for some $u$ and
$yu = z$,
thus because of the induction hypothesis $u\epsilon D_0$ and therefore $x\epsilon D_0$.

DEFINITION.
1) We write $x$ instead of $x_1...x_n$ resp. $x_1,...,x_n$.
2) Let be $\mathbb{N}_S$ the set of natural numbers of $\mathbb{N}_1$, having only the digit 1 in the
binary representation (series of strokes).
3) A predicate on $\mathbb{N}_1$ belongs to $RE^*$ iff it can be generated from equations and
negated equations by a finite number of the following steps:
a) the composition of predicates with $\land$ or $\lor$,
b) the use of an unbounded $\exists$-quantifier,
c) the use of a bounded $\forall$-quantifier, that means the transition from $Q$ to
$Pbx \leftrightarrow \forall a END^+ b Qxa$
where
$xEND^+ y \leftrightarrow x=y \lor \exists z \ y=\ nu^*(z,x)$,
d) permutation or identification of variables.
In the equations and negated equations function terms are permitted only if they
consist just of variables, $1^*$, $f_0^*$ and $u^*$.

ABBREVIATIONS.

$x\text{BE Gy}$ for $x = y \lor \exists z \ xz = y$ \quad ($x$ is a beginning of $y$)
10 for $f_0(1)$
100 for $f_0(f_0(1))$
101 for $f_1(f_0(1))$
$\neg 100\text{BE Gz}$ for $z = 10 \lor 11\text{BE Gz} \lor 101\text{BE Gz}$
$\mathbb{N}_S z$ for $1\text{EN Dz} \land \forall u \ u\text{EN Dz} \ \exists v \ (z = vu \land 1\text{EN Dv})$ \quad ($z$ is a series of strokes)
x$\text{PWy}$ for $x\text{BE Gy} \lor x\text{EN Dy} \lor \exists uv \ y = uvx$ \quad ($x$ is part of $y$)
$<x,y>$ for $f_0(x)f_0(y)$

For the corresponding abbreviations in an arbitrary model $I$ on $D$ we choose usual
letters instead of bold letters. If $I$ is the standard model $I^*$ on $\mathbb{N}_1$, we add
sometimes the symbol $^*$ in order to distinguish two models.

THEOREM 1.
Every recursively enumerable predicate on $\mathbb{N}_S$ is semirepresented by a $\forall$-bounded
formula in $\mathbb{N}_S^*$.

PROOF.
We have to show that the graph of any primitive recursive function on $\mathbb{N}_S$ belongs
to $RE^*$. The demand 5(a) in the definition ahead of Lemma 1 is a consequence of
5(b) according to Lemma 2.2 because we use $\forall$-bounded formulas.
We omit here the symbol $^*$. 
1) \( y=1 \) belongs to \( \text{RE}^* \).
2) The graph of the successor function on \( \mathbb{N}_{SR} \) has on \( \mathbb{N}_1 \) the representation
\[ \text{SR} x \land y = f_1(x). \]
3) The graph of \( \lambda x_1 \ldots x_n \, x_i \) on \( \mathbb{N}_{SR} \) has on \( \mathbb{N}_1 \) the representation
\[ \text{SR} \, \ldots \, \text{SR} x_n \land y = x_i. \]
4) The graph of \( f(x) = h(g_1(x), \ldots, g_k(x)) \) on \( \mathbb{N}_{SR} \) has the representation
\[ z = f(x) \iff \exists y_1 \ldots y_m \ (z = h(y_1, \ldots, y_m) \land y_1 = g_1(x) \land \ldots \land y_m = g_m(x)). \]
5) Let \( f(x, y) = h(x, y, f(x, y)) \) on \( \mathbb{N}_{SR} \). The graph of \( f \) has on \( \mathbb{N}_1 \) the representation
\[ z = f(x, y) \iff \exists a \ (100 < y, z, a) \land b \land \text{BEa} \land \exists d \ (100 < c, d) \land \text{BEc} \land (c=p \land d=f(x) \land \exists uvw \ (c=f_1(u) \land d=h(x, u, v) \land w \land \text{BEc} \land 100 < u, v, w)). \]

We choose \( a = 100 < y, f(x, y) > 100 < y-1, f(x, y-1) > 100 < 2, f(x, 2) > 100 < 1, f(x, 1) > . \)

We show by induction on the length of the binary representation of \( b \)
\[ \text{BEa} \land 100 < p, q > \text{BEb} \land \text{SRp} \land \text{SRq} \land q = f(x, p). \]

Let \( \text{BEa} \land 100 < p, q > \text{BEb} \). (*)
Then there are \( c, d \) with \( 100 < c, d > \text{BEc} \), hence \( c=p \) and \( d=q \).

\( \alpha \) For the shortest \( b \) with (*) we have
\( c=1 \land d=g(x), \) hence \( d=f(x, 1) \).

\( \beta \) Induction step:
For \( c=1 \) we get again \( d=g(x) \). Otherwise we have for some \( u, v, w \)
\( c=f_1(u) \land d=h(x, u, v) \land w \land \text{BEa} \land 100 < u, v > \text{BEb} \).
Because of \( 100 < u, v > \text{BEb} \) we have \( w \neq b \). According to the induction hypothesis we have
\[ \text{SRu} \land \text{SRv} \land v = f(x, u). \]
Thus we get
\[ \text{SRc} \land \text{SRd} \land d = f(x, c). \]q.e.d.

For the proof of Gödel's two incompleteness theorems we assume any Gödel-numbering of the C-formulas with Gödelnumbers in \( \mathbb{N}_{SR} \). Furthermore we define a Gödel-numbering of all proofs of C-formulas from a set \( U \) of C-formulas.

We choose a correct and complete first order calculus where any rule has not more than two premises and where the following two predicates \( H \) and \( K \) are recursive:
\( H \) is the set of Gödelnumbers of C-formulas being logical axioms of the calculus.
\( K \) is the set of Gödelnumbers of C-formulas.
Proofs of C-formulas in (from) a set \( U \) of C-formulas are finite series \( C_1, \ldots, C_k \)
of C-formulas where for any \( i \) with \( 1 \leq i \leq k \):
\( C_i \in U \), or \( C_i \) is a logical axiom, or there are \( r, s \) with \( 1 \leq r, s, i-1 \), such that \( C_i \) is gained from \( C_r \) and \( C_s \) by a rule of the calculus.
As Gödelnumber of the proof $C_1, ..., C_k$ we choose the natural number
\[100f_0^*(a_k)100f_0^*(a_{k-1})100.........100f_0^*(a_1)1,\]
where $a_1, ..., a_k$ are the Gödelnumbers of $C_1, ..., C_k$ in $\mathbb{N}_{SR}$.

The following predicates $S, F$ are decidable and therefore recursive according to Church’s thesis. If we want to prove this without using Church’s thesis, we have to specify the Gödelnumbering of the $C$-formulas. We omit these details.

Let $M$ be the set of Gödelnumbers of $C$-formulas belonging to $U$.

**DEFINITION.**
1) $S_M(a)$ is the Gödelnumber of a $C$-formula and $c$ is the Gödelnumber of a proof in $U$ of the formula belonging to $a$.

We use this predicate only for a recursive $M$.

In case of $U=\mathbb{N}^*$, we write $S_c$ instead of $S_M$.

Let $S(x, y)$ be a $\forall$-bounded formula semirepresenting the predicate $S$. (Because the Gödelnumbers of a proof do not belong to $\mathbb{N}_{SR}$, we cannot use Theorem 1 for getting such a formula. But later on we will define this formula.

2) $B_c$ is the Gödelnumber of a $C$-formula and $c$ is the Gödelnumber of a proof in $\mathbb{N}^*$ of the diagonal formula of $a$.

The diagonal formula of a one-place $C$-formula $A(x)$ with the Gödelnumber $a$ is the formula $A(a^*)$. The diagonal formula of any other formula is the formula itself.

3) If $x$ is the Gödelnumber of a one-place $C$-formula, let $f(x)$ be the Gödelnumber of the diagonal formula. Otherwise, we choose $f(x) = x$.

Let $F(x, w)$ be a $\forall$-bounded formula semirepresenting the predicate $F$.

4) $B(x, y) \equiv \forall w (F(x, w) \to S(w, y))$.

The formula $B$ is not $\forall$-bounded. But in the standard model $I^*$ on $\mathbb{N}$ we have:

$B(x, y)$ is valid for $I^*$ iff $B^{I^*}(x)I^*(y)$.

However, we have not a semirepresentation.

5) We call $b$ the Gödelnumber of the formula $\neg\exists y B(x, y)$ and $d$ the Gödelnumber of the diagonal formula $\neg\exists y B(b^*, y)$.

6) We call $e$ the Gödelnumber of the formula $\exists x \neg x = x$.

**THEOREM 2 (GÖDEL'S FIRST INCOMPLETENESS THEOREM FOR $\mathbb{N}^*$):**

If $\mathbb{N}^*$ is consistent, there is no proof in $\mathbb{N}^*$ of the formula $\neg\exists y B(b^*, y)$.

However, in the standard model $I^*$ on $\mathbb{N}$, the formula $\neg\exists y B(b^*, y)$ is valid, i.e. $\neg\exists y Bby$.

**PROOF.** For any proof of $\neg\exists y B(b^*, y)$, we get $\exists y Bby$. But the formula is true in $I^*$ when it is provable, i.e. $\neg\exists y Bby$. (According to [1] and Lemma 2.2 $\mathbb{N}^*$ is valid in $I^*$ if it is consistent.)

$\neg\exists y Bby$ means that there is no proof of $\exists x \neg x = x$. This is a very natural formulation of the consistency of $\mathbb{N}^*$.

Gödel’s second incompleteness theorem asserts for a natural choice of the $C$-formula $S$ that also $\neg\exists y S(e^*, y)$ is not provable in $\mathbb{N}^*$ (if $\mathbb{N}^*$ is consistent), that means a formula expressing the consistency in the standard model on $\mathbb{N}$ in a natural way. For the second incompleteness theorem we need a proof in $\mathbb{N}^*$ of $\neg\exists y S(e^*, y) \to \neg\exists y B(b^*, y)$.

We first show that there is a proof in $\mathbb{N}^*$ of $\neg\exists y S(d^*, y) \to \neg\exists y B(b^*, y)$.
For otherwise we would have in a model $I$ both $\neg \exists y S(d^*,y)$ and $\exists y B(b^*,y)$.
Because of the semirepresentation of $F$ by $F$ we have $F(b^*,d^*)$ in $I$, because of
$\exists y B(b^*,y)$ therefore $\exists y S(d^*,y)$. Contradiction.
Thus we have to prove in $\mathfrak{M}^*$
$$\neg \exists y S(e^*,y) \rightarrow \neg \exists y S(d^*,y),$$
i.e.
$$\exists y S(d^*,y) \rightarrow \exists y S(e^*,y). \tag{*}$$
Because of the First Incompleteness Theorem we have $\neg \exists y S e y \rightarrow \neg \exists y S d y$, i.e.
$$\exists y S d y \rightarrow \exists y S e y. \tag{**}$$
To prove (**) we have so show the validity of (*) in any model of $\mathfrak{M}^*$, whereas (***)
asserts the validity only in the standard model.

LEMMA 3. If $I$ is the standard model $I^*$ on $\mathfrak{M}_1$, we have:

$S_{\mathbf{Mac}} \rightarrow \text{Ga} \land 100f_0(a)\text{BEGc}$
$$\land \ \forall b \ \exists! b \ \exists! (\neg 100\text{BEGb})$$
$$\lor b = 100f_0(u)1 \land (\text{Mu} \lor \text{Hu})$$
$$\lor b = 100f_0(u)100f_0(p)q$$
$$\land (\text{Mu} \lor \text{Hu})$$
$$\lor \exists vw(100f_0(v)PW100f_0(p)q \land 100f_0(w)PW100f_0(p)q \land Kwvu))$$

PROOF.
"$\rightarrow$": Because of $S_{\mathbf{Mac}}$ we have
$$c = 100f_0(a_k)100f_0(a_{k-1})100\ldots 100f_0(a_1)1,$$ where
$$a_k, \ldots a_1$$ is a series of Gödel-numbers of C-formulas with $a_k=a$ and for $0 \leq i \leq k$:
$$a_i \in \text{EM}, \text{ or } H_1. \text{ or there are } r,s \text{ with } 1 \leq r,s \leq -1 \text{ and } Ka_rA_sA_i.$$
The proposition is verified easily.
"$\leftarrow$":
Let
$$b = 100f_0(u)1 \land (\text{Mu} \lor \text{Hu}),$$
then there are $u,p,q$ with
$$b = 100f_0(u)100f_0(p)q$$
$$\land (\text{Mu} \lor \text{Hu}) \lor \exists vw(100f_0(v)PW100f_0(p)q \land 100f_0(w)PW100f_0(p)q \land Kwvu))$$
By induction on the length (number of digits) of $b$ we show
$$S_{\mathbf{Mac}}.$$ (1)

LEMMA 4. In $\mathfrak{M}^*$ the following formulas are provable:
1. $\forall \text{abcde} (\text{abcde = cde} \rightarrow \text{ab Begc} \land \text{dendb} \lor \text{c Bega} \land \text{b endd})$
2. $\forall \text{ab} \neq 100$

PROOF. We show that the formulas are valid in any model of $\mathfrak{M}^*$ by using $A_5$.
1. a) Let $ab = cd \land d \text{Endb}$.
For $b=d$ we get with $A_2 a=c$, thus $ab \text{Begc}$.
For $b \neq d$ we get for a certain $u$
b = ud, thus
aud = cd, hence with A₂
au = c, thus again aBEGc.
b) Let ab = cd \land \neg d\text{END}b, thus according to A₅
b\text{END}d \land b \neq d, thus for some u
d = ub, thus
ab = cub, thus with A₂
a = cu, thus c\text{BEG}a.
2. With ab = f₀(f₀(1)) we get for some u
b = f₀(u), thus
au = f₀(1), thus for some v
av = 1, contradicting Lemma 1.1.

LEMMA 5. In \mathcal{M}ⁿ the following formula is provable:
b\text{END}y₁y₂ \land 100\text{BEG}b
\rightarrow b\text{END}y₂ \lor \exists p (b=py₂ \land p\text{END}y₁ \land 100\text{BEG}p)

PROOF. We show the validity of the formula in any model of \mathcal{M}ⁿ.
Case 1: Let b = y₁y₂.
We choose p = y₁ and show 100\text{BEG}p:
Because of 100\text{BEG}y₁y₂ and Lemma 4.2 we have for some v
100v = y₁y₂.
With Lemma 4.1 we get
1.1 100\text{BEG}y₁ \land y₂\text{END}v 
or
1.2 y₁\text{BEG}100 \land v\text{END}y₂.
Case 1.1: 100\text{BEG}p is evident.
Case 1.2: According to Lemma 4.2 we have y₁ = 100, thus again 100\text{BEG}p.
Case 2: Let ub = y₁y₂.
According to Lemma 4.1 we get
u\text{BEG}y₁ \land y₂\text{END}b \lor y₁\text{BEG}u \land b\text{END}y₂.
For b\text{END}y₂ the proposition is proved. Therefore we can assume
u\text{BEG}y₁ \land b = py₂ for some p.
Because of 100\text{BEG}b and Lemma 4.2 we have
py₂ = 100w.
Like in case 1 we get
100\text{BEG}p.
With ub = y₁y₂ and b = py₂ we get also
upy₂ = y₁y₂, thus with A₂
up = y₁, thus
p\text{END}y₁.
q.e.d.

Now we choose \forall-bounded formulas G, M, H, K semirepresenting the predicates G, M, H, K (compare Theorem 1).
$S_{M}[x,y] \equiv G(x) \land 100f_0(x)BEGy$

$\land \forall b \ bENDy \ \exists upq (\neg 100BEGb$

$\lor b = 100f_0(u)1 \land (M(u) \lor H(u))$

$\lor b = 100f_0(u)100f_0(p)q$

$\land (M(u) \lor H(u))$

$\lor \exists vw(100f_0(v)PW100f_0(p)q \land 100f_0(w)PW100f_0(p)q \land K(w,v,u)))$

According to Lemma 3 the predicate $S_{Mac}$ is semirepresented by the $\forall$-bounded formula $S_{M}[x,y]$. Special cases:

1) $M$ is the set $M^*$ of Godel numbers of the formulas of $\mathbb{M}^*$. Then we have $S_{Mac} \leftrightarrow Sac$. Instead of $S_{M^*}[x,y]$ we write $S[x,y].$

2) $M$ is the set $M^{**} = M^*u\{d\}$; as $M[p]$ we chose $M^*[p] \lor p = d^*$, where the $\forall$-bounded formula $M^*[p]$ is semirepresenting the set $M^*$.

THEOREM 3 (GOEDEL’S SECOND INCOMPLETENESS THEOREM FOR $\mathbb{M}^*$):

If $\mathbb{M}^*$ is consistent, there is no proof of $\neg \exists y S(e^*,y)$ in $\mathbb{M}^*$.

PROOF. The denotations $I,D,D_0$ are used as in Lemma 2.2.

We have to show the validity of (*) for $I$. Because of (**) we have $S_{M^*}(e^*,y)$ in the standard model $I^*$ on $\mathbb{N}_1$ for a certain choice of $I^*(y)$, therefore also for the restriction of $I$ to $D_0$ with a certain choice of $I(y)=y_1$ (compare Lemma 2.2), therefore also for $I$ on $D$, because the formula is $\forall$-bounded. Furthermore we assume the validity of $S(d^*,y)$ in $I$ for a certain choice of $I(y)=y_2$.

Let $y_3 = y_1y_2$.

Proposition: $S(e^*,y)$ is valid in $I$ for $I(y)=y_3$. Obviously $e$ is Goedel number of a $C$-formula, and we have $100f_0(e^*)BEGy_1y_2$ because of $100f_0(e^*)BEGy_1$.

Therefore we have just to prove for any $b$ with $bENDy_1y_2 \land 100BEGb$ for some $u,p,q$:

$b = 100f_0(u)1 \land (M^*(u) \lor H(u))$ (1)

$\lor b = 100f_0(u)100f_0(p)q$

$\land (M^*(u) \lor H(u)) \lor \exists vw(100f_0(v)PW100f_0(p)q \land 100f_0(w)PW100f_0(p)q \land K(w,v,u)))$ (2)

According to Lemma 5 we have one of the following two cases:

1) $bENDy_2$

2) $b=py_2 \land pENDy_1 \land 100BEGp$ for some $p$.

In the first case we get (1) or (2) from the validity of $S(d^*,y)$ in $I$ for $I(y)=y_2$.

In the second case we get from the validity of $S_{M^{**}}(e^*,y)$ in $I$ for $I(y)=y_1$ some $r,s$ with

$p = 100f_0(t)1 \land (M^*(t) \lor t=d^* \lor H(t))$ or

$p = 100f_0(t)100f_0(r)s \land (M^*(t) \lor t=d^* \lor H(t))$

$\lor \exists vw(100f_0(v)PW100f_0(r)s \land 100f_0(w)PW100f_0(r)s \land K(w,v,u)))$

For $u = t$ we get (2) because of $b=py_2$, using in case of $t=d^*$ again the validity of $S(d^*,y)$ in $I$ for $I(y)=y_2$. 


REFERENCES.


